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PERTURBATION OF ORDINARY DIFFERENTIAL OPERATORS[1]

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1. Introduction.

In the present paper L will denote an operator in the Hilbert space $\mathscr{L}^2(a,b)$ derived from a formally selfadjoint, in general singular, ordinary differential operator of order $2n$ and B will denote an operator of the form $By=\beta y^{(k)}$ of order k, $0 \leqq k \leqq 2n-1$.

We find conditions under which, in the terminology of Wolf [6] the operator B is L-compact. The study of such conditions is of interest in view of the fact that perturbation of the operator L by an L-compact operator leaves the essential spectrum unchanged (the spectrum of L is divided into the set of isolated eigenvalues of finite multiplicity and the rest of the spectrum, which is called the essential spectrum). For the basic results concerning L-compact perturbations we refer to the paper by Gokhberg and Krein [2] in which general Banach spaces are considered, and to the paper by Wolf [6], where a simpler and more detailed treatment is given for the case of Hilbert spaces.

As was pointed out to the author by professor Kuroda (see Kuroda [4]), it is of importance for some questions of quantum mechanics to know whether B is of L-Hilbert–Schmidt type (see definition 3.5), and in the theorems 5I,1, 5II,2 and 5II,3 we give necessary and sufficient conditions for this.

It turns out that a necessary condition that B be defined on the domain of L is that $\beta \in \mathscr{L}^2_{\mathrm{loc}}(a,b)$, and the sufficient conditions for B to be L-compact or of L-Hilbert–Schmidt type are growth conditions on

$$\int_\alpha^\beta |\beta(x)|^2 dx$$

for $\alpha \to a$ and $\beta \to b$.

The main results are formulated in the theorems of section 5: I 1,3,4; II 2,3,5,6 and III 2,3,5,7.

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2. Definition of the operators.

a) *The unperturbed operators.* The terminology of Neumark [5, Kap. V] will be used.

Let (a,b) be an interval, where $a=-\infty$ and $b=+\infty$ are allowed as boundary points, and let

$$p_0, p_1, \ldots, p_n$$

be real-valued functions on (a,b), such that $1/p_0, p_1, \ldots, p_n$ are locally integrable. The quasi-derivatives $y^{[k]}$ of a complex-valued function y on (a,b) are defined by

$$y^{[k]} = \frac{d^k y}{dx^k} \qquad \text{for } k = 0,1,\ldots,n-1 ,$$

$$y^{[n]} = p_0 \frac{d^n y}{dx^n},$$

$$y^{[n+k]} = p_k \frac{d^{n-k} y}{dx^{n-k}} - \frac{d}{dx} y^{[n+k-1]} \qquad \text{for } k = 1,2,\ldots,n .$$

The formal differential operator l is defined by

$$D(l) = \{y \mid y^{[k]} \text{ exist, loc. a.c., for } 0 \leqq k \leqq 2n-1\}$$

and

$$l(y) = y^{[2n]} \quad \text{for} \quad y \in D(l) .$$

Corresponding to l we consider the following operators in $\mathscr{L}^2(a,b)$: The maximal operator L defined by

$$D_L = D = \{y \mid y \in D(l) \cap \mathscr{L}^2(a,b),\, l(y) \in \mathscr{L}^2(a,b)\}$$

and

$$Ly = l(y) \quad \text{for} \quad y \in D .$$

L_0' is the restriction of L to the set D_0' of functions in D with compact support. The minimal operator L_0 is the closure of L_0', its domain is denoted by D_0. Finally, L_c will denote a closed extension of L_0, D_c its domain, L_s a self-adjoint extension of L_0 and D_s its domain.

b) *The perturbing operators.*

The theory of L_c-compactness requires that B is a closable operator defined on D_c.

In the following, when $k = n+\nu+1$ with $0 \leqq \nu \leqq n-1$, we assume, that $(1/p_0)^{(\nu)}$ and $p_r^{(\nu-r)}$ exist, loc. a.c., for $1 \leqq r \leqq \nu-1$. This implies, that $y^{(n+\nu)}$ exists, loc. a.c., and hence the existence of $y^{(k)}$ for all $y \in D$. Then we can define for every complex-valued function β on (a,b) the formal differential operator b_k by

$$D(b_k) = D$$

and

$$b_k(y) = \beta y^{(k)} \quad \text{for} \quad y \in D(b_k) .$$

If $b_k(y) \in \mathscr{L}^2(a,b)$ for $y \in D_c$ and for some fixed k, $0 \leqq k \leqq 2n-1$, we define the operator $B_k{}'$ by

$$D_{B_k{}'} = D_c$$

and

$$B_k{}'y = b_k(y) \quad \text{for} \quad y \in D_{B_k{}'} .$$

LEMMA 2.1. *Suppose that $B_k{}'$ is defined on D_c, and that β is* a.e. *equal to a function β_1 with the following properties:*

(1) *$S = \{x \mid x \in (a,b), \beta_1(x) = 0\}$ is closed.*
(2) *For every interval $[\alpha,\beta] \subset (a,b)\setminus S$ there exists a $K_{\alpha,\beta} > 0$ such that $1/\beta_1(x) < K_{\alpha,\beta}$ for $\alpha \leqq x \leqq \beta$.*

Then $B_k{}'$ is closable.

This form of the conditions is due to conversations with T. Gamelin.

PROOF. a) We consider first the case $\beta(x) \equiv 1$. Let B'_{k0} be the restriction of $B_k{}'$ to functions with compact support. It is easy to prove that $B_k{}'$ is contained in B'^{*}_{k0} and hence closable.

b) In the general case let
(i) $y_r \to_{r\to\infty} 0$ in $\mathscr{L}^2(a,b)$,
(ii) $B_k{}'y_r \to_{r\to\infty} z$ in $\mathscr{L}^2(a,b)$.

