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ALGEBRAIC EXTENSIONS OF RELATIONAL SYSTEMS

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Introduction.

This paper is concerned with a class K of relational systems, subject to two conditions of a rather general nature. In particular, these conditions are satisfied if K is the class of all (commutative) fields. The notion of an algebraic extension of a system in K is introduced, as well as several other related notions, and a series of results are obtained that generalize many of the basic theorems concerning algebraic field extensions.

Our first assumption concerning K is:

(I) *K is the class of all models of a set of universal sentences.*

It is well known that the class of all fields satisfies this condition, provided we regard a field as a system with two binary operations, two distinguished elements, and two unary operations, defining 0^{-1} in any way whatsoever, say $0^{-1}=0$. Throughout most of the paper we shall actually assume, in place of (I), three simple conditions of a purely mathematical nature:

(I_1) *Every system that is isomorphic to a member of K belongs to K.*
(I_2) *Every system that is a subsystem of a member of K belongs to K.*
(I_3) *Every system that is the union of a non-empty directed family of members of K belongs to K.*

These conditions are easily seen to be consequences of (I). A fourth condition,—also a consequence of (I), although this is somewhat less obvious,—is introduced in Section 9. It is used only in the proofs of two of our results, the transitivity of the relation of being an algebraic extension, and the existence of algebraically closed algebraic extensions.

In addition to the conditions (I_1) we assume:

(II) *K has the amalgamation property.*

By this we mean that the following is true: *If A, B_1 and B_2 are any systems in K, and if f_1 and f_2 are isomorphisms of A into B_1 and B_2, respectively,*

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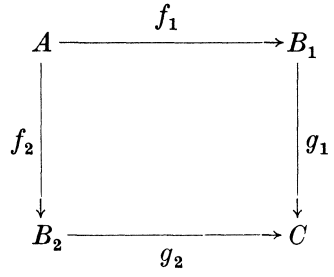
then there exist a system C in K and isomorphisms g_1 and g_2 of B_1 and B_2, respectively, into C, such that $g_1 f_1 = g_2 f_2$. In the terminology of homological algebra this means that if in the diagram below the systems A, B_1 and B_2 in K and the monomorphisms f_1 and f_2 are given, then there exist a system C in K and monomorphisms g_1 and g_2 such that the diagram commutes.



It is not difficult to show that the class K of all fields has the amalgamation property. It may be assumed that A is a subfield of both B_1 and B_2, and that both f_1 and f_2 are the identity automorphism of A. There then exist algebraically closed extensions C_1 and C_2 of B_1 and B_2, such that C_1 and C_2 have the same transcendence degree over A. From this it follows that C_1 and C_2 are equivalent extensions of A, and we can take C to be either one of them.

The proof outlined above makes use of certain facts, concerning algebraically closed algebraic extensions of fields, that are special cases of results proved below, as well as of properties of purely transcendental extensions that have no counterpart in the general development. It is therefore of some interest to observe that it is possible to give a more elementary proof that uses only some basic facts concerning simple extensions. Such a proof will be given in Section 10.

It should be noted that, although many important classes K of relational systems satisfy the conditions (I) and (II), and the results presented here therefore apply to them, these results will in many cases yield nothing of interest. The reason for this is that our basic definition, the definition of an algebraic element, is very strong. Thus it will be shown in Section 10 that for certain classes K, such as the class of all groups, every member of K is algebraically closed.

It is to be hoped that some new and interesting applications will be found, but this is not our object at the present. The fact that some of the most beautiful results from classical algebra can be obtained in such a general setting seems in itself to be of sufficient interest to make these investigations worth while.

The amalgamation property was first introduced into the literature by R. Fraïssé (cf. [2]) in connection with certain embedding properties for relational systems, and it has played a role in connection with the investigations of other such problems in [3], [4], [5], [6] and [8]. In the classical treatment of field extensions the amalgamation property is not explicitly mentioned, but the condition which we use in Section 10 to give an elementary proof of the amalgamation property for the class of all fields is either explictly or implicitly involved. In the literature on differential fields of characteristic 0, for instance in [7], the amalgamation property is not formulated in its full generality, but a special case (Theorem 2.5 in [7]) plays an important role.

There exist in the literature several generalizations of the classical theory of field extensions, or of certain portions of that theory, but most of these have little connection with the investigations reported on in this paper. It should be mentioned that in a paper by Shoda [10] some portions of the theory of algebraic extensions are derived from assumptions concerning the structure of the class K of systems under consideration rather than from specific axiom systems. However, with his definition of an algebraic element he is forced to make some rather strong assumptions concerning K, and in fact several of the basic properties that we derive have to be explicitly assumed. Observe also that although the amalgamation property is not mentioned in [10], this or some other related properties is actually needed in one of the proofs there. This fact is pointed out in [9].

2. Notation and terminology.

By a *relational system* or, more briefly, a *system*, we here mean a (possibly transfinite) sequence

$$\mathfrak{A} = \langle A, F_0, F_1, \ldots, F_\xi, \ldots, R_0, R_1, \ldots, R_\eta, \ldots \rangle_{\xi<\alpha,\, \eta<\beta},$$

where A is a set, α and β are ordinals and, for each $\xi < \alpha$ and $\eta < \beta$, F_ξ is an operation of some finite rank μ_ξ over A (a map of A^{μ_ξ} into A) and R_η is a relation of some finite rank ν_η over A (a subset of A^{ν_η}). Some of the operations F_ξ may be of rank 0, in which case the domain of F_ξ consists of the null sequence ϕ alone. In this case F_ξ is usually identified with the element $F_\xi(\phi)$ of A, and this element is referred to as a distinguished element of $\mathfrak{A}$.

The ordinals α and β and the ranks μ_ξ and ν_η are assumed to be the same for all the systems $\mathfrak{A}$ under consideration. Usually all reference to the operations F_ξ and to the relations R_η is omitted, and the system $\mathfrak{A}$

is identified with its underlying set A. By a *subsystem* of A is meant a subset B of A that is closed under all the operations F_ξ. Of course it is understood that the operations and relations over B are the restrictions to B of the operations and relations over A. Observe that if A has distinguished elements, then these are members of every subsystem, but if no such elements are present, then the empty set is regarded as a subsystem of A. If B is a subsystem of A, then A is said to be an *extension* of B, and in symbols this is expressed by writing $B \leqq A$ and $A \geqq B$. We also write $f\colon A \to B$ to mean that f is an isomorphism of A into B.

If L is a non-empty family of subsystems of a given system A, then their intersection, $\cap L$, is also a subsystem of A. A family L of systems is said to be directed if and only if any two members of L have a common extension that also belongs to L. If L is a non-empty directed family of systems, then the union, $\cup L$, of all the members of L, with its operations and relations defined in an obvious manner, is a system and an extension of all the members of L. If, in addition, all the members of L are subsystems of some given system A, then $\cup L$ is also a subsystem of A.

Throughout Sections 3–9 we consider a fixed class K of relational systems, and assume that it satisfies the conditions (I_1), (I_2), (I_3) and (II). It is agreed that all systems under consideration are members of K, and this fact will in general not be explicitly mentioned. It is further agreed that the letters A, B, C, D and E, with or without subscripts and primes, will always denote systems (members of K).

3. The amalgamation property.

In this section we prove a generalized form of the amalgamation property and two theorems concerning extensions of isomorphisms.

