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RECURSIVE ARITHMETIC OF SKOLEM II

H. A. POGORZELSKI

1. Introduction.

In our previous note [5], Part I, we extended the Skolem arithmetic through the unique resolution theorem with respect to exponent chains of Mycielski numbers. In the present note, Part II, we further extend the Skolem arithmetic in the following two ways: (1) In section 2, we show that certain word-arithmetical methods in certain denumerable alphabets are valid in the Skolem arithmetic. (2) In section 3, we introduce a class of new unique resolution theorems in the Skolem arithmetic.

We shall assume the notation and definitions of Part I. In addition, we shall make use of the following. The class of natural numbers shall be denoted by N. We denote the recursive arithmetic of Skolem by $\sum(1)$ so as to emphasize the fact that it is also a word system in an alphabet consisting of one sign, namely, $\{1\}$. We shall denote the classes of numbers defined by scheme (20) of Part I as $\mathsf{P}^{(1)}, \mathsf{P}^{(2)}, \mathsf{P}^{(3)}, \ldots$, respectively, where $\mathsf{P}^{(1)}$ is the class of consecutive Mycielski numbers, and so on. With respect to the Hilbert–Ackermann class of primitive recursive functions defined by scheme (17) of Part I, we add the following two obvious properties concerning $\mathsf{P}^{(k)}$:

$$\xi_k(m_\mu{}^{(k)},x)^{non} = \xi_k(m_\nu{}^{(k)},x) \vee m_\mu{}^{(k)} = m_\nu{}^{(k)}; \tag{1.1}$$

$$\xi_k(m_\mu{}^{(k)},x)^{non} = \xi_k(m_\mu{}^{(k)},y) \vee x = y\ . \tag{1.2}$$

In these formulas as in all formulas and mathematical symbols in the following, $k=1,2,\ldots$ and $\mu,\nu \in N$.

2. Word theory and the Skolem arithmetic.

We begin this section with some pertinent facts about the word theory in question. However, we shall not dwell upon the results already treated in the author's note [6], referred to as note A, on the formal word $\Omega(\mathsf{A})$ in the denumerable alphabet $\mathsf{A}=\{a_1,a_2,\ldots\}$, neither shall we elaborate on those results which carry over easily in the obvious way from the results of note A to the subject-matter of this section.

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On the strength of (1.1) and (1.2), we can obviously construct the classes of functions

(2.1) $$F^{(k)} = \{F_1^{(k)}(x), F_2^{(k)}(x), \ldots\} ,$$

where, for $\mu, \nu \in N$,

(2.2) $$F_\mu^{(k)}(x) = \xi_k(m_\mu^{(k)}, x) ,$$

(2.3) $$F_\mu^{(k)}(x)^{non} = F_\nu^{(k)}(x) \vee \mu = \nu ,$$

(2.4) $$F_\mu^{(k)}(x)^{non} = F_\mu^{(k)}(y) \vee x = y .$$

Next, following the recursive methods of R. Péter [4] in the form given in note A, we can easily in the obvious way construct word systems $\Omega(\mathsf{P}^{(k)})$ in denumerable alphabets $\mathsf{P}^{(k)}$ with 1 as the empty word in each case. Clearly, the word systems $\Omega(\mathsf{P}^{(k)})$ are arithmetizations or rather interpretations of the formal word system $\Omega(\mathsf{A})$ in the denumerable alphabet A. We point out here that as a consequence of the R. Péter construction of $\Omega(\mathsf{P}^{(k)})$ and the fact that they are arithmetizations of $\Omega(\mathsf{A})$ it follows that $\Omega(\mathsf{P}^{(k)})$ are isomorphic.

Several easy consequences of the above constructions are the following theorems:

(2.5) $$F_\mu^{(k)}(x)^{non} = F_\nu^{(k)}(y) \vee (m_\mu^{(k)} = m_\nu^{(k)} \wedge x = y);$$

(2.6) $$\begin{cases} 1 \in S \\ \dfrac{x^{non} \in S \vee F_\mu^{(k)}(x) \in S}{S = \Omega(\mathsf{P}^{(k)})} \end{cases}$$

From this point of the section, we shall relate the above subject-matter to the Skolem arithmetic $\sum(1)$.

