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DYNAMICAL SYSTEMS WITH A CERTAIN LOCAL CONTRACTION PROPERTY

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1.

We will consider dynamical systems (i.e. autonomous systems of differential equations) having a certain local contraction property (defined in sec. 2), and will investigate the existence of periodic solutions and the stability properties of such systems.

These questions have previously been treated, in a somewhat more special case, by G. Borg [1].

2.

Let M^n be an n-dimensional Riemannian manifold ($n \geqq 2$), i.e. a differentiable (C^∞) manifold with a metric defined by a positive definite, symmetric, second order covariant differentiable (C^∞) tensor field on M^n. Let X be an open, connected subset of M^n such that its closure $\bar{X}$ in M^n is compact and its boundary ∂X is a differentiable $(n-1)$-manifold in M^n.

The Riemannian distance between two points x and y in M^n is denoted by $d(x,y)$, and the Riemannian inner product between tangent vectors a and b is denoted by (a,b).

Let f be a contravariant C^1 vector field on $\bar{X}$. The field f defines a system S (using local coordinates)

$$\frac{dx^i}{dt} = f^i(x^1,\ldots,x^n), \qquad i = 1,\ldots,n ,$$

of differential equations on $\bar{X}$. From now on it will be assumed that f satisfies the two following conditions:

A. f penetrates the boundary ∂X of $\bar{X}$ inwards, that is, $(f(x),n_x) > 0$ for every $x \in \partial X$, where n_x is the inner normal to ∂X at x.

B. at each point $x \in X$ we have:
$(\nabla_a f,a) < 0$ for every tangent vector $a(\neq 0)$ at x with $(f(x),a)=0$.
($\nabla_a f$ denotes the covariant derivative of f in the direction a).

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In local coordinates, (B) becomes (writing g_{ij} for the fundamental tensor field):

B. at each point $x \in X$ we have:

$$\sum_{i,j,k,m} \left(\frac{\partial f^i}{\partial x^j} + \Gamma_{jk}{}^i f^k\right) a^j a^m g_{im} < 0$$

for every $a = (a^i) \neq 0$ with

$$\sum_{i,j} f^i a^j g_{ij} = 0 .$$

3.

Consider the solution curve of S starting at $x \in \bar{X}$ at $t=0$. Denote by $F(x,\tau)$ the point on the curve corresponding to $t=\tau$. The mappings $x \to F(x,t)$ form a one-parameter $(0 \leqq t < \infty)$ semigroup of transformations. Let R^+ be the set of non-negative real numbers.

Definition. The *ε-tube* N_ε around a trajectory $F(p,\mathsf{R}^+)$, where $f(p) \neq 0$, is $N_\varepsilon = \{x \mid x \in \bar{X}$ and $d(x, F(p,\mathsf{R}^+)) \leqq \varepsilon\}$. An *$\varepsilon$-sphere* around a singular point p, that is, $f(p)=0$, is defined analogously. The ε-tube resp. ε-sphere is *normal* if the ε-neighborhood U of every point $F(p,t)$ satisfies: any two points in U can be joined by a unique geodesic in U; this geodesic is the unique shortest geodesic in M^n joining the two points.

Normal ε-tubes (ε-spheres) always exist when $\bar{X}$ is compact, as is well known. In case $\overline{F(p,\mathsf{R}^+)}$ does not contain any singular points, we require N_ε to satisfy also the following condition:

$(f(x), \Pi_{yx} f(y)) > 0$ for all pairs x and y with $x \in F(p,\mathsf{R}^+)$ and $d(x,y) \leqq \varepsilon$.

Π_{yx} denotes the parallel displacement of tangent vectors from y to x. This can be required since $(f(x), \Pi_{yx} f(y))$ is >0 on the diagonal of the compact subset $\overline{F(p,\mathsf{R}^+)} \times \overline{F(p,\mathsf{R}^+)}$ of $\bar{X} \times \bar{X}$, and therefore is >0 in a neighborhood of the diagonal.

Definition. The *section* at $x \in F(p,\mathsf{R}^+)$ of a normal ε-tube N consists of those y in N that can be reached from x along a geodesic in N of length $\leqq \varepsilon$ perpendicular to $f(x)$.

Theorem 1. (i) *Let N be a normal ε-sphere around a singular point p. A solution starting in N approaches p with monotonously decreasing distance from p.*

(ii) *Let N be a normal ε-tube around $F(p,\mathsf{R}^+)$, p non-singular, and suppose $y \in N$. If $F(p,\cdot)$ tends to a singular point q, then also $F(y,\cdot)$*

tends to q; *if* $F(p,\cdot)$ *does not tend to any singular point, then* $F(y,\cdot)$ *approaches* $F(p,\mathsf{R}^+)$ *with monotonously decreasing distance.*

PROOF. Let $x \in F(p,\mathsf{R}^+)$ and consider any geodesic L starting at x. For each point $y \in L$, let a_y be the tangent vector to L at y, pointing in the direction away from x and of unit length. We will show that if $(f(x),a_x)=0$, then $(f(y),a_y)<0$ for all $y \in L$ $(y \neq x)$. So suppose $(f(y_1)a_{y1}) \geqq 0$ for some $y_1 \in L$. The function $(f(g),a_y)$ is a C^1-function on L, and

$$\nabla_a(f(y),a_y) = (\nabla_a f(y),a_y)+(f(y),\nabla_a a_y) .$$

But obviously $\nabla_a a_y=0$, so by cond. (B) of sec. 2 we have $(f(y),a_y)<0$ in a neighborhood of x $(y \neq x)$ on L. By continuity there exists a nearest point y_2 to x on L where $(f(y)_2,a_{y2})=0$ and then $(f(y),a_y) \leqq 0$ between x and y_2 on L. But repeating this argument for y_2 instead of x, we find that $(f(y),-a_y)<0$ in a neighborhood of y_2 $(y \neq y_2)$ on L, which gives a contradiction.