From (ii) it follows, that $z(x) = 0$ a.e. for $x \in S$. Also for any interval $[\alpha,\beta] \subset (a,b)\setminus S$, the conditions (2) and (ii) imply

$$\int_\alpha^\beta \left| \frac{z(x)}{\beta(x)} - y_r{}^{(k)}(x) \right|^2 dx \underset{r\to\infty}{\longrightarrow} 0$$

From a) it follows that $z(x) = 0$ a.e. on $[\alpha,\beta]$. Hence $z = 0$ in $\mathscr{L}^2(a,b)$, and the lemma is proved.

In the following we shall assume that β is a.e. equal to a function β_1, having the properties (1) and (2) stated in lemma 2.1, so that $B_k{}'$, whenever defined on D_c, is closable.

3. Formulation of the problem.

DEFINITION 3.1. For any closed operator A in a Hilbert space H with norm $\|\cdot\|$, we define the A-norm of $x \in D_A$ by

$$\|x\|_A^2 = \|x\|^2 + \|Ax\|^2 .$$

Then D_A is a Hilbert space with the A-norm.

DEFINITION 3.2. A set $S \subset D_A$ is said to be A-bounded, if $\|x\|_A < K$ for $x \in S$. An operator B defined on D_A is said to be A-defined. When B maps every A-bounded set into a bounded set, B is called A-bounded. When B maps every A-bounded set into a precompact set, B is said to be A-compact.

REMARK 3.3. When A is closed, and B is a closable A-defined operator, B is A-bounded.

LEMMA 3.4. *B is A-compact if, and only if, $B(A-\lambda)^{-1}$ is compact for some λ in $\varrho(A)$, the resolvent set of A (or, equivalently, for all $\lambda \in \varrho(A)$).*

DEFINITION 3.5. B is said to be of A-Hilbert–Schmidt type, if $B(A-\lambda)^{-1}$ is a Hilbert–Schmidt operator for some $\lambda \in \varrho(A)$ (for all $\lambda \in \varrho(A)$).

For every L_c and k, $0 \leqq k \leqq 2n-1$, we shall consider the following problem: For which functions β is B_k' an L_c-compact operator, resp. of L_c-Hilbert–Schmidt type?

Instead of treating this problem directly we consider the corresponding problem for the quasi-derivatives: Let the operator B_k be defined by

$$B_k y = \beta y^{[k]} .$$

For which functions β is B_k an L_c-defined and L_c-compact operator, resp. of L_c-Hilbert–Schmidt type?

For $0 \leqq k \leqq n-1$ we have $B_k = B_k'$; for $k=n$ and $p_0(x) \neq 0$ a.e. the solution of the problem for B_k can immediately be applied to B_k'. For $n+1 \leqq k \leqq 2n-1$ the derivatives $y^{(k)}$ can be expressed linearly by the $y^{[s]}$, $s = 2n-k, \ldots, k$, with certain functions of the $p_r^{(k-n-q)}$, $q = r, \ldots, k-n$, $r = 0, \ldots, k-n$, as coefficients; then the results for the B_s can be applied to B_k', at least to give sufficient conditions on β in order that B_k' be L_c-compact.

4. Local conditions and reduction of the main problem.

LEMMA 4.1. *In the regular case, i.e. when (a,b) is finite, and $1/p_0$, $p_1, \ldots, p_n$ are integrable on (a,b), for any set of complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{2n-1}, \beta_0, \beta_1, \ldots, \beta_{2n-1}$, there exists a function $y \in D$ such that*

$$y^{[k]}(a) = \alpha_k,\ y^{[k]}(b) = \beta_k, \qquad k = 0,1,\ldots,2n-1\ .$$

PROOF. See Neumark [5, § 17.3, lemma 2].

LEMMA 4.2. *For every L and k a necessary condition in order that B_k be L_0'-defined is that $\beta \in \mathscr{L}^2_{\text{loc}}(a,b)$, that is*

$$\int_c |\beta(x)|^2\, dx < \infty$$

for every compact subset c of (a,b).

PROOF. By means of lemma 4.1 it is simple to construct for any $x_0 \in (a,b)$ a function $y_0 \in D_0'$ such that

$$y_0^{[k]}(x_0) = 1\ .$$

From this and the continuity of $y_0^{[k]}$ the conclusion of the lemma follows.

LEMMA 4.3. *In the regular case $\beta \in \mathscr{L}^2(a,b)$ implies that B_k is L-compact for $k=0,1,\ldots,2n-1$.*

PROOF. Let $y_1,y_2,\ldots,y_{2n}$ be a system of linearly independent solutions of the equation

$$(l-\lambda)y = 0 \qquad \text{for some non-real } \lambda\ ,$$

normed such that the Wronskian is 1. Set

$$\text{(1)} \qquad v_k(x) = \begin{vmatrix} y_1(x) & \ldots & y_{k-1}(x) & y_{k+1}(x) & \ldots & y_{2n}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{[2n-2]}(x) & \ldots & y_{k-1}^{[2n-2]}(x) & y_{k+1}^{[2n-2]}(x) & \ldots & y_{2n}^{[2n-2]}(x) \end{vmatrix}\ .$$

Then the solution of the equation

$$(l-\lambda)z = f$$

and its quasi-derivatives are given by

$$\text{(2)} \quad z^{[k]}(x) = \sum_{i=1}^{2n} c_i y_i^{[k]}(x) + \sum_{i=1}^{2n} y_i^{[k]}(x) \int_0^x v_i(\xi) f(\xi)\, d\xi, \qquad k = 0,1,\ldots,2n-1\ .$$

Let $\{z_s\}$ be an L-bounded sequence. Then by (2) for $k=0$ and Schwarz' inequality

$$\left\{ \sum_{i=1}^{2n} c_{is} y_i \right\} \text{ is bounded in } \mathscr{L}^2(a,b)\ ,$$

and hence

$$|c_{is}| < K \quad \text{for} \quad i = 1,2,\ldots,2n, \; s = 1,2,\ldots .$$

We choose a subsequence $\{z_{s_t}\}$ such that

$$c_{is_t} \underset{t\to\infty}{\longrightarrow} k_i, \qquad i = 1,2,\ldots,2n$$

and

$$f_{s_t} \underset{t\to\infty}{\longrightarrow} f, \qquad \text{weakly} .$$

Then by Lebesgue's dominated convergence-theorem

$$B_k y_{s_t} \underset{t\to\infty}{\longrightarrow} \beta(x)\left\{\sum_{i=1}^{2n} k_i y_i^{[k]}(x) + \sum_{i=1}^{2n} y_i^{[k]}(x) \int_0^x v_i(\xi) f(\xi)\, d\xi\right\}$$

in $\mathscr{L}^2(a,b)$.