On the strength of the recursive apparatus given in Part I and theorems (1.1) and (1.2) of section 1, it is not difficult to see that theorems (2.5) are also theorems in $\sum(1)$. On the other hand, the fact that the stage induction theorems (2.6) are also valid in $\sum(1)$ is not so obvious. We shall therefore verify this fact. However, since the proof of the above-mentioned validity requires amongst other things a considerable deal of new notation and definitions, we shall expedite matters by simply assuming certain established results, definitions and notation of Asser [1], Goodstein [2], Skolem [8] and Vučković [10] to be pointed out as we go along. Finally, we shall also abbreviate some of our terminology in what follows by referring to mathematical induction and primitive number-theoretic recursion as induction and primitive recursion in $\sum(1)$ respectively, and to the relevant forms of stage induction and primitive word recursion as induction and primitive recursion in $\Omega(\mathsf{P}^{(k)})$ respectively.

We now outline the verification that induction in $\Omega(\mathsf{P}^{(k)})$ is valid in $\sum(1)$. (I) On the strength of the results of Goodstein [2] and Skolem [8], induction in $\sum(1)$ is equivalent to the principle of uniqueness of primitive recursion in $\sum(1)$, that is, the principle that every function defined by the primitive recursive scheme in $\sum(1)$ is uniquely determined. (II) On the other hand, Vučković [10] recently proved that stage induction follows from the principle of uniqueness of primitive word recursion, and his results easily carry over in the obvious way to the same in $\Omega(\mathsf{P}^{(k)})$. In turn, the fact that the principle of uniqueness of primitive recursion in $\Omega(\mathsf{P}^{(k)})$ follows from induction in $\Omega(\mathsf{P}^{(k)})$ can be shown in the simplest form as follows: If

$$f(1) = \alpha, \qquad \varphi(1) = \alpha\,,$$
$$f(F_\mu{}^{(k)}(x)) = \beta_\mu(x, f(x)), \qquad \varphi(F_\mu{}^{(k)}(x)) = \beta_\mu(x, \varphi(x))\,,$$

and $f(x) = \varphi(x)$ for some x, then

$$f(F_\mu{}^{(k)}(x)) = \beta_\mu(x, f(x)) = \beta_\mu(x, \varphi(x)) = \varphi(F_\mu{}^{(k)}(x))\,,$$

and, by induction in $\Omega(\mathsf{P}^{(k)})$, $f(x) = \varphi(x)$. Consequently, by virtue of the above results of Vučković, induction in $\Omega(\mathsf{P}^{(k)})$ is equivalent to the principle of uniqueness of primitive recursion in $\Omega(\mathsf{P}^{(k)})$. (III) Finally, we show that if a function is primitive recursive in $\Omega(\mathsf{P}^{(k)})$, then it can be defined by primitive recursion in $\sum(1)$. The proof of this theorem runs parallel to the proof given by Asser [1] of his Hauptsatz [Beweis a), page 264] for primitive recursive word functions. In particular, the proof that in $\Omega(\mathsf{P}^{(k)})$ the null word function, successor word function, identity word function and the substitution scheme (for definitions, see Asser [1]) reduce to the primitive recursive definition in $\sum(1)$ carries over in the obvious way from the proof given by Asser [1]. With respect to the primitive recursive scheme in $\Omega(\mathsf{P}^{(k)})$ using $F_\mu{}^{(k)}(x)$ as the successor function, it is not difficult to see that the primitive recursive definition in $\Omega(\mathsf{P}^{(k)})$ reduces to the primitive recursive definition in $\sum(1)$ by applying some form of course-of-value recursion [3].

From (I), (II) and (III) above, it follows that the principle of uniqueness in $\Omega(\mathsf{P}^{(k)})$ implies the principle of uniqueness in $\sum(1)$, and therefore induction in $\Omega(\mathsf{P}^{(k)})$ implies induction in $\sum(1)$.