THEOREM 3.1. *If $f_i\colon A \to B_i$ for all i in I, then there exist C and functions g_i, associated with all the elements i of I, such that $g_i\colon B_i \to C$ for all i in I and $g_i f_i = g_j f_j$ for all i, j in I.*

PROOF. We may assume that I coincides with the set of all ordinals less than a fixed ordinal γ. We then obtain systems $C_\varkappa$ and functions $g_\varkappa$, associated with all the ordinals $\varkappa < \gamma$, such that

$$(1) \qquad g_\varkappa\colon B_\varkappa \to C_\varkappa, \qquad C_\varkappa \leqq C_\tau, \qquad g_\varkappa f_\varkappa = g_\tau f_\tau$$

whenever $\varkappa \leqq \tau < \gamma$. In fact, if we take $C_0 = B_0$ and let g_0 be the identity automorphism of B_0, then these conditions obviously hold for $\varkappa \leqq \tau < 1$.

Assuming that $\lambda < \gamma$, and that $C_\varkappa$ and $g_\varkappa$ have been so chosen for all $\varkappa < \lambda$ that (1) holds whenever $\varkappa \leqq \tau < \lambda$, consider the system

$$C_\lambda' = \bigcup\{C_\varkappa | \varkappa < \lambda\}$$

which, according to (I_3), belongs to K. Then $g_\varkappa: B_\varkappa \to C_\lambda'$ for all $\varkappa < \lambda$, and all the isomorphisms $g_\varkappa f_\varkappa$, with $\varkappa < \lambda$, of A into C_λ' coincide. Since $f_\lambda: A \to B_\lambda$, there exist by (II) a system C_λ and functions g_λ and h_λ such that

$$g_\lambda: B_\lambda \to C_\lambda, \qquad h_\lambda: C_\lambda' \to C_\lambda, \qquad g_\lambda f_\lambda = h_\lambda g_0 f_0 .$$

Because of (I_1) we may assume that C_λ is an extension of C_λ' and that h_λ is the identity map. It then readily follows that (1) holds whenever $\varkappa \leqq \tau < \lambda + 1$. By an application of Zorn's Lemma we infer that $C_\varkappa$ and $g_\varkappa$ can be so chosen for all $\varkappa < \gamma$ that (1) holds whenever $\varkappa \leqq \tau < \gamma$, and the conclusion of the theorem is therefore seen to holds with

$$C = \bigcup\{C_\varkappa | \varkappa < \gamma\} .$$

THEOREM 3.2. *If $A_i \leqq B_i$ and $f_i: A_i \to C$ for all i in I, then there exists $D \geqq C$ such that for each i in I there is an isomorphism of B_i into D that agrees with f_i on A_i.*

PROOF. Observe that the following special case of the theorem is an easy consequence of 3.1 and (I_1):

If $A \leqq B_i$ for all i in I, and if $f: A \to C$, then there exists $D \geqq C$ such that for each i in I there is an isomorphism of B_i into D that agrees with f on A.

In fact, for each i in I we have $f_i: A \to B_i$ where f_i is the identity automorphism of A. Applying 3.1 to these isomorphisms together with the given isomorphism f, we obtain a system D, an isomorphism g of C into D and, for each i in I, an isomorphism g_i of B_i into D such that $gf = g_i f_i$ for all i in I. Because of (I_1) we may assume that D is an extension C and $g(x) = x$ for all x in C. Then for all x in A, $g_i(x) = g_i f_i(x) = gf(x) = f(x)$.

To prove the theorem we shall apply the above special case twice.

First, consider a fixed i in I, and apply the special case with I replaced by the one-element set $\{i\}$, and with $A = A_i$ and $f = f_i$. This yields an extension D_i of C and an isomorphism f_i' of B_i into D_i that agrees with f_i on A_i.

Next, apply the special case with A and B_i replaced by C and D_i, and taking for f the identity automorphism of C. We then obtain an extension D of C and for each i in I an isomorphism h_i of D_i into D such

that $h_i(x)=x$ for all x in C. The proof is then completed by checking that the isomorphism $g_i=h_if_i'$ of B_i into D agrees with f_i on A_i.

THEOREM 3.3. *If, for all i in I, $A_i \leqq B$ and $f_i\colon A_i \to B$, then there exists $C \geqq B$ such that for each i in I there is an automorphism g_i of C that agrees with f_i on A_i.*

PROOF. By repeated application of 3.2 we obtain a commutative diagram

$$\begin{array}{ccccccccccc} A_i & \to & B & \to & C_0 & \to & C_1 & \to & C_2 & \to & \dots \\ \downarrow f_i & & \downarrow h_{0,i} & & \downarrow h_{1,i} & & \downarrow h_{2,i} & & \downarrow h_{3,i} & & \\ B & \to & C_0 & \to & C_1 & \to & C_2 & \to & C_3 & \to & \dots \end{array}$$

where the horizontal maps are identity isomorphisms (injections) and the vertical maps are isomorphisms. Letting D_0 be the union of the systems C_i, we infer that all the maps f_i extend to isomorphisms $g_{0,i}$ of D_0 into itself.

This is a weaker form of the theorem; by applying it repeatedly we obtain another commutative diagram

$$\begin{array}{ccccccccccc} A & \to & D_0 & \to & D_1 & \to & D_2 & \to & D_3 & \to & \dots \\ \downarrow f_i & & \downarrow g_{0,i} & & \downarrow g_{1,i} & & \downarrow g_{2,i} & & \downarrow g_{3,i} & & \\ B & \to & D_0 & \to & D_1 & \to & D_2 & \to & D_3 & \to & \dots . \end{array}$$

In fact, each $(g_{0,i})^{-1}$ maps a subsystem $D_{0,i}$ of D_0 isomorphically onto D_0, and by the result just obtained there exists an extension D_1 of D_0 such that all the functions $(g_{0,i})^{-1}$ extend to isomorphisms $g_{1,i}$ of D_1 into D_1. D_2 and $g_{2,i}$ are then obtained by applying the same result to the maps $(g_{1,i})^{-1}$, etc.

Let C be the union of the systems D_n. The maps $g_{2n,i}$, $n=0,1,\dots$, have a common extension g_i that maps C isomorphically into itself, and g_i is an extension of f_i. Also, the maps $g_{2n+1,i}$, $i=0,1\dots$, have a common extension g_i' that maps C isomorphically into itself, and it is easy to check that $g_i'=g_i^{-1}$. Therefore g_i is an automorphism of C.

4. Algebraic extensions.

We now introduce some of the basic concepts of this paper.

DEFINITION 4.1. *If $A \leqq B$ and U is a subset of B, then by the extension of A by U,—in symbols $A(U)$—we mean the intersection of all those subsystems of B that contain $A \cup U$ as a subset.*

The extension of A by U does of course depend on the system B, but the notational ambiguity will lead to no confusion. This is largely

due to the obvious fact that if $A \leqq B \leqq C$, and if U is a subset of B, then the extension of A by U in B coincides with the extension of A by U in C.

If the set U consists of just one element, $U = \{u\}$, then we write $A(u)$ for $A(U)$. More generally, if $U_1, U_2, \ldots$ are subsets of B and $u_1, u_2, \ldots$ are elements of B, then we write $A(U_1, U_2, \ldots, u_1, u_2, \ldots)$ for $A(U_1 \cup U_2 \cup \ldots \cup \{u_1, u_2, \ldots\})$.

DEFINITION 4.2. *Suppose* $A \leqq B$.

