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A NOTE ON CONTRAPRODUCTION DOMAINS[1]

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1.

Our discussion concerns the character of the domains of partial recursive functions which are contraproductive (see section 2 below) for certain natural number sets. In section 3, we precisely formulate and prove this claim: if β is a contraproductive set which is not productive, and α is the domain of partial recursive function contraproductive for β, then α is either a creative set or a mesoic set of a certain special kind.

2.

Before proceeding to the results of the note, we shall list, for the convenience of the reader, the meanings of various terms and notations employed in the paper.

1. "ω_i" denotes the i-th r.e. (recursively enumerable) set of natural numbers, in an enumeration of *all* the r.e. sets according to the Kleene Enumeration Theorem ([2, p. 281], or [4, p. 89, Theorem 8]):

$$\omega_i = \{x \mid (\exists y) T_1(i,x,y)\} ,$$

$T_1(z,x,y)$ being a certain fixed, 3-place primitive recursive predicate.

2. Let σ be a set of natural numbers. σ is said to be *productive* just in case there is a partial recursive function p such that, for all i, $\omega_i \subseteqq \sigma$ implies that i is in the domain of p and $p(i) \in \sigma - \omega_i$. σ is called *contraproductive* just in case there is a partial recursive function p such that, for all i, $\sigma \subseteqq \omega_i$ implies that i is in the domain of p and $p(i) \in \omega_i - \sigma$.

3. An infinite r.e. set β of natural numbers is termed: (i) *simple*, just in case $\tilde{\beta}$ is infinite but without any infinite r.e. subset; (ii) *creative*, just in case $\tilde{\beta}$ is productive; and (iii) *mesoic*, just in case it is neither recursive nor simple nor creative. (The existence of simple and creative r.e. sets was proven by Post in 1944; the existence of mesoic r.e. sets was first observed by Dekker, in 1953.) Other terminology which may be unfamiliar to some readers will be explained where it is introduced.

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3.

It is known that every productive set of numbers admits a *general* recursive *contra*productive function. (See, for example, the discussion of this in [3], and the concluding remark on p. 149 of [1].) On the other hand, although the proof which we shall present is very like one of the key phases of the usual proof of the contraproductivity of any productive set, the following partial converse seems not to have been observed in the literature:

THEOREM A. *If β is a contraproductive number set which admits a total (i.e., a general recursive) contraproductive function, then β is productive.*

To prove this result, we will apply an elegant general form of the recursion theorem proved by Smullyan ([4, p. 72, Theorem 4]). We here reproduce, as a lemma, a statement of as much of this theorem as we shall need.

LEMMA. *For any recursively enumerable relation $R(z,x,y)$, there is a recursive function, r, such that, for all i, $\omega_{r(i)}=\{x \mid R(i,x,r(i))\}$.*

PROOF OF THEOREM A. Let f be a general recursive function contraproductive for β. In order to apply the above lemma, we take as $R(z,x,y)$ the predicate: $x \in \omega_z \vee x \neq f(y)$. This predicate is certainly recursively enumerable; and, by the lemma, there is a recursive function g such that, for all numbers i, $\omega_{g(i)}=\omega_i \cup \{f(g(i))\}^{\sim}$. We verify that $f\circ g$ is a *productive* function for β. Suppose that $\omega_i \subseteq \beta$. If $f(g(i)) \notin \beta$, then $\beta \subseteq \omega_{g(i)}$; hence, since f is contraproductive for β, we have

$$f(g(i)) \in \omega_{g(i)} = \omega_i \cup \{f(g(i))\}^{\sim} .$$

But, $f(g(i)) \notin \omega_i$ (since $f(g(i)) \notin \beta$ and $\omega_i \subseteq \beta$); and certainly $f(g(i)) \notin \{f(g(i))\}^{\sim}$. From this contradiction, we conclude $f(g(i)) \in \beta$. If $f(g(i)) \in \omega_i$, then

$$\omega_{g(i)} = \omega_i \cup \{f(g(i))\}^{\sim} = N ,$$

N the set of all natural numbers. Therefore, we find that $f(g(i)) \in N-\omega_i$: contradiction. Thus, $f(g(i)) \in \beta-\omega_i$, and the verification is complete.

