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ON THE DISTRIBUTION OF LINEAR FUNCTIONS OF SPACINGS FROM A UNIFORM DISTRIBUTION[1]

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Suppose $X_1,\ldots,X_{n-1}$ are independent random variables, each with a uniform distribution over $[0,1]$. Let $Y_1 \leqq Y_2 \leqq \ldots \leqq Y_{n-1}$ denote the ordered values of $X_1,\ldots,X_{n-1}$, and define

$$T_1 = Y_1, \qquad T_k = Y_k - Y_{k-1} \text{ for } k = 2,\ldots,n-1, \qquad T_n = 1 - Y_{n-1}.$$

For each positive integer n, let $\{h_{1n},\ldots,h_{nn}\}$ be a given sequence of constants, and let $\bar{h}_n$ and Z_n be defined by

$$\bar{h}_n = \frac{1}{n}\sum_{i=1}^{n} h_{in}, \qquad Z_n = \frac{n\sum_{i=1}^{n} h_{in}(T_i - 1/n)}{[\sum_{i=1}^{n}(h_{in}-\bar{h}_n)^2]^{\frac{1}{2}}}.$$

Blom in [1] showed that the asymptotic distribution of Z_n is standard normal if the following condition is satisfied:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(h_{in}-\bar{h}_n)^r}{[\sum_{i=1}^{n}(h_{in}-\bar{h}_n)^2]^{r/2}} = 0, \qquad r = 3,4,\ldots .$$

Here it will be shown that the asymptotic distribution of Z_n is standard normal under the much weaker condition that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}|h_{in}-\bar{h}_n|^{2+\delta}}{[\sum_{i=1}^{n}(h_{in}-\bar{h}_n)^2]^{1+\delta/2}} = 0$$

for some positive δ. (Note that Blom's condition for $r=4$ implies this latter condition with $\delta=2$.)

To prove the asymptotic normality of Z_n under the weaker condition, we define

$$g_{in} = \frac{n(h_{in}-\bar{h}_n)}{[\sum_{i=1}^{n}(h_{in}-\bar{h}_n)^2]^{\frac{1}{2}}}$$

and note that Z_n can be written as $\sum_{i=1}^{n} g_{in}T_i$, since $\sum_{i=1}^{n} T_i = 1$. We also note that $\sum_{i=1}^{n} g_{in} = 0$, and $\sum_{i=1}^{n} {g_{in}}^2 = n^2$.

Next we define $U_1,\ldots,U_n$ as independent random variables, each with probability density function e^{-u} for $u>0$, zero for $u<0$. Define S_n

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as $U_1 + \ldots + U_n$. It is a well-known and easily verifiable fact that the joint distribution of $U_1/S_n, \ldots, U_n/S_n$ is exactly the same as the joint distribution of $T_1, \ldots, T_n$. Defining W_n as $\sum_{i=1}^n g_{in}[U_i/S_n]$, we see that the distribution of W_n is exactly the same as the distribution of Z_n, for each n. Therefore we investigate the distribution of W_n.

Since S_n/n converges stochastically to unity as n increases, if $(1/n) \sum_{i=1}^n g_{in} U_i$ has asymptotically a standard normal distribution, so does

$$W_n = \left[\frac{S_n}{n}\right]^{-1} \left\{\frac{1}{n} \sum_{i=1}^n g_{in} U_i\right\},$$

by a theorem to be found in Cramér [2, p. 254]. Since $\sum_{i=1}^n g_{in} = 0$,

$$\frac{1}{n} \sum_{i=1}^n g_{in} U_i = \frac{1}{n} \sum_{i=1}^n g_{in}(U_i - E\{U_i\}).$$

Applying the classical Liapounoff criterion to the sum of the independent random variables with zero means

$$\frac{1}{n} g_{1n}(U_1 - E\{U_1\}), \ldots, \frac{1}{n} g_{nn}(U_n - E\{U_n\}),$$

cf. Uspensky [3, p. 284], we find that $(1/n) \sum_{i=1}^n g_{in} U_i$ has asymptotically a standard normal distribution if, for some $\delta > 0$,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n E\{|(1/n) g_{in}(U_i - 1)|^{2+\delta}\}}{[\sum_{i=1}^n E\{(1/n) g_{in}(U_i - 1)\}^2]^{1+\delta/2}} = 0.$$

It is easily verified that this last equality holds if and only if

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n |h_{in} - \bar{h}_n|^{2+\delta}}{[\sum_{i=1}^n (h_{in} - \bar{h}_n)^2]^{1+\delta/2}} = 0.$$

This completes the proof.

The device of using the fact that the joint distribution of $T_1, \ldots, T_n$ is exactly the same as the joint distribution of $U_1/S_n, \ldots, U_n/S_n$ has been used to prove asymptotic joint normality of several functions of $T_1, \ldots, T_n$ by Weiss [4].

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