Definition 4.4. Unmixed boundary conditions are conditions of the form

$$\sum_{i=0}^{2n-1} \alpha_i y^{[i]}(a) = 0 \qquad \text{and} \qquad \sum_{i=0}^{2n-1} \beta_i y^{[i]}(b) = 0 .$$

Definition 4.5. When l is a formal differential operator applied to functions on (a,b), we shall denote by $L(\alpha,\beta)$ the maximal operator corresponding to l applied to functions on $[\alpha,\beta] \subset (a,b)$. If L_c is defined by certain unmixed boundary condtions at the endpoints a and b, and $a < \alpha < \beta < b$, we shall denote by $L_c(a,\alpha)$ the operator corresponding to l applied to functions on (a,α) and with the same boundary conditions at the point a as L_c, but with no boundary conditions at α. $L_c(\beta,b)$ is defined in the same way, and the same notation is used for B_k. If there are no boundary conditions at the point a, then $L_c(a,\alpha)$ shall mean $L(a,\alpha)$, and similarly for b.

Theorem 4.6. *Let the operator L_c be defined by certain unmixed boundary conditions at the endpoints a and b. Then a necessary and sufficient condition for the operator B_k to be L_c-bounded or L_c-compact is that*

(1) $\beta \in \mathscr{L}^2_{\text{loc}}(a,b)$

and

(2) *$B_k(a,\alpha)$ and $\beta_k(\beta,b)$ are L_c-bounded, resp. L_c-compact with respect to $L_c(a,\alpha)$ and $L_c(\beta,b)$ for some α,β with $a < \alpha < \beta < b$ (equivalently, for all such α,β).*

Proof. For every L_c-bounded sequence $\{y_n\}$ the restrictions of y_n to the intervals (a,α), (α,β) and (β,b) form $L_c(a,\alpha)$-bounded, $L_c(\alpha,\beta)$-bounded and $L_c(\beta,b)$-bounded sequences. From this follows the sufficiency.

By means of lemma 4.1 it is simple to construct to a given $L_c(a,\alpha)$-bounded sequence $\{z_n\}$ an L_c-bounded sequence $\{y_n\}$ such that z_n is the restriction of y_n to (a,α), and similarly for the intervals (α,β) and (β,b). From this follows the necessity.

By theorem 4.6 the problem concerning compactness is reduced to the following main cases:

I. The interval $[0,1]$ with both endpoints regular, boundary conditions at 0 and no boundary conditions at 1.

II. L and L_s on $[0,\infty)$ with 0 regular.

III. L on $[0,1)$ with 0 regular, 1 singular.

REMARK 4.7. Every theorem concerning L_c-compactness of the B_k remains valid if the operator L_c is changed by addition of a bounded function r to p_n. For, obviously, a set $S \subset D_c$ is L_c-bounded if, and only if, it is (L_c+r)-bounded.

5. Investigation of the main cases.

Case I: The interval $[0,1]$ with both endpoints regular, boundary conditions at 0 and no boundary conditions at 1.

THEOREM 5I,1. *Suppose that* $y^{[k]}(0)=0$ *is not a boundary condition for* L_c *for some* k *with* $0 \leqq k \leqq 2n-1$. *Then* $\beta \in \mathscr{L}^2(0,1)$ *is necessary for* B_k *to be* L_c*-bounded and sufficient for* B_k *to be* L_c*-compact. Also* $\beta \in \mathscr{L}^2(0,1)$ *is sufficient in order that* B_k *be of* L_s*-Hilbert–Schmidt type.*

PROOF. a) Suppose that B_k is L_c-bounded. The existence of a function $y \in D_c$, such that $y^{[k]}(0) \neq 0$, implies

$$\int_0^{\varepsilon} |\beta(x)|^2 \, dx < \infty \quad \text{for some } \varepsilon > 0 ,$$

and then by lemma 4.2 $\beta \in \mathscr{L}^2(0,1)$.

b) By lemma 4.3 $\beta \in \mathscr{L}^2(0,1)$ implies that B_k is L_c-compact.

For any L_s the c_i in the expression (2) for $z^{[k]}$ used in the proof of lemma 4.3 are bounded linear functionals of f for $i=1,2,\ldots,2n$. Then there exist functions $h_i \in \mathscr{L}^2(0,1)$, $i=1,2,\ldots,2n$, such that

$$c_i(f) = \int_0^1 h_i(\xi) f(\xi) \, d\xi .$$

By substitution of these expressions in (2) of lemma 4.3 we obtain the following representation of $(B_k-\lambda)^{-1}$:

$$(B_k-\lambda)^{-1}f(x) = \int_0^1 K_k(x,\xi)\,f(\xi)\,d\xi$$

with

$$K_k(x,\xi) = \begin{cases} \beta(x) \sum_{i=1}^{2n} y_i^{[k]}(x)\,[h_i(\xi)+v_i(\xi)]\,, & 0 \leqq \xi \leqq x\,, \\ \beta(x) \sum_{i=1}^{2n} y_i^{[k]}(x)\,h_i(\xi)\,, & x<\xi\,. \end{cases}$$

Since the $y_i^{[k]}$ and the v_i (see lemma 4.3 (1)) are continuous on $[0,1]$, it follows that $\beta \in \mathscr{L}^2[0,1]$ implies

$$\int_0^1\int_0^1 |K_k(x,\xi)|^2\,dx\,d\xi < \infty \quad \text{for} \quad k = 0,1,\ldots,2n-1\,.$$