(i) *We say that* B *is a finite extension of* A *if and only if* $B = A(U)$ *for some finite subset* U *of* B.

(ii) *We say that* B *is a simple extension of* A *if and only if* $B = A(u)$ *for some element* u *of* B.

DEFINITION 4.3. *Suppose* $A \leqq B$ *and* $A \leqq C$.

(i) *By an* A*-isomorphism of* B *onto, respectively into,* C *we mean an isomorphism of* B *onto, respectively into,* C *such that* $f(x) = x$ *for all* x *in* A.

(ii) *By an* A*-automorphism of* B *we mean an* A*-isomorphism of* B *onto* B.

(iii) *We say that* B *and* C *are equivalent extensions of* A *if and only if there exists an* A*-isomorphism of* B *onto* C.

(iv) *An element* u *in* B *is said to be* A*-equivalent to an element* v *in* C *if and only if there exists an* A*-isomorphism of* $A(u)$ *onto* $A(v)$ *that maps* u *onto* v.

DEFINITION 4.4. *Suppose* $A \leqq B$.

(i) *An element* u *in* B *is said to be algebraic over* A *if and only if in any extension* C *of* A *there are only finitely many distinct elements that are* A*-equivalent to* u.

(ii) *A subset* U *of* B *is said to be algebraic over* A *if and only if every element in* U *is algebraic over* A.

(iii) *We say that* B *is an algebraic extension of* A *if and only if* B *is algebraic over* A.

LEMMA 4.5. *Suppose* $A \leqq B$. *An element* u *in* B *is algebraic over* A *if and only if there exists a positive integer* n *with the property that in any extension* C *of* A *there are at most* n *distinct elements that are* A*-equivalent to* u.

PROOF. Clearly the existence of such an integer n implies that u is algebraic over A. On the other hand, if no such integer exists, then for each positive integer n there exists an extension C_n of A such that there are n distinct elements $v_{n,1}, v_{n,2}, \ldots, v_{n,n}$ in C_n that are A-equivalent to u.

We now apply 3.2, taking for I the set of all positive integers, replacing

A_i, B_i and C by A, C_i and A, and taking for f_i the identity automorphism of A. This yields an extension D of A, and for each positive integer n an A-isomorphism of C_n into D. Each isomorphism f_n maps $A(v_{n,j})$ onto $A(w_{n,j})$ where $w_{n,j}=f_n(v_{n,j})$, and it follows that the elements $w_{n,j}$ are A-equivalent to u. Since, for each n, the elements $w_{n,1}, w_{n,2}, \ldots, w_{n,n}$ are distinct, we conclude that there are infinitely many distinct elements in D that are A-equivalent to u, and that u is therefore not algebraic over A.

DEFINITION 4.6. *Suppose $A \leqq B$. If the element u in B is algebraic over A, then by the reduced degree of u over A we mean the smallest positive integer n that satisfies the condition in Lemma* 4.5.

LEMMA 4.7. *Suppose $A \leqq B$, $B \leqq C$, and $B \leqq D$. If an element u in C is B-equivalent to an element v in D, then u is A-equivalent to v.*

PROOF. The B-isomorphism of $B(u)$ onto $B(v)$ that maps u onto v also maps $A(u)$ onto $A(v)$.

COROLLARY 4.8. *If $A \leqq B \leqq C$, and if an element u in C is algebraic and of reduced degree n over A, then u is algebraic and of reduced degree at most n over B.*

THEOREM 4.9. *If $A \leqq B$, and if the subset U of B is algebraic over A, then $A(U)$ is an algebraic extension of A.*

PROOF. We wish to show that if v is any element in $A(U)$ and if C is any extension of A, then the number of distinct elements of C that are A-equivalent to v is finite or, in other words, the number of distinct A-isomorphism of $A(v)$ into C is finite.

There exists a finite subset V of U such that v is in $A(V)$, and by 3.2 there exists an extension D of C with the property that every A-isomorphism of $A(v)$ into C can be extended to an A-isomorphism of $A(V)$ into D. Since the number of distinct A-isomorphisms of $A(V)$ into D is obviously finite,—in fact, it is at most equal to the product of the reduced degrees of the elements in V,—the conclusion follows.

COROLLARY 4.10. *If $A \leqq B$, then the elements in B that are algebraic over A form a subsystem of B (and obviously an algebraic extension of A).*

PROOF. If U is the set of all those elements in B that are algebraic over A, then by 4.9 the system $C=A(U)$ is an algebraic extension of A. Thus every element of C is algebraic over A, and we infer that $C=A(U)$.

5. Splitting extensions and normal extensions.

Lacking a counterpart to the notion of a polynomial, we formulate the definitions of splitting extensions and of normal extensions entirely in terms of elements. The term "splitting extension" is perhaps not very natural, but is used in order to stay as close as possible to the classical terminology.

DEFINITION 5.1. *Suppose $A \leqq B$ and u is an element in B.*

(i) *u is said to split in B over A if and only if u is algebraic over A, and the number of elements in B that are A-equivalent to u is equal to the reduced degree of u over A.*

(ii) *B is said to be a splitting extension of u over A if and only if u splits in B over A, and there exists no extension C of A such that $C \leqq B$, $C \neq B$, and u splits in C over A.*

DEFINITION 5.2. *Suppose $A \leqq B$ and U is a subset of B.*

(i) *U is said to split in B over A if and only if every member of U splits in B over A.*

(ii) *B is said to be a splitting extension of U over A if and only if U splits in B over A, and there exists no extension C of A such that $C \leqq B$, $C \neq B$, and U splits in C over A.*

Given an element u in B that is algebraic and of reduced degree n over A, in order for B to be a splitting extension of u over A it is obviously necessary and sufficient that $B = A(u_1, u_2, \ldots, u_n)$ where $u_1, u_2, \ldots, u_n$ are distinct elements in B and are A-equivalent to u. Similarly, given a subset U of B, B is a splitting extension of U over A if and only if U splits in B over A and $B = A(U')$ where U' is the set of all those elements in B that are A-equivalent to some element in U.

COROLLARY 5.3. *Suppose $A \leqq B$, and suppose U is a subset of B that is algebraic over A. Then:*

(i) *There exists an extension C of B such that U splits in C over A.*

(ii) *There exists an extension D of $A(U)$ such that D is a splitting extension of U over A.*

PROOF. For each u in U, if $n(u)$ is the reduced degree of u over A, then there exists an extension C_u of A such that there are $n(u)$ distinct elements in C_u that are A-equivalent to u.

We now apply 3.2 with I replaced by U, with A_i, B_i and C replaced by A, C_u and B, and with all the functions f_i replaced by the identity automorphism of A. This yields an extension C of B and, for each u in U, an A-isomorphism of C_u into C. Consequently, for each u in U

there are $n(u)$ distinct elements in C that are A-equivalent to u, and U therefore splits in C.

The second statement immediately follows the first.

DEFINITION 5.4. *Suppose $A \leqq B$. We say that B is a normal extension of A if and only if every member of B splits in B over A.*

LEMMA 5.5. *If B is an algebraic extension of A, then the following conditions are equivalent:*

(i) *B is a normal extension of A.*

(ii) *For every extension C of B, every A-isomorphism of B into C maps B into itself.*

(iii) *For every extension C of B, every A-automorphism of C maps B onto itself.*

PROOF. That (i) implies (ii) follows from the fact that every A-isomorphism of B into an extension C of B maps each member u of B onto an element v in C that is A-equivalent to u, and from the observation that if B is a normal extension of A, then every member of C that is A-equivalent to a member of B is itself a member of B. That (ii) implies (iii) is obvious.