Now let $s(t)$ be the distance of $F(y,t)$, $y \in N$, from $x \in F(p,\mathsf{R}^+)$. When $F(y,t)$ is in the ε-sphere around x, we find, using normal local coordinates around x, that

$$\frac{ds}{dt} = \sum_i \frac{\partial s}{\partial x^i}\frac{dx^i}{dt} = \sum_{i,j} g_{ij}a^j f^i = (f,a) .$$

It follows that no solution can leave the ε-tube (resp. the ε-sphere) N. It is now easy to see that the theorem holds.

4.

By the *limit set* of the system S is meant the union of the limit sets of all solutions of the system. One easily proves:

LEMMA 1. *The limit set* L *of* S *is closed in* M^n.

LEMMA 2. L *is connected.*

PROOF. Suppose we had $L=L_1 \cup L_2$ with L_1 and L_2 closed in L, thus closed in M^n by lemma 1, and $L_1 \cap L_2=\emptyset$. Let d_0 be the distance between L_1 and L_2.

Obviously no solution can have a limit set that intersects both L_1 and L_2. Put

$$P_i = \{p \mid \text{the limit set of } F(p,\cdot) \text{ is contained in } L_i\} ,$$

$i=1,2$. Then $P_1 \cup P_2=\bar{X}$ and $P_1 \cap P_2=\emptyset$. But P_i is open in $\bar{X}$. For if $p \in P_i$, then $F(p,t)$ belongs to the $\frac{1}{4}d_0$-neighborhood of L_i for sufficiently large t and so $F(q,t)$ belongs to the $\frac{1}{2}d_0$-neighborhood of L_i for large t

if q is sufficiently near p. But $\bar{X}$ was supposed to be connected, so either P_1 or P_2 must be empty, that is, L_1 or L_2 is empty.

Theorem 1 (i) and lemma 2 gives:

THEOREM 2. *If S has a singular point, then this point is the limit set of S.*

THEOREM 3. *If S has no singular point, then it has a periodic solution, which is the limit set of S.*

PROOF. Let p be a point in the limit set of the system. Choose a normal ε-tube N around $F(p, \mathsf{R}^+)$ and let E be the ε-sphere around p. If ε is small enough, every solution starting in E stays in N and we can find $\varepsilon_1 < \varepsilon$ such that after some time all these solutions are in the ε_1-tube around $F(p, \mathsf{R}^+)$.

Since p is in the limit set of S, there is a point x with $d(x,p) < \frac{1}{3}(\varepsilon - \varepsilon_1)$ such that $F(x, \cdot)$ returns to the $\frac{1}{3}(\varepsilon - \varepsilon_1)$-neighborhood of p for arbitrarily large t. Since $F(x,t)$ is in the $\frac{1}{3}(\varepsilon - \varepsilon_1)$-tube around $F(p, \mathsf{R}^+)$, this means that $F(p,T)$ is in the $\frac{2}{3}(\varepsilon - \varepsilon_1)$-neighborhood of p for some arbitrarily large T. Then the section at $F(p,T)$ of the ε_1-tube around $F(p, \mathsf{R}^+)$ is contained in E. Every solution starting at $t=0$ in E reaches this section at a time nearly equal to T, which gives a continuous map of E into the section, i.e. into E. By the Brouwer theorem this map has a fixed point. Thus every neighborhood of p contains a starting point for a periodic solution. It is then obvious from Theorem 1 that $F(p, \cdot)$ is the unique periodic solution.

5.

The following topological characterization of X is easily proved:

THEOREM 4. *The system S has a singular point if and only if X is homeomorphic with R^n, and has a periodic solution if and only if X is homeomorphic with either a solid torus or a solid Klein bottle.* (The solid n-torus is $\mathsf{R}^{n-1} \times S^1$, while the solid Klein bottle is the non-trivial fiber bundle of R^{n-1} over S^1.)

6.

In the Euclidean case the obtained results specialize to:

THEOREM 5. *Consider a system $dx/dt = f(x)$, $f \in C^1$, on an open, connected and bounded subset of R^n with the vector field f penetrating ∂X*

inwards. Suppose there exists a constant, positive definite, symmetric $n \times n$*-matrix* G *such that for each* $x \in X$,

$$(V(x)y, Gy) < 0 \quad \textit{for all} \quad y \neq 0 \textit{ with } (f, Gy) = 0 .$$

Then X *is homeomorphic with either* R^n *or a solid torus. The system has as its limit set, in the first case a singular point and in the second case a periodic solution.*

PROOF. Apply the previous results with the flat metric on R^n defined by G.

7.

Finally, we state the following result:

THEOREM 6. *The system* S *is structurally stable on* $\overline{X}$.

For definition of structural stability, see [2]. A proof of the theorem is obtained by a modification of the proof of theorem 2 in [2] to apply to our global situation.

REFERENCES

1. G. Borg, *A condition for the existence of orbitally stable solutions of dynamical systems*, K. Tekniska Högskolans Handl. 153 (1960).
2. L. Markus, *Structurally stable differential systems*, Ann. of Math. 73 (1961), 1–19.

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