Lemma 5I,2. *For* $0<a\leqq\infty$ *let* $f\in\mathscr{L}^2(0,a)$ *and set*

$$F(x) = \int_0^x f(t)\,dt\,.$$

Let ψ *be a complex-valued function on* $[0,a)$ *and let* T *be the operator in* $\mathscr{L}^2(0,a)$ *defined by*

$$D_T = \{f \mid f\in\mathscr{L}^2(0,a),\ \psi F\in\mathscr{L}^2(0,a)\}$$

and

$$Tf(x) = \psi(x)F(x) \quad \text{for} \quad f\in D_T\,.$$

Then T *is a compact operator with* $D_T=\mathscr{L}^2(0,a)$ *if, and only if*

(1) $$\Psi(x) = \int_x^a |\psi(t)|^2 dt<\infty, \quad 0<x<a;$$

(2) $$x\Psi(x) \to_{x\to 0} 0;$$

(3) $$x\Psi(x) \to_{x\to\infty} 0\,.$$

This lemma goes back to a corresponding statement concerning boundedness, which for $a<\infty$, is due to J. Odhnoff (private communication): T is a bounded operator with $D_T=\mathscr{L}^2(0,a)$ if, and only if, (1) holds and

(2a) $$x\Psi(x) < K \quad \text{for} \quad 0<x<a\,.$$

The idea of the proof given here is due to E. Thue Poulsen.

Proof. We prove the lemma for $a=\infty$; for $a<\infty$ it follows easily (in that case also (3) is trivially implied by (1)).

(a) Suppose that T is compact with $D_T=\mathscr{L}^2(0,\infty)$ and consider for $0<x<\infty$ the functions f_x defined by

$$f_x(t) = \begin{cases} x^{-\frac{1}{2}} & 0 \leqq t \leqq x , \\ 0 & x < t < \infty . \end{cases}$$

The family $\{f_x\}_{0<x<\infty}$ is bounded in $\mathscr{L}^2(0,\infty)$, hence (1) follows. Since $f_x(t) \to 0$ for $x \to 0$ or $x \to \infty$ and T is compact, it follows that

$$x\Psi(x) \leqq \|Tf_x\|^2 \to 0 \quad \text{for} \quad x \to 0 \text{ or } x \to \infty .$$

(b) We now prove that (1), (2) and (3) imply that T is compact.

(i) Let $x\Psi(x) \leqq c$ for $0<x<\infty$. For $f \in \mathscr{L}^2(0,\infty)$, $f(x) \geqq 0$, $0<\alpha<\beta<\infty$, we have

$$\int_\alpha^\beta |\psi(x)|^2 F^2(x)\, dx = -\Psi(\beta)F^2(\beta) + \Psi(\alpha)F^2(\alpha) + 2\int_\alpha^\beta x\Psi(x)\frac{F(x)}{x}f(x)\, dx .$$

By Schwarz' and Hardy's inequalities

$$\left|\int_\alpha^\beta |\psi(x)|^2 F^2(x)\, dx\right| \leqq \beta\Psi(\beta)\|f\|_2{}^2 + \alpha\Psi(\alpha)\|f\|_2{}^2 + 4c\|f\|_2{}^2$$
$$\leqq 6c\|f\|_2{}^2 .$$

(For Hardy's inequality, see e.g. Hardy, Littlewood and Polya [3]). This implies, that T is a bounded operator with $D_T = \mathscr{L}^2(0,\infty)$, and $\|T\|^2 \leqq 6c$.

(ii) For a function ψ with compact support satisfying (1) the result follows from Schwarz' inequality and Lebesgue's theorem on dominated convergence.

(iii) For any ψ satisfying (1), (2), and (3), let $\psi_n = \psi x_n$, where x_n is the characteristic function of $[1/n,n]$. By (i) and (ii) the corresponding operators T_n form a sequence of compact operators converging uniformly to T, which proves, that T is compact.

THEOREM 5I,3. *When $y^{[2n-1]}(0)=0$ is a boundary condition for L_c, and $p_n \in \mathscr{L}^2(0,1)$, a necessary and sufficient condition in order that B_{2n-1} be L_c-compact is that*

$$\text{(1)} \qquad \int_x^1 |\beta(t)|^2 dt < \infty \quad \textit{for} \quad 0<x<1 ,$$

$$\text{(2)} \qquad x\int_x^1 |\beta(t)|^2\, dt \to_{x\to 0} 0 .$$

PROOF. (a) Suppose, that β satisfies (1) and (2). For $f = L_c y$ we have

$$y^{[2n-1]}(x) = \int_0^x [p_n(t)y(t) - f(t)]\, dt .$$

If $\{y_s\}$ is an L_c-bounded sequence, then

$$|y(x)| < K \quad \text{for} \quad 0 \leqq x \leqq 1, \quad s = 1, 2, \ldots .$$

Hence $\{p_n y_s - f_s\}$ is bounded in $\mathscr{L}^2(0,1)$, and by lemma 5I,2 the sequence $\{\beta y_s^{[2n-1]}\}$ is compact.

(b) Let B_{2n-1} be L_c-compact and consider a set $\{y_\varepsilon\}_{0<\varepsilon\leqq 1}$ such that

$$\frac{d}{dx}\left(y_\varepsilon^{[2n-1]}(x)\right) = \begin{cases} \varepsilon^{-\frac{1}{2}}, & 0 \leqq x \leqq \varepsilon, \\ 0, & \varepsilon < x \leqq 1 . \end{cases}$$

It is easy to see, that we can choose $\{y_\varepsilon\}_{0<\varepsilon\leqq 1}$ to be an L_c-bounded set, and then by the proof of lemma 5I,2a,

$$\varepsilon \int_\varepsilon^1 |\beta(x)|^2\, dx \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

THEOREM 5I,4. *If $y^{[k+\nu]}(0) = 0$ are boundary conditions for L_c for some k, $0 \leqq k \leqq n-2$, and $\nu = 0, 1, \ldots, p$ with $0 \leqq p \leqq n-2-k$, and $y^{[k+p+1]}(0) = 0$ is not a boundary condition for L_c, then it is necessary for B_k to be L_c-bounded and sufficient for B_k to be L_c-compact, that*

$$\beta(x) x^{p+1} \in \mathscr{L}^2(0,1) .$$

PROOF. By means of the expression (2) of lemma 4.3 for $y^{[k+p+1]}$ the proof is straightforward.