This theorem, which we believe is of some interest in itself, is the basis of the following result on contraproduction domains:

THEOREM B. *Let β be a contraproductive set which is not productive; and let p be a partial recursive function contraproductive for β, with $\alpha=$ domain of p. Then, α is either creative or is a mesoic set which is pseudo-creative with no set simple in its complement. (A mesoic set α is called pseudo-creative just in case, for any* r.e. *set $\lambda \subseteq \tilde{\alpha}$, there is another* r.e. *subset*

τ of $\tilde{\alpha}$ such that $\tau - \lambda$ is infinite. We further say that α has no set simple in its complement just in case there is no (r.e.) *subset λ of $\tilde{\alpha}$ such that $\alpha \cup \lambda$ is simple.*)

Proof. It follows from Theorem A that α is not recursive; for, if it were, p could then be extended to a *total* recursive contraproductive function for β, implying productivity of β. Again, α is not simple. We verify this by showing that for some number i, and some infinite recursive set $\beta(i)$ of indices of ω_i, $\beta(i) \cap \alpha = \emptyset$. Suppose no such $\beta(i)$ exists. Following Myhill (see [4, p. 69]), let $h(x,y)$ be a 2-place recursive function such that, for all numbers i and m, $h(i,m) > m$ and $\omega_{h(i,m)} = \omega_i$. From the sequence

$$s(q,1) = h(q,0), \qquad s(q,2) = h(q,s(q,1)),$$
$$s(q,3) = h(q,s(q,2)), \quad \text{etc.},$$

one gets an infinite recursive set $\beta(q)$ of indices of ω_q, q being arbitrarily given. Thus let q be any number: generate α and $\beta(q)$ simultaneously; by hypothesis, we will come at length upon a number $m \in \alpha \cap \beta(q)$; let the first such m encountered be called "$m(q)$", and define $p^*(q) = p(m(q))$. The function p^* thus obtained is contraproductive for β (since p is contraproductive for β and $\omega_{m(q)} = \omega_q$ for all q); and p^* is *total* recursive. We thus have a contradiction to Theorem A, and so α cannot be simple. Now suppose that $\omega_j \subseteq \tilde{\alpha}$. Then, defining a partial recursive function p^*, with domain $\alpha \cup \omega_j$, by $p^*(x) = p(x)$ for $x \in \alpha$ and $p^*(x) = 0$ for $x \in \omega_j$, we see that p^* is another contraproductive function for β. Hence, by what we have already proved, $\alpha \cup \omega_j$ must be either creative or mesoic. So, α has no r.e. set simple in $\tilde{\alpha}$; and, if α is mesoic, it follows that α must be pseudocreative. The proof is complete.

Corollary. If β is a simple or a mesoic r.e. set, then $\tilde{\beta}$ is contraproductive, and if α is the domain of any partial recursive function contraproductive for $\tilde{\beta}$, then α is either creative or pseudo-creative mesoic with no set simple in its complement.

Proof. It is shown in [1] that $\tilde{\beta}$ is contraproductive but not productive. Now apply Theorem B.

Remarks. 1. Let β, p, α be as in Theorem B. Then p can be extended to a function p^* such that p^* is contraproductive for β and has a creative domain. For, since α is not recursive or simple, $\tilde{\alpha}$ must contain a creative subset λ; then $\alpha \cup \lambda$ is creative, and, proceeding in the evident manner previously indicated, we extend p to a function p^*, with domain $\alpha \cup \lambda$, which is still contraproductive for β. We might remark that it can be shown that the domains of those functions *shown* by Dekker, in [1], to be contraproductive for complements of simple and mesoic sets, are all of them creative.

2. It is easy to show that if β is simple, then there are $2^{\aleph_0}$ number sets α such that $\tilde{\beta} \subseteqq \alpha$ and α is contraproductive but not productive. By Theorem A, then, each of these $2^{\aleph_0}$ contraproductive sets α fails to admit any *total* contraproductive function.

ADDED IN PROOF. Since this paper was submitted, the author has proved the following stronger result in connection with Theorem B: If β, p, α are as in Theorem B, e is a number such that $\omega_e \subseteqq \beta$, and $\sum_e = \{x \mid \omega_x = \omega_e$ and $x \notin \alpha\}$, then $\sum_e$ contains an infinite r.e. subset. Hence, in particular, the i of the proof of Theorem B, satisfying $\beta(i) \cap \alpha = \emptyset$, can be taken an index of $\emptyset$.

REFERENCES

1. J. C. E. Dekker, *Productive sets*, Trans. Amer. Math. Soc., 78 (1955), 129–149.
2. S. C. Kleene, *Introduction to metamathematics*, Amsterdam and New York, 1952.
3. Hartley Rogers, Jr., *Theory of recursive functions and effective computability*, vol. I, Massachusetts Institute of Technology, 1956.
4. R. M. Smullyan, *Theory of formal systems*, Princeton University Press, 1961.

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