Assume that (iii) holds. By 5.3 there exists an extension C' of B such that C' is a splitting extension of B over A. Hence $C' = A(U)$ where U is the set of all those elements in C' that are A-equivalent to some member of B. For each element u in U there exists an element v in B that is A-equivalent to u, and hence there exists an A-isomorphism f of $A(v)$ onto $A(u)$ that maps v onto u. By 3.3, f can be extended to an A-automorphism g of some extension C of C'. Since, by (iii), g maps B onto itself, it follows that u belongs to B. Thus U is a subset of B, $C' = B$, every member of B splits in B over A, and B is a normal extension of A.

COROLLARY 5.6. *If $A \leqq B \leqq C$, and if C is a normal extension of A, then C is a normal extension of B.*

COROLLARY 5.7. *If $A \leqq B \leqq C$, and if C is a normal extension of A, then every A-isomorphism of B into C can be extended to an A-automorphism of C*

PROOF. If f is an A-isomorphism of B into C, then by 3.3 f can be extended to an A-automorphism g of some extension D of C. By 5.5, g maps C onto itself.

COROLLARY 5.8. *If B is a normal extension of A, and if the elements*

u and v in B are A-equivalent to each other, then there exists an A-automorphism of B that maps u onto v.

COROLLARY 5.9. *Suppose $A \leqq B \leqq C$, and C is a normal extension of A. Then B is a normal extension of A if and only if every A-automorphism of C maps B onto itself.*

THEOREM 5.10. *Suppose $A \leqq B$, U is a subset of B, and $B = A(U)$. Then B is a normal extension of A if and only if U splits in B over A.*

PROOF. If U splits in B over A, then U is algebraic over A, and it follows from 4.9 that B is an algebraic extension of A. Also, every A-isomorphism of B into an extension C of B must map U into B, and must therefore map B into itself. Consequently, by 5.5, B is a normal extension of A.

COROLLARY 5.11. *If $A \leqq B$, and if B is a splitting extension over A of some subset of B, then B is a normal extension of A.*

PROOF. If U is a subset of B such that B is a splitting extension of U over A, then $B = A(U')$, where U' is the set of all those elements in B that are A-equivalent to some member of U. Clearly U' splits in B, whence it follows by 5.10 that B is a normal extension of A.

COROLLARY 5.12. *If $A \leqq B$, then the elements in B that split in B over A form a subsystem C of B, and C is a normal extension of A.*

PROOF. If U is the set of all those elements in B that split in B over A, then every member of U splits in $C = A(U)$ over A. Hence, by 5,10, C is a normal extension of A, whence it follows that every member of C splits in C over A, and therefore that $C = U$.

THEOREM 5.13. *If B and C are splitting extensions over A of their subsets U and V, respectively, then every A-isomorphism of $A(U)$ onto $A(V)$ can be extended to an A-isomorphism of B onto C.*

PROOF. If f is an A-isomorphism of $A(U)$ onto $A(V)$, then by 3.2 there exists an A-isomorphism g of B into some extension D of C such that g agrees with f on $A(U)$.

We have $B = A(U')$ and $C = A(V')$, where U' is the set of all those elements in B that are A-equivalent to some member of U, and V' is the set of all those elements in C that are A-equivalent to some member of V. If u' is in U', then u' is A-equivalent to some member u of U, and the elements $v = g(u)$ and $v' = g(u')$ are therefore A-equivalent to each other. Since $v = g(u) = f(u)$ is in $A(V)$, and therefore in C, and since

by 5.10 C is a normal extension of A, it follows that v' is in C. Thus g maps B into C.

Similarly, there exists an A-isomorphism of C into B, and since an algebraic extension of A cannot be A-equivalent to a proper subsystem of itself, we infer that g must map B onto C.

COROLLARY 5.14. *If B and C are splitting extensions over A of their elements u and v, respectively, and if u is A-equivalent to v, then there exists an A-isomorphism of B onto C that maps u onto v.*

THEOREM 5.15. *If B and C are splitting extensions over A of their subsets U and V, respectively, if every member of U is A-equivalent to some member of C, and if every member of V is A-equivalent to some member of B, then B and C are equivalent extensions of A.*

PROOF. By essentially the same argument as was used to prove 5.13.

6. Iterated finite algebraic extensions.

We do not know how to prove, on the basis of the assumptions (I_1), (I_2), (I_3) and (II), that an algebraic extension of an algebraic extension of a system in K is an algebraic extension of that system. The principal result of this section shows, however, that finite algebraic extensions do have this property.

DEFINITION 6.1. *If $A \leqq B$, then by the Galois group of B over A,—in symbols $G(B/A)$,—we mean the group of all A-automorphisms of B.*

LEMMA 6.2. *If $A \leqq B \leqq C$, and if each member of $G(C/A)$ maps B onto itself, then $G(C/B)$ is a normal subgroup of $G(C/A)$. In fact, the function that maps each member of $G(C/A)$ onto its restriction to B is a homomorphism of $G(C/A)$ into $G(B/A)$, and its kernel is $G(C/B)$.*

PROOF. This is essentially just an elementary property of groups of permutations, and the fact that the permutations are automorphisms of relational systems is quite irrelevant. More precisely, if H is a group of permutations of some set U, and if V is a subset of U that is mapped onto itself by each member of H, then the function that maps each member of H onto its restriction to V is a homomorphism of H onto a group H' of permutations of V, and the kernel of this homomorphism is the set of all those members of H that leave V pointwise fixed. In the present case, with $U = C$, $V = B$ and $H = G(C/A)$, H' is clearly a subgroup of $G(B/A)$, and the kernel of the homomorphism is $G(C/B)$.

COROLLARY 6.3. *If $A \leqq B \leqq C$, and if B and C are normal extensions of A, then the homomorphism in Lemma 6.2 maps $G(C/A)$ onto $G(B/A)$.*

PROOF. By 5.5, each member of $G(C/A)$ maps B onto itself, and Lemma 6.2 therefore applies. By 5.7 every member of $G(B/A)$ is the restriction to B of some member of $G(C/A)$, and the given homomorphism is therefore onto $G(B/A)$.

LEMMA 6.4. *Suppose $A \leqq B \leqq C$, and assume that each member of $G(C/A)$ maps B onto itself. Let R and S be the binary relations over C such that, for all x, y and C,*

$$xRy \text{ if and only if } y = f(x) \text{ for some } f \text{ in } G(C/A),$$
$$xSy \text{ if and only if } y = f(x) \text{ for some } f \text{ in } G(C/B).$$

Then:

(i) *R and S are equivalence relations over C, and $S \leqq R$.*

(ii) *For all x,y in C and f in $G(C/A)$, if $y=f(x)$, then f maps the equivalence class x/S onto the equivalence class y/S.*

(iii) *For all x,y in C, if xRy, then x/S and y/S have the same number of elements.*

(iv) *For all x in C, the number of elements in x/R is less than or equal to the number of elements in x/S times the order of the group $G(C/A)/G(C/B)$.*

PROOF. The first statement is obvious. Under the hypothesis of (ii), if x' is in x/S, then $x'=g(x)$ for some g in $G(C/B)$. Therefore $f(x')=fgf^{-1}(y)$, and since $G(C/B)$ is a normal subgroup of $G(C/A)$, it follows that $f(x')$ is in y/S. Thus f maps x/S into y/S. Similarly, f^{-1} maps y/S into x/S, and we infer that f must map x/S onto y/S. This proves (ii), which in turn implies (iii).