REMARK 5I,5. Similar, more complicated relations hold for $n-1 \leqq k \leqq 2n-2$, $0 \leqq p \leqq 2n-2-k$.

Case II: L and L_s on the interval $[0, \infty]$ with the endpoint 0 regular.

THEOREM 5II,1. *Let $A(x)$ be an $n \times n$ matrix, whose coefficients are complex-valued functions on $[0, \infty)$ such that for sufficiently large x, say $x > x_0$,*

$$A(x) = A_0(x) + A_1(x) ,$$

where the elements of $A_0(x)$ are loc. a.c., *and the elements of $A_0'(x)$ and $A_1(x)$ are integrable on (x_0, ∞). Let*

$$w_1(x), w_2(x), \ldots, w_n(x)$$

be the eigenvalues of $A_0(x)$ arranged such that $w_i(x)$ is continuous for $i = 1, 2, \ldots, n$ and suppose further, that

$$\lim_{x\to\infty} \operatorname{Re}(w_i(x)-w_k(x)) \neq 0 \qquad \text{for} \quad i\neq k,\ i,k=1,2,\ldots,n\,.$$

Then the system of equations

$$\frac{dy}{dx} = A(x)y(x)$$

has n linearly independent solutions y_j, $j=1,2,\ldots,n$, *such that*

$$y_{ik}(x) = c_{ij}(x)\exp\left(\int_0^x w_k(\xi)\,d\xi\right),$$

where

$$c_{jk}(x) \xrightarrow[k\to\infty]{} c_{jk}\,.$$

PROOF. We refer to Neumark [5, §22,1, Satz 2, Folgerung 1].

THEOREM 5II,2. *Suppose that the coefficients of l satisfy the conditions*

$$1/p_0(x) = a_0(x)+b_0(x), \qquad p_i(x) = a_i(x)+b_i(x), \qquad i = 1,2,\ldots,n\,,$$

such that for $x>x_0$ *the functions* a_i *are* loc. a.c.,

$$\int_{x_0}^{\infty} |a_i'(x)|\,dx < \infty \qquad \text{and} \qquad \int_{x_0}^{\infty} |b_i(x)|\,dx < \infty$$

for $i=0,1,\ldots,n$, *and*

$$\lim_{x\to\infty} a_0(x) \neq 0\,.$$

Then, for $k=0,1,\ldots,2n-1$, $\beta\in\mathscr{L}^2(0,\infty)$ *implies that* B_k *is L-compact and that* B_k *is of* L_s*-Hilbert–Schmidt type.*

If $y^{[k]}(0)=0$ *is not a boundary condition for* L_s, *then* $\beta\in\mathscr{L}^2(0,\infty)$ *is also a necessary condition in order that* B_k *be of* L_s*-Hilbert–Schmidt type.*

PROOF. (a) Suppose, that $\beta\in\mathscr{L}^2(0,\infty)$. The equation

$$l(y)-\lambda y=0 \tag{1}$$

is equivalent to a system

$$\frac{d\tilde{y}}{dx} = A(x)\,\tilde{y}(x), \tag{2}$$

where

$$\tilde{y} = (y, y^{[1]}, \ldots, y^{[2n-1]})$$

and

$$A(x) = A_0(x)+A_1(x)$$

with

$$A_0(x) = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & & & & 0 & 0 \\ 0 & 0 & 0 & \ddots & & & 0 & 0 \\ \cdot & & & 1 & 0 & 0 & & \cdot \\ \cdot & & & 0 & a_0 & 0 & & \cdot \\ \cdot & & & a_1 & 0 & -1 & & \cdot \\ & & & & \ddots & & \ddots & \\ 0 & a_{n-1} & 0 & & & & 0 & -1 \\ a_n-\lambda & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}$$

(a_0 is in the n'th row and $(n+1)$'th column of the $n \times n$ matrix $A_0(x)$). The elements of A_0' exist and are integrable on (x_0, ∞), and the coefficients of A_1 are integrable on (x_0, ∞). The eigenvalues $w_1(x), \ldots, w_{2n}(x)$ of $A_0(x)$ are the roots of the equation

$$\frac{1}{a_0(x)}\varrho^{2n} - a_1(x)\varrho^{2n-2} + a_2(x)\varrho^{2n-4} - \ldots + (-1)^n\big(a_n(x) - \lambda\big) = 0 .$$

We choose λ such that

$$\lim_{x\to\infty} \operatorname{Re}\big(w_i(x) - w_k(x)\big) \neq 0 \quad \text{for} \quad i \neq k, \quad i,k = 1,2,\ldots,n .$$

Let $r_i(x) = \operatorname{Re} w_i(x)$, $i = 1,2,\ldots,2n$, and choose the order of the w_i such that

$$r_1(\infty) < r_2(\infty) < \ldots < r_{2n-1}(\infty) < r_{2n}(\infty) .$$

Then there exist $x_1 > 0$ and $\varepsilon > 0$ such that

$$r_1(x) < r_2(x) < \ldots < r_n(x) < -\varepsilon < 0 < \varepsilon < r_{n+1}(x) < \ldots < r_{2n}(x)$$

for $x > x_1$, and

$$r_i(x) = -r_{2n-i+1}(x), \qquad i = 1,2,\ldots,n .$$

Application of theorem 5II,1 to (2) shows, that (1) has $2n$ linearly independent solutions $y_1, \ldots, y_{2n}$ such that

$$\text{(3)} \qquad y_k^{[j]}(x) = c_{jk}(x) \exp\left(\int_0^x w_k(\xi)\, d\xi\right) = c_{jk}(x) W_k(x),$$

where

$$c_{jk}(x) \xrightarrow[x\to\infty]{} c_{jk} \quad \text{for} \quad j = 1,\ldots,2n, \quad k = 0,\ldots,2n-1 ,$$

and

$$W_k(x) = \exp\left(\int_0^x w_k(\xi)\,d\xi\right), \qquad k = 1,2,\ldots,2n\,.$$