3. Unique resolution theorems in the Skolem arithmetic.

In this section, making use of the results of section 2, we introduce a class of unique resolution theorems in $\sum(1)$ of which one of the theorems of the class is the unique resolution theorem proved in Part I.

In particular, we shall assume definitions (17), (18), (19) and (20) of Part I. Furthermore, we shall make use of the arithmetical chains $\Xi^{(k)}_{1\leqq r\leqq\mu} x_r$ defined as follows

$$(3.1)\qquad \begin{cases} \Xi^{(k)}_{1\leqq r\leqq 1} x_r = x_1\,, \\ \Xi^{(k)}_{1\leqq r\leqq \mu+1} x_r = \xi_k(x_{\mu+1}, \Xi^{(k)}_{1\leqq r\leqq\mu} x_r)\,, \end{cases}$$

where $\Xi^{(1)}_{1\leqq r\leqq\mu} x_r$ is the exponent chain defined by (7) in Part I.

Recalling the results in section 2, it is not difficult to see that the following theorems are corollaries of theorems (2.5) and consequently theorems in $\sum(1)$:

$$(3.2)\qquad {}^{non}[\mathsf{R}_k(\Xi^{(k)}_{1\leqq r\leqq\mu} q_r, m) \wedge \Lambda r \leqq \mu\{\mathsf{M}_k(q_r)\} \wedge \mathsf{M}_k(m)] \vee q_\mu = m\,.$$

On the strength of theorems (3.2), we prove the following Unique Resolution Theorems in $\sum(1)$:

$$(3.3)\qquad \Xi^{(k)}_{1\leqq r\leqq\mu} p_r \,{}^{non}\!= \Xi^{(k)}_{1\leqq s\leqq\nu} q_s \vee \mathsf{V} r \leqq \mu\{{}^{non}\mathsf{M}_k(p_r)\} \vee \mathsf{V} s \leqq \nu\{{}^{non}\mathsf{M}_k(q_s)\} \vee \mathsf{E}(p,q;\mu,\nu)\,.$$

The proof runs parallel to the one given originally by Skolem [9]. The case $\mu=1$ is easy to prove. Assume that the theorem is true for some μ. Then from the assumption of the theorem we have

$$\Xi^{(k)}_{1\leqq r\leqq\mu+1} p_r = \Xi^{(k)}_{1\leqq s\leqq\nu} q_s \wedge \Lambda r \leqq \mu+1\{\mathsf{M}_k(p_r)\} \wedge \Lambda s \leqq \nu\{\mathsf{M}_k(q_s)\}\,,$$

and consequently

$$\mathsf{R}_k(\Xi^{(k)}_{1\leqq s\leqq\nu} q_s, p_{\mu+1}) \wedge \Lambda s \leqq \nu\{\mathsf{M}_k(q_s)\} \wedge \mathsf{M}_k(p_{\mu+1})\,,$$

from which it follows by virtue of theorem (3.2) that $p_{\mu+1}=q_\nu$. Furthermore, on the strength of definition (3.1) we have $\Xi^{(k)}_{1\leqq r\leqq\mu} p_r = \Xi^{(k)}_{1\leqq s\leqq\nu-1} q_s$, from which on the grounds of the inductive hypothesis, we obtain

$$(p_{\mu+1} = q_\nu) \wedge \mathsf{E}(p,q;\mu,\nu-1) \Leftrightarrow \mathsf{E}(p,q;\mu+1,\nu)\,.$$

In conclusion, we should like to offer the remark that it is possible by virtue of the fact that $\Omega(\mathsf{P}^{(k)})$ are arithmetizations of $\Omega(\mathsf{A})$ and by means of the recursive word-arithmetical methods of Vučković [10] to show further that the class of unique word resolution theorems (see, author [7]) with respect to $\Omega(\mathsf{P}^{(k)})$ are also theorems of the Skolem arithmetic $\sum(1)$.

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