Given x in C, choose representatives x_i, i in I, from the equivalence classes mod S that are contained in x/R. By (iii), the number of elements in x/R is equal to the number of elements in x/S times the order of the set I. For each i in I let H_i be the set of all those members of $G(C/A)$ that map x/S onto x_i/S. The sets H_i are pairwise disjoint, and their union is $G(C/A)$. Since the members of $G(C/B)$ map x/S onto itself, we see that if f is in H_i, then the coset of $f \bmod G(C/B)$ is a subset of H_i. Consequently the order of I is at most equal to the index of $G(C/B)$ in $G(C/A)$, whence the conclusion of (iv) follows.

THEOREM 6.5. *If B is a finite algebraic extension of A, and if C is an algebraic extension of B, then C is an algebraic extension of A.*

PROOF. By hypothesis, $B=A(U)$ for some finite subset U of B. By 5.3 there exists an extension C' of C such that U splits in C' over A.

Let V be the set of all those elements in C' that are A-equivalent to some member of U. Then V is finite, and V splits in $A(V)$ over A. Consequently, by 5.10, $A(V)$ is a normal extension of A.

Consider any element u in C. Since $B \leqq A(V)$, it follows from 4.8 and the hypothesis that u is algebraic over $A(V)$. In order to prove that u is algebraic over A, it suffices to show that in any extension D of C' there exist only finitely many elements that are A-equivalent to u. By 3.3 there exists an extension E of D with the property that if v is any element of D that is A-equivalent to u, then $v = f(u)$ for some f in $G(E/A)$. Let R and S be the equivalence relations over E such that, for all x,y in E,

$$xRy \text{ if and only if } y = f(x) \text{ for some } f \text{ in } G(E/A)\,,$$
$$xSy \text{ if and only if } y = f(x) \text{ for some } f \text{ in } G(E/A(V))\,.$$

Then all the elements in D that are A-equivalent to u belong to a single equivalence class mod R. On the other hand, the equivalence class u/S consists of elements that are $A(V)$-equivalent to u, and this class is therefore finite. By 5.5, every member of $G(E/A)$ maps $A(V)$ onto itself, and Lemmas 6.2 and 6.4 therefore apply. Since the group $G(A(V)/A)$ is finite, it follows that u/R must be finite, and hence that there are only finitely many elements in D that are A-equivalent to u.

LEMMA 6.6. *Suppose C is a normal extension of A. If $u_1, u_2, \ldots, u_k$ are elements of C, if the reduced degree of u_1 over A is n_1, and if, for $i = 2, 3, \ldots, k$, the reduced degree of u_i over $A(u_1, u_2, \ldots, u_{i-1})$ is n_i, then the number of distinct A-isomorphisms of $A(u_1, u_2, \ldots, u_k)$ into C is $n_1 n_2 \ldots n_k$.*

PROOF. It clearly suffices to show that if f is an A-isomorphism of the system $B = A(u_1, u_2, \ldots, u_{i-1})$ into C, then there are exactly n_i A-isomorphisms of $B(u_i)$ into C that agree with f on B.

By 5.7 there exists an A-automorphism g of C that agrees with f on B. Also, since by 5.6 C is a normal extension of B, and u_i therefore splits in C over B, there exist exactly n_i distinct B-isomorphisms g_p, $p = 1, 2, \ldots, n_i$, of $B(u_i)$ into C. The functions gg_p are distinct A-isomorphisms of $B(u_i)$ into C, and they all agree with f on B. The number of distinct extensions of f is therefore at least n_i.

On the other hand, if $h_1, h_2, \ldots$ are distinct A-isomorphisms of $B(u_i)$ into C that agree with f on B, then $g^{-1}h_1, g^{-1}h_2, \ldots$ are distinct B-isomorphisms of $B(u_i)$ into C, and the number of such functions h_q therefore cannot exceed n_i.

THEOREM 6.7. *Suppose C is a normal extension of A. If $C = A(u_1, u_2, \ldots, u_k)$ where the reduced degree of u_1 over A is n_1 and, for*

$i=2,3,\ldots,k$, the reduced degree of u_i over $A(u_1,u_2,\ldots,u_{i-1})$ is n_i, then the order of $G(C/A)$ is $n_1n_2\ldots n_k$.

PROOF. By 6.6.

Lemma 6.6 shows that we can define the reduced degree of B over A, —in symbols $(B:A)_0$,—where B is a finite algebraic extension of A, in such a way that the following conditions hold:

(1) If $B=A(u)$, then $(B:A)_0$ is the reduced degree of u over A.
(2) If $A\leqq B\leqq C$, then $(B:A)_0(C:B)_0=(C:A)_0$.

7. Separable extensions and the Galois correspondence.

Assuming that C is an extension of A, for each subgroup H of $G(C/A)$ let H^φ be the fixpoint set of H (that is, the set of all x in C such that $f(x)=x$ for all f in H), and for each intermediate system B between A and C let $B^\tau=G(C/B)$. The maps $H\to H^\varphi$ and $B\to B^\tau$ form a Galois correspondence (see e.g. [1, p. 56]) between the lattice of all subgroups of $G(C/A)$ and the lattice of all systems between A and C. This means that, for any subgroups H and H_1 of $G(C/A)$, and for any systems B and B_1 between A and C, the following statements hold:

$$\text{If } H\leqq H_1, \text{ then } H_1{}^\varphi\leqq H^\varphi.$$
$$\text{If } B\leqq B_1, \text{ then } B_1{}^\tau\leqq B^\tau.$$
$$H\leqq H^{\varphi\tau} \text{ and } B\leqq B^{\tau\varphi}.$$

From this it follows that

$$H^\varphi = H^{\varphi\tau\varphi} \quad \text{and} \quad B^\tau = B^{\tau\varphi\tau}.$$

It also follows that the maps $H\to H^{\varphi\tau}$ and $B\to B^{\tau\varphi}$ are closure operations on the two lattices, that the closed subgroups and intermediate systems form complete lattices under set-inclusion, and that restricted to these new lattices the two original maps are anti-isomorphisms and are inverses of each other.

A complete analogue of the classical Galois theory cannot be expected here, but assuming that C is a normal extension of A, we shall obtain a characterization of the closed intermediate systems and show that the set of all fixpoint of a normal subgroup of $G(C/A)$ is a normal extension of A.

DEFINITION 7.1. *We say that B is a separable extension of A if and only if B is an algebraic extension of A and every element in B of reduced degree 1 over A belongs to A.*

DEFINITION 7.2. *Suppose $A \leqq B$. An element u in B is said to be separable over A if and only if $A(u)$ is a separable extension of A.*

LEMMA 7.3. *Suppose $A \leqq B \leqq C$, and suppose every member of B is algebraic and of reduced degree 1 over A. Then:*

(i) $G(C/A) = G(C/B)$.

(ii) *Two elements in C are B-equivalent to each other if and only if they are A-equivalent to each other.*

(iii) *Any element in C that is algebraic over B is also algebraic over A, and has the same reduced degree over A as over B.*

PROOF. The first statement follows from the fact that a member of $G(C/A)$ maps each element u of B onto an element that is A-equivalent to u, and the only such element is u itself.