We choose λ such that $c_{jk} \neq 0$ for $j=1,\ldots,2n$, $k=0,\ldots,2n-1$. Then

$$y_i \in \mathscr{L}^2(0,\infty) \quad \text{for} \quad i = 1,\ldots,n\,,$$
$$y_i \notin \mathscr{L}^2(0,\infty) \quad \text{for} \quad i = n+1,\ldots,2n\,.$$

Let $y_1,\ldots,y_{2n}$ be normed such that the Wronskian is 1. Then for $f \in \mathscr{L}^2(0,\infty)$ the solution of the equation

$$Ly - \lambda y = f$$

together with its derivatives are given by

$$(4)\ y^{[k]}(x) = \sum_{i=1}^{n} k_i y_i^{[k]}(x) + \sum_{i=1}^{n} y_i^{[k]}(x) \int_0^x v_i(\xi) f(\xi)\,d\xi - \sum_{i=n+1}^{2n} y_i^{[k]}(x) \int_x^\infty v_i(\xi) f(\xi)\,d\xi$$

for $k=0,1,\ldots,2n-1$, where

$$v_i(\xi) = (-1)^i \begin{vmatrix} c_{0,1}(\xi)W_1(\xi) & \ldots c_{0,i-1}(\xi)W_{i-1}(\xi) & c_{0,i+1}(\xi)W_{i+1}(\xi) & \ldots c_{0,2n}(\xi)W_{2n}(\xi) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_{2n-2,1}(\xi)W_1(\xi) & \ldots c_{2n-2,i-1}(\xi)W_{i-1}(\xi) & c_{2n-2,i+1}(\xi)W_{i+1}(\xi) & \ldots c_{2n-2,2n}W_{2n}(\xi) \end{vmatrix}$$

$$= c_i(\xi)W_i(\xi) = c_i(\xi)\exp\left(\int_0^\xi w_i(t)\,dt\right),$$

and

$$c_i(\xi) \xrightarrow[\xi\to\infty]{} c_i\,,$$

$i=1,2,\ldots,2n$. (For the definition of the v_i, see lemma 4.3 (1).) By the choice of λ we obtain $c_i \neq 0$ for $i=1,\ldots,2n$.

Substitution of the asymptotic expressions for $y_i^{[k]}$ and v_i in (4) gives

$$(5) \qquad y^{[k]}(x) = \sum_{i=1}^{n} k_i c_{ki}(x) W_i(x) + \sum_{i=1}^{n} c_{ki}(x) W_i(x) \int_0^x c_i(\xi) W_i(\xi) f(\xi)\,d\xi -$$
$$- \sum_{i=n+1}^{2n} c_{ki}(x) W_i(x) \int_x^\infty c_i(\xi) W_i(\xi) f(\xi)\,d\xi$$

for $k=0,1,\ldots,2n-1$. Then

$$(6) \qquad B_k y(x) = \beta(x) \sum_{i=1}^{n} k_i c_{ki}(x) W_i(x) + \int_0^\infty K_k(x,\xi) f(\xi)\,d\xi$$
$$= B_{k1} y(x) + B_{k2} y(x)$$

where

$$K_k(x,\xi) = \begin{cases} \beta(x)\sum_{i=1}^{n} c_{ki}(x)c_i(\xi)\exp\left(\int_\xi^x w_i(t)\,dt\right), & 0 \leqq \xi \leqq x, \\ -\beta(x)\sum_{i=n+1}^{2n} c_{ki}(x)c_i(\xi)\exp\left(-\int_x^\xi w_i(t)\,dt\right), & x < \xi. \end{cases}$$

Since $|c_{ki}(x)| < K$ for $0 \leqq x < \infty$, $i = 1,2,\ldots,2n$, $k = 0,1,\ldots,2n-1$, we have

$$\int_{x_1}^{x} |c_{ki}(x)|^2 |c_i(\xi)|^2 \exp\left(2\int_\xi^x r_i(t)\,dt\right) d\xi \leqq K^4(2\varepsilon)^{-1}\int_{x_1}^{x}(-2r_i(\xi))\exp\left(2\int_\xi^x r_i(t)\,dt\right) d\xi < K_1$$

for $i = 1,2,\ldots,n$, and similarly, for $i = n+1,\ldots,2n$, $k = 0,\ldots,2n-1$,

$$\int_x^\infty |c_{ki}(x)|^2 |c_i(\xi)|^2 \exp\left(-2\int_x^\xi r_i(t)\,dt\right) \leqq K^4(2\varepsilon)^{-1}\int_x^\infty 2r_i(\xi)\exp\left(-2\int_x^\xi r_i(t)\,dt\right) d\xi < K_1 .$$

Also $\int_0^\infty |K_k(x,\xi)|^2 d\xi < K|\beta(x)|^2$ for $0 \leqq x \leqq x_1$ and hence

$$\int_0^\infty\int_0^\infty |K_k(x,\xi)|^2\, d\xi\, dx \leqq K\int_0^\infty |\beta(x)|^2\, dx < \infty ,$$

and B_{k2} is a Hilbert–Schmidt operator for $k = 0,\ldots,2n-1$. From the expression (5) for $y = y^{[0]}$ it is seen, that if $\{y_s\}$ is an L-bounded sequence, then $\{k_{is}\}$ is bounded for $i = 1,\ldots,n$. Since

$$\left| c_{ki}(x)\exp\left(\int_0^x r_i(t)\,dt\right)\right| < K \qquad \text{for} \quad 0 \leqq x < \infty,\ i = 1,\ldots,n ,$$

it follows from Lebesgue's theorem on dominated convergence, that B_{k1} is L-compact. Thus, we have proved that B_k is L-compact.