If the elements u and v of C are A-equivalent, then by 3.3 there exists an extension D of C such that u is mapped onto v by some member of $G(D/A)$. Since, by (i), $G(D/A) = G(D/B)$, it follows that u is B-equivalent to v. Thus (ii) holds.

If u is an element in C, and if D is any extension of C, then we can apply (ii) with C replaced by D to infer that the number of elements in D that are A-equivalent to u is equal to the number of elements in D that are B-equivalent to u. From this (iii) follows.

THEOREM 7.4. *If C is a normal extension of A, then the fixpoint set of $G(C/A)$ is the set of all elements in C whose reduced degree over A is 1.*

PROOF. Clearly an element u in C whose reduced degree over A is 1 is mapped onto itself by every member of $G(C/A)$. Conversely, it follows from 5.8 that if the reduced degree of u over A is greater than 1, then $f(u) \neq u$ for some $G(C/A)$.

COROLLARY 7.5. *If C is a normal extension of A, then in order for the fixpoint set of $G(C/A)$ to be equal to A it is necessary and sufficient that C be a separable extension of A.*

COROLLARY 7.6. *If C is an algebraic extension of A, then the elements in C whose reduced degree over A is 1 form a subsystem B of C, and C is a separable extension of B.*

PROOF. By 5.3 and 5.10 there exists an extension C' of C such that C' is a normal extension of A. By 7.4, the elements in C' whose reduced degree over A is 1 form a subsystem of C', in fact, the fixpoint set B' of $G(C'/A)$. Consequently the elements in C whose reduced degree over

A is 1 form the subsystem $B = B' \cap C$ of C. The second statement in the conclusion of the corollary follows by 7.3 (iii).

THEOREM 7.7. *If C is a normal extension of A, and if H is a normal subgroup of $G(C/A)$, then the fixpoint set of H is a normal extension of A.*

PROOF. Since every element in the fixpoint set B of H splits in C, it suffices to show that if an element u in C is A-equivalent to an element v in B, then u is in B. By 5.8 there exists f in $G(C/A)$ such that $f(v) = u$. For any h in H, $f^{-1}hf$ is also in H, and therefore $f^{-1}hf(v) = v$, $hf(v) = f(v)$, $h(u) = u$. Thus u is in B, as was to be shown.

8. Maximal algebraic extensions.

It is an open question whether, for an arbitrary class K that satisfies the conditions (I_1), (I_2), (I_3) and (II), Theorem 6.5 can be generalized by dropping the finiteness assumption for the first extension. In the next section we shall impose on K one further condition which will be shown to hold whenever K is the class of all models of a set of first order universal sentences, and which will at once yield the desired generalization. This will permit us to prove the existence of algebraically closed algebraic extensions, but in order to carry the investigations as far as possible on the basis of the present assumptions, we now introduce and investigate the (apparently) more general concept of a maximal algebraic extension.

DEFINITION 8.1. *We say that B is a maximal algebraic extension of A if and only if B is an algebraic extension of A, and there exists no algebraic extension C of A such that $B \leqq C$ and $B \neq C$.*

LEMMA 8.2. *If B is an algebraic extension of A, then the following conditions are equivalent:*

(i) *B is a maximal algebraic extension of A.*

(ii) *For every algebraic extension C of A, there exists an A-isomorphism of C into B.*

(iii) *For every finite algebraic extension C of A, there exists an A-isomorphism of C into B.*

PROOF. Assume (i). For any algebraic extension C of A there exists, by 3.2, an A-isomorphism of C into some extension C' of B. Since the images of the elements in C are algebraic over A, it follows from 4.9 and the maximality of B that the isomorphism must map C into B. Thus (ii) holds.

Assume (iii), suppose D is an algebraic extension of A with $B \leqq D$, and consider an element u in D. If the reduced degree of u over A is n, then there exists an extension C of A such that there are n distinct elements $v_1, v_2, \ldots, v_n$ in C that are A-equivalent to u. We may assume that $C = A(v_1, v_2, \ldots, v_n)$, so that C is a finite algebraic extension of A. By (iii) it follows that there are n distinct elements in B that are A-equivalent to u. On the other hand there are at most n distinct elements in D with this property, and they must therefore all belong to B. In particular, it follows that u is in B. Thus (i) holds.

Since (iii) is a special case of (ii), this completes the proof.

COROLLARY 8.3. *If B and C are maximal algebraic extensions of A, then B is A-equivalent to C.*

PROOF. By 8.2 and the fact that no algebraic extension B of A is A-equivalent to a proper subsystem of itself.

COROLLARY 8.4. *If B is a maximal algebraic extension of A, then B is a normal extension of A.*

PROOF. By 5.3 there exists an extension C of B such that C is a splitting extension of B over A. By the maximality of B it follows that $B = C$. Thus every element in B splits in B over A, and B is a normal extension of A.

THEOREM 8.5. *For any system A there exists a maximal algebraic extension B of A.*

PROOF. There exist finite algebraic extensions B_i of A, associated with all the elements i in some set I, such that every finite algebraic extension C of A is A-equivalent to some B_i. We now apply 3.2, taking for A_i and C the system A, and for f_i the identity automorphism of A. This yields an extension B of A such that each of the systems B_i is A-equivalent to some subsystem B_i' of C. Since each B_i' is an algebraic extension of A, it follows from 4.9 that we can take B to be an algebraic extension of A. The maximality of B is now an easy consequence of 8.2.

DEFINITION 8.6. *A system A is said to be perfect if and only if every algebraic extension C of A is a separable extension of A.*

LEMMA 8.7. *Suppose B is a maximal algebraic extension of A. Then A is perfect if and only if B is a separable extension of A.*

PROOF. By 8.2, every algebraic extension C of A is A-equivalent to some subsystem C' of B. If B is a separable extension of A, then so is C', and hence so is C. The converse is obvious.

THEOREM 8.8. *For any system A there exists an extension B of A such that B is perfect, and every element in B is algebraic and of reduced degree 1 over A. Furthermore, if two extensions of A have these properties, then they are A-equivalent to each other.*

PROOF. By 8.5 there exists a maximal algebraic extension C of A, and by 7.6 the elements in C whose reduced degree over A is 1 form a subsystem B of C, and C is a separable extension of B. By 7.3, C is a maximal algebraic extension of B, and we conclude by 8.7 that B is perfect.

Now suppose the extension B' of A is also perfect, and suppose every element in B' is algebraic and of reduced degree 1 over A. By 8.2 there exists an A-isomorphism of B' onto a subsystem B'' of C. Since every element in B'' has reduced degree 1 over A, we have $B'' \leqq B$. On the other hand, B'' is perfect, and C is therefore a separable extension of B''. From this it follows that every element in C whose reduced degree over A is 1 belongs to B''. Thus $B \leqq B''$, $B = B''$.

9. Algebraically closed algebraic extensions.

If K is the class of all fields, then the following condition obviously holds:

(I_4) *Suppose $A \leqq B$, U is a subset of B, and u is an element in B. If u is algebraic over $A(U)$, then there exists a finite subset V of U such that u is algebraic over $A(V)$.*

It is not known whether this property is a consequence of the conditions (I_1), (I_2), (I_3) and (II). However, it does hold whenever K is the class of all models of a set of universal sentences, and we shall now outline a proof of this fact.

THEOREM 9.1. *If* (I) *holds, then so does* (I_4).