For any L_s the k_i in (6) considered as functions of f are bounded linear functionals on $\mathscr{L}^2(0,\infty)$. Hence there exist functions $h_i \in \mathscr{L}^2(0.\infty)$, $i = 1,\ldots,n$, such that

$$B_k(L_s - \lambda)^{-1}f(x) = \int_0^\infty H_k(x,\xi)f(\xi)\,d\xi ,$$

where

$$H_k(x,\xi) = \begin{cases} \beta(x)\left[\sum_{i=1}^{n}\left(h_i(\xi)\exp\left(\int_0^\xi w_i(t)\,dt\right) + c_i(\xi)\right) c_{ki}(x)\exp\left(\int_\xi^x w_i(t)\,dt\right)\right], & 0 \leqq \xi \leqq x , \\ \beta(x)\left[\sum_{i=1}^{n} h_i(\xi)c_{ki}(x)\exp\left(\int_0^x w_i(\xi)\,d\xi\right) - \sum_{i=n+1}^{2n} c_{ki}(x)c_i(\xi)\exp\left(-\int_x^\xi w_i(t)\,dt\right)\right], & x < \xi . \end{cases}$$

It is evident that, when $\beta \in \mathscr{L}^2(0,\infty)$, the terms involving $h_i(\xi)$ are also square-integrable, so that $B_k(L_s-\lambda)^{-1}$ is a Hilbert–Schmidt operator.

(b) Suppose now that

$$\int_0^\infty \int_0^\infty |H_k(x,\xi)|^2 \, d\xi \, dx < \infty$$

for some k, $0 \leqq k \leqq 2n-1$, and that $y^{[k]}(0)=0$ is not a boundary condition for L_s. We choose x_0 and x_1 such that $0<x_1<x_0$ and, for some $\delta>0$,

$$r_n(x) > r_n(\infty)-\delta > r_{n-1}(\infty)+\delta > r_i(x) \quad \text{for} \quad x > x_0,\ i = 1,2,\ldots,n-1,$$

and

$$|c_n(\xi)c_{kn}(x)| - \sum_{i=1}^{n-1} |c_i(\xi)c_{ki}(x)| \exp(-2\delta x_1) > K > 0 \qquad \text{for} \quad x>x_0 .$$

This is possible, since all the c_i and c_{ki} have finite limit values different from 0 as $x \to \infty$. Then, for $x>x_0+x_1$,

$$\int_0^x \left| \sum_{i=1}^n c_i(\xi)c_{ki}(x) \exp\left(\int_\xi^x w_i(t)\,dt \right) \right|^2 d\xi \geqq K^2 \int_{x_0}^{x-x_1} \exp\left(2 \int_\xi^x r_n(t)\,dt \right) d\xi > K_1 > 0 .$$

Finally we can find $x_2>x_0+x_1$ such that

$$\int_0^x \left| \sum_{i=1}^n h_i(\xi)c_{ki}(x) \exp\left(\int_0^x w_i(t)\,dt \right) \right|^2 d\xi < K_1/4 \qquad \text{for} \quad x \geqq x_2 .$$

Then

$$\int_0^\infty |H_k(x,\xi)|^2 \, d\xi > |\beta(x)|^2 K_1/4 \qquad \text{for} \quad x \geqq x_2 ,$$

hence

$$\int_{x_2}^\infty |\beta(x)|^2 \, dx < \infty .$$

Also by theorem 4.6 and theorem 5I,1

$$\int_0^{x_2} |\beta(x)|^2 \, dx < \infty ,$$

and we have proved, that $\beta \in \mathscr{L}^2(0,\infty)$.

Theorem 5II,3. *If, on the interval* $(-\infty,\infty)$, *the coefficients satisfy the asymptotic relations required in theorem* 5II,2 *both for* $x \to \infty$ *and for* $x \to -\infty$, *then the index of deficiency is* $(0,0)$, *and* L *is self-adjoint. A* *necessary and sufficient condition in order that B_k be of L_s-Hilbert–Schmidt type is that $\beta \in \mathscr{L}^2(-\infty, \infty)$.*

PROOF. The proof is similar to the proof of theorem 5II,2.

REMARK 5II,4. For $p_0 \equiv 1$, $p_i \equiv 0$ for $i = 1, 2, \ldots, n$, the result of Agudo and Wolf [1] follows.

THEOREM 5II,5. *The following conditions are sufficient in order that B_{2n-1} be L-compact:*

$$\beta \in \mathscr{L}^2(0, \infty) \tag{1}$$

$$x \int_x^\infty |\beta(t)|^2 \, dt \to_{x \to \infty} 0 \tag{2}$$

$$\beta(x) \left(\int_0^x |p_n(t)|^2 \, dt \right)^{\frac{1}{2}} \in \mathscr{L}^2(0, \infty). \tag{3}$$

PROOF. For $f = Ly$,

$$\begin{aligned} B_{2n-1} y(x) &= \beta(x) y^{[2n-1]}(0) - \beta(x) \int_0^x f(t) dt + \beta(x) \int_0^x p_n(t) y(t) \, dt \\ &= B'_{2n-1} y(x) + B''_{2n-1} y(x) + B'''_{2n-1} y(x) \, . \end{aligned}$$

Since $\{y_s^{[2n-1]}(0)\}$ is bounded for an L-bounded sequence $\{y_s\}$, it follows from (1) that B'_{2n-1} is L-compact. By lemma 5I,2, conditions (1) and (2) imply that B''_{2n-1} is L-compact. By lemma 4.3 the operator $B_{2n-1}(0, K)$ is $L(0, K)$-compact, and since this evidently holds for $B'_{2n-1}(0, K)$ and $B''_{2n-1}(0, K)$, also $B'''_{2n-1}(0, K)$ is $L(0, K)$-compact. Then an L-bounded sequence $\{y_s\}$ has a subsequence $\{y_{s_k}\}$ such that

$$B'''_{2n-1} y_{s_k}(x) \xrightarrow[k \to \infty]{} z(x) \quad \text{a.e. on } (0, K) \, .$$