OUTLINE OF PROOF.[1] Assume, as in the hypothesis of (I_4), that $A \leqq B$, that U is a subset of B, and that u is an element of B.

By hypothesis, K is the class of all models of a set Σ of first order universal sentences. These sentences are assumed to be formulated in a language L that contains the usual logical symbols, including the identity symbol, as well as operation symbols 0_ξ of rank μ_ξ and predicates P_η

[1] This proof of Theorem 9.1 is essentially due to E. Engeler. Actually, what Professor Engeler proved was Theorem 11.1, and originally that result was used in conjunction with Lemma 6.4 to prove Theorem 9.2. Later, by an obvious modification of Engeler's argument, the present proof of Theorem 9.1 was obtained.

of rank ν_η, associated with all the ordinals $\xi < \alpha$ and $\eta < \beta$. We now form a new language L' by adjoining to L as constants (operation symbols of rank 0), distinct symbols a' associated with all the elements a in $A(U,u)$, and by also adjoining distinct operation symbols $\theta_1, \theta_2, \ldots$ of rank 1.

Let Σ_U be the set obtained from Σ as follows: First, we add all sentences of the types

$$\theta_n(a') \neq \theta_n(b'), \quad \theta_n(c') = c', \quad \theta_m(u') \neq \theta_n(u'),$$

where m and n are distinct natural numbers, a and b are distinct members of $A(U,u)$, and c is a member of $A(U)$. Secondly, we add all sentences of the types

$$0_\xi(a_0', a_1', \ldots a'_{\mu_\xi - 1}) = a'_{\mu_\xi}, \qquad P_\eta(b_0', b_1', \ldots, b'_{\nu_\eta - 1}),$$

where $\xi < \alpha$ and $\eta < \beta$, and where $a_0, a_1, \ldots, a_{\mu_\xi}$, $b_0, b_1, \ldots, b_{\nu_\eta - 1}$ are elements in $A(U,u)$ such that the conditions

$$0_\xi(a_0, a_1, \ldots, a_{\mu_\xi - 1}) = a_{\mu_\xi}, \qquad P_\eta(b_0, b_1, \ldots, b_{\nu_\eta - 1}),$$

hold in $A(U,u)$. Finally, we add all sentences of the types

$$0_\xi\big(\theta_n(a_0'), \theta_n(a_1'), \ldots, \theta_n(a'_{\mu_\xi - 1})\big) = \theta_n\big(0_\xi(a_0', a_1', \ldots, a'_{\mu_\xi - 1})\big),$$

$$P_\eta\big(\theta_n(b_0'), \theta_n(b_1'), \ldots, \theta_n(b'_{\nu_\eta - 1})\big) \leftrightarrow P_\eta(b_0', b_1', \ldots, b'_{\nu_\eta - 1}),$$

where $\xi < \alpha$, $\eta < \beta$, n is a natural number, and $a_0, a_1, \ldots, a_{\mu_\xi - 1}$, $b_0, b_1, \ldots$, $b_{\nu_\eta - 1}$ are elements in $A(U,u)$.

It is now easy to see that Σ_U is consistent if and only if u is not algebraic over $A(U)$. For, a model B of Σ_U would be a model of Σ with certain distinguished elements a^* associated with all the elements a in $A(U,u)$, and enriched by certain unary operations $\theta_n{}^*$. The axioms of the first type imply that the map $a \to a^*$ is one-to-one, and the axioms in the second group assert that this correspondence is an isomorphism with regard to the old operations and relations. Hence we may identify a^* with a and regard $A(U,u)$ as a subsystem of B. The axioms in the first group and in the last group then assert that the functions $\theta_n{}^*$ are $A(U)$-isomorphisms of $A(U,u)$ into B, and that no two of them map u onto the same element.

Similarly, if V is a finite subset of U, and if we let Σ_V be the set obtained by adding to Σ precisely those formulas of the above types that have the additional property that all the new constants occurring in them are associated with elements of $A(V,u)$, then we find that Σ_V is consistent if and only if u is not algebraic over $A(V)$.

Furthermore, every finite subset of Σ_U is contained in Σ_V for some

finite subset V of U. Consequently Σ_U is consistent if and only if Σ_V is consistent for every finite subset V of U. From this the proof readily follows. For, if u is algebraic over $A(U)$, then Σ_U is not consistent, and there must exist a finite subset V of U such that Σ_V is not consistent and, therefore, u is algebraic over $A(V)$.

THEOREM 9.2. *Suppose* (I_4) *holds. If B is an algebraic extension of A, and if C is an algebraic extension of B, then C is an algebraic extension of A.*

PROOF. Every element u in C is, by hypothesis, algebraic over the system $B=A(B)$. Therefore, by (I_4), u is algebraic over $A(V)$ for some finite subset V of B, and it follows by 6.5 that u is algebraic over A.

DEFINITION 9.3. *A system A is said to be algebraically closed if and only if there exists no algebraic extension B of A with $A \neq B$.*

THEOREM 9.4. *Suppose* (I_4) *holds, and suppose B is an algebraic extension of A. Then B is algebraically closed if and only if B is a maximal algebraic extension of A.*

PROOF. Suppose B is a maximal algebraic extension of A. By 9.2, if C is an algebraic extension of B, then C is also an algebraic extension of A, whence it follows that $B=C$. Thus B is algebraically closed. The converse is obvious.

10. Examples.

We begin this section by showing that if K satisfies a certain condition (II′) which is a stronger form of the amalgamation property, then every system in K is algebraically closed. We then consider a special case of the amalgamation property, and show that it is equivalent to the original property. This result is used, first, to give an elementary proof of the fact that the class of all fields has the amalgamation property and, second, to give another example of a class K that satisfies (I) and (II), but does not satisfy (II′). This class, while probably of no great interest in itself, is useful for providing counterexamples to various questions that arise naturally in connection with the results in the earlier sections.

We now drop the blanket assumption, in effect since the end of Section 2, that K satisfies the conditions (I_1), (I_2), (I_3) and (II), and shall instead state explicitly in each case what is being assumed.

THEOREM 10.1. *Suppose* (I_1), (I_2) *and* (I_3) *hold, and also the following condition*:

(II′) *If the systems B_1 and B_2 in K have a common subsystem A, and if B_1 and B_2 have no element in common except those in A, then B_1 and B_2 have a common extension that also belongs to K.*

Under these hypotheses every member of K is algebraically closed.

PROOF. The property (II′) can easily be extended to arbitrarily many systems B_i. More precisely, we have: If the systems B_i with i in I belong to K and have a common subsystem A, and if, for any two distinct members i and j of I, the systems B_i and B_j have no element in common except those in A, then all the systems B_i with i in I have a common extension that also belongs to K.

From this the conclusion readily follows. For suppose A and B are two systems in K with $A \leqq B$. Choosing an infinite set I, we can associate with each member i of I an extension B_i of A that is A-isomorphic to B, in such a way that for any two distinct members i and j of I the systems B_i and B_j have no elements in common except those in A. By the above generalization of (II′) it follows that there exists a system C in K that is an extension of all the systems B_i. If u is any member of B, then there exists for each i in I an element u_i in B_i that is A-equivalent to u. If, further, u is not in A, then all the elements u_i must be distinct, and since they are all members of the same extension C of A, it follows that u is not algebraic over A. Thus A is algebraically closed.