Since this holds for any $K > 0$, there exists a subsequence $\{y_{s_l}\}$ such that

$$B'''_{2n-1} y_{s_l}(x) \xrightarrow[l \to \infty]{} z(x) \quad \text{a.e. on } (0, \infty) \, .$$

By Lebesgue's dominated convergence theorem, this together with (3) implies

$$B'''_{2n-1} y_{s_l} \xrightarrow[l \to \infty]{} z \quad \text{in} \quad \mathscr{L}^2(0, \infty) \, ,$$

and B'''_{2n-1} is L-compact.

THEOREM 5II,6. *We consider the case, where $n = 1$, $p(x) = p_0(x) \geqq 0$, $1/p \in \mathscr{L}^1(0, \infty)$, and*

$$xP(x) = x\int_x^\infty dt/p(t) < K \qquad \textit{for} \quad 0 \leqq x < \infty \ \textit{and}\ p_1(x) \equiv 0.$$

Then B_1 is L-compact if, and only if, β satisfies the conditions

$$\beta \in \mathscr{L}^2(0,\infty), \tag{1}$$

$$x\int_x^\infty |\beta(t)|^2\,dt \to_{x\to\infty} 0 \tag{2}$$

PROOF. By theorem 5II,5 conditions (1) and (2) imply that B_1 is L-compact. Suppose, on the other hand, that B_1 is L-compact. Define the functions y_a for $0 < a < \infty$ by

$$y_a(x) = \begin{cases} a^{-\frac12}xP(x) + a^{-\frac12}\displaystyle\int_x^a P(t)\,dt - \frac{a^{-\frac12}\int_0^a P(t)\,dt}{P(0)}P(x)\,, & 0 \leqq x \leqq a\,, \\ a^{\frac12}P(x)\left(1 - \dfrac{a^{-1}\int_0^a P(t)\,dt}{P(0)}\right), & a < x < \infty\,. \end{cases}$$

Then

$$Ly_a(x) = \begin{cases} a^{-\frac12}, & 0 \leqq x \leqq a\,, \\ 0\ , & a < x < \infty; \end{cases}$$

$$B_1y_a(x) = \begin{cases} \beta(x)\left(\dfrac{a^{-\frac12}\int_0^a P(t)\,dt}{P(0)} - a^{-\frac12}x\right), & 0 \leqq x \leqq a\,, \\ \beta(x)a^{\frac12}\left(-1 + \dfrac{a^{-1}\int_0^a P(t)\,dt}{P(0)}\right), & a < x < \infty\,. \end{cases}$$

It follows from simple inequalities that $\{y_a\}_{a_0<a<\infty}$ is an L-bounded set for some $a_0 > 0$, and

$$B_1y_a(x) \xrightarrow[a\to\infty]{} 0 \qquad \text{pointwise}\,.$$

This together with the L-compactness of B_1 implies

$$\|B_1y_a\|_2 \xrightarrow[a\to\infty]{} 0\,,$$

and since

$$a^{-1}\int_0^a P(t)\,dt \xrightarrow[a\to\infty]{} 0\,,$$

(1) and (2) follow.

Case III: L on the interval $[0,1)$ with 0 regular and 1 singular.

We suppose that $p_n \in \mathscr{L}^2(0,1)$ and consider only the operator B_{2n-1}. The following theorems can be proved by the methods used in I and II.

REMARK 5III,1. For every $y \in D$ the function $y^{[2n-1]}(x)$ has a limit as $x \to 1$.

THEOREM 5III,2. *If* $\beta \in \mathscr{L}^2(0,1)$, *then* B_{2n-1} *is L-compact.*

THEOREM 5III,3. *If there exists a* $y \in D$ *such that* $y^{[2n-1]}(1) \neq 0$, *then* $\beta \in \mathscr{L}^2(0,1)$ *is necessary in order that* B_{2n-1} *be L-bounded.*

COROLLARY 5III,4. *If* $n=1$, $p_1 \equiv 0$ *and* $\int_0^x dt/p_0(t) \in \mathscr{L}^2(0,1)$, *then* $\beta \in \mathscr{L}^2(0,1)$ *is necessary in order that* B_1 *be L-bounded and sufficient in order that* B_1 *be L-compact.*

THEOREM 5III,5. *If* $p_1(x) \equiv 0$ *and* $y^{[2n-1]}(0) = 0$ *for all* $y \in D$, *then* B_{2n-1} *is L-compact if*

$$\varepsilon \int_0^{1-\varepsilon} |\beta(x)|^2 \, dx \underset{\varepsilon \to 0}{\longrightarrow} 0 \, .$$

COROLLARY 5III,6. *If* $n=1$, $p_1(x) \equiv 0$, $p_0(x) \geqq 0$ *and* $\int_0^x dt/p_0(t) \notin \mathscr{L}^2(0,1)$, B_1 *is L-compact, if*

$$\varepsilon \int_0^{1-\varepsilon} |\beta(x)|^2 \, dx \underset{\varepsilon \to 0}{\longrightarrow} 0 \, .$$

THEOREM 5III,7. *If* $n=1$, $p_1(x) \equiv 0$, $p_0(x) \geqq 0$, $P(x) = \int_0^x dt/p_0(t) \notin \mathscr{L}^2(0,1)$ *and* $(1-x)P(x) < K$ *for* $0 \leqq x < 1$, *then* B *is L-compact if, and only if,*

$$\varepsilon \int_0^{1-\varepsilon} |\beta(x)|^2 \, dx \underset{\varepsilon \to 0}{\longrightarrow} 0 \, .$$

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