THEOREM 10.2. *Suppose* (I_1), (I_2) *and* (I_3) *hold, and also the following condition*:

(II_0) *For any systems A, B_1 and B_2 in K, if B_1 and B_2 are simple extensions of A, then for some extension C of A, also in K, there exist A-isomorphisms of B_1 and of B_2 into C.*

Under these hypotheses K has the amagamation property.

PROOF. It clearly suffices to show that (II_0) implies the corresponding property with the word "simple" omitted from the hypothesis.

First consider the case in which B_1 is a simple extension of A, $B_1 = A(u)$, but B_2 is an arbitrary extension of A. Consider the family L of all four-termed sequences $\langle B, C, f_1, f_2 \rangle$ such that $A \leqq B \leqq B_2$, C is a member of K and an extension of A, f_1 is an A-isomorphism of B_1 into C, and f_2 is an A-isomorphism of B onto a subsystem $\bar{C}$ of C, such that $C = \bar{C}(f_1(u))$. (To avoid the set-theoretic anomalies the choice of C can be restricted by requiring it to be a subset of some sufficiently large set that is fixed in advance.) The family L is partially ordered by the condition that

$$\langle B', C', f_1', f_2' \rangle \prec \langle B'', D'', f_1'', f_2'' \rangle$$

if and only if $B' \leqq B''$, $C' \leqq C''$, $f_1' = f_1''$, and f_2'' agrees with f_2' on B'. A routine argument shows that every simply ordered subfamily of L has an upper bound that belongs to L. This implies the existence of a maximal member $\langle B, C, f_1, f_2 \rangle$, and we infer with the aid of (II_0) and (I_1) that $B = B_2$.

In the general case with B_1 and B_2 arbitrary extensions of A, we use essentially the same argument, except that in place of the condition $C = \bar{C}(f_1(u))$ we now require $C = \bar{C}(\bar{B})$ where $\bar{B}$ is the image of B_1 under f_1, and in proving that the maximal sequence satisfies the condition $B = B_2$ we now use in place of (II_0) the more general property derived in the first part of the proof.

It is easy to prove, using only the most elementary properties of simple extensions of fields, that the class K of all fields has the property (II_0). Actually, we verify directly the more general property considered in the first part of the proof of Theorem 10.2. In fact, suppose A, B_1, B_2 are fields, B_1 is a simple extension of A, $B_1 = A(u)$, and B_2 is an arbitrary extension of A. If u is an indeterminate over A, then we let $C = B_2(v)$ where v is an indeterminate over B_2, but if u is algebraic over A, and therefore a root of a polynomial p that is irreducible over A, then we consider a factor q of p that is irreducible over B_2, and let $C = B_2(v)$ where v is a root of q. In either case the isomorphism of B_1 into C maps $r(u)$ onto $r(v)$, for every rational function r over A, and as the isomorphism of B_2 into C we take simply the identity automorphism of B_2.

We now give another simple example of a class K that satisfies the conditions (I) and (II), but does not satisfy (II').

THEOREM 10.3. *Suppose n is an integer greater than* 1, *and let K be the class of all algebraic systems $\langle A, F \rangle$ such that F is a unary operation over A, and the following conditions hold*:

(i) *For any positive integer k, and for all a in A, if $F^k(a) = a$, then $F(a) = a$.*

(ii) *For any a in A, there exist at most n distinct elements x in A such that $F(x) = a$.*

Then K satisfies (II), *but does not satisfy* (II').

OUTLINE OF PROOF. Suppose $\langle B_1, G_1 \rangle$ is a simple extension of $\langle A, F \rangle$, obtained by adjoining the element u, and suppose $\langle B_2, G_2 \rangle$ is an arbitrary extension of $\langle A, F \rangle$. We wish to prove that there exists an $\langle A, F \rangle$-isomorphism of $\langle B_1, G_1 \rangle$ into some extension $\langle C, H \rangle$ of $\langle B_2, G_2 \rangle$. It may be assumed without loss of generality that the only elements common to B_1 and B_2 are those in A.

If there exists no positive integer k such that $G_1^k(u)$ is in A, then we let $C = B_1 \cup B_2$, and so define the operation H that it agrees with G_1 on B_1 and with G_2 on B_2. The alternative case easily reduces to the subcase in which the element $a = G_1(u)$ is in A, and therefore $B_1 = A \cup \{u\}$. If the equation $G_2(x) = a$ has a solution $x = v$ that is in B_2 but not in A, then we can take $\langle C, H\rangle = \langle B_2, G_2\rangle$, for the function that takes u into v and each element of A into itself is an $\langle A, F\rangle$-isomorphism of $\langle B_1, G_1\rangle$ into $\langle B_2, G_2\rangle$. Finally, if the equation $F_2(x) = a$ has no solution that is in B_2 but not in A, then it is easy to see that the construction applied in the first case yields a system with the desired properties.

That K does not have the property (II′) is obvious: We take for B_1 and B_2 two n-element sets with only one element a in common, let $A = \{a\}$, and let $F(a) = a$, $G_1(x) = a$ for all x in B_1, and $G_2(x) = a$ for all x in B_2. Then the systems $\langle B_1, F_1\rangle$ and $\langle B_2, F_2\rangle$ have $\langle A, F\rangle$ as a common subsystem, but they have no common extension that belongs to K, for if such an extension $\langle C, H\rangle$ did exist, then the equation $H(x) = a$ would have $2n - 1$ distinct solutions.

This example can be used to settle by means of counterexamples various questions that arise in connection with the results in the preceding sections. As examples we consider the following statements which are known to hold if K is the class of all fields:

(a) *If A and B are systems in K, and if B is a finite, separable extension of A, then B is a simple extension of A.*

(b) *If A, B and C are systems in K, if $A \leqq B \leqq C$, and if C is a separable extension of A, then C is a separable extension of B.*

(c) *If A, B and C are systems in K, if $A \leqq B \leqq C$, and if u is an element in C that is separable over A, then u is separable over B.*

(d) *If A and B are systems in K, if A is perfect, and if B is an algebraic extension of A, then B is perfect.*

(e) *If A and C are systems in K, if C is a finite normal extension of A, and if H is a subgroup of $G(C/A)$, then there exists a system B in K such that $A \leqq B \leqq C$, and $H = G(C/B)$.*

Before showing that none of these statements hold if K is defined as in Theorem 10.3, we must find out what some of the relevant concepts mean in this case. It will be convenient to denote the operations in all the systems under consideration by the same letter F, and to identify in the usual manner the system $\langle A, F\rangle$ with its underlying set A.

Suppose $A \leqq B$ and u is an element in B. It is easy to show that u is algebraic over A if and only if $F^k(u)$ is in A for some positive integer

k. Assuming that u is algebraic over A, let $\delta(u)$ be the smallest positive integer k for which $F^k(u)$ is in A. Two elements u and v in B that are algebraic over A but are not in A are then A-equivalent if and only if $\delta(u)=\delta(v)$ and $F^{\delta(u)}(u)=F^{\delta(u)}(v)$.

We are now ready to construct the counterexamples to the statements (a)–(e). For convenience we take $n=4$.

EXAMPLE (a). B consists of three distinct elements a, u_1, u_2 with $F(a)=F(u_1)=F(u_2)=a$, and A consists of the element a alone. Since u_1 and u_2 are A-equivalent, B is a separable extension of A, but it is not a simple extension since $A(u_i)=\{a,u_i\}$ for $i=1,\ 2$.

EXAMPLE (b). C consists of four distinct elements a, u_1,u_2,u_3 with $F(a)=F(u_1)=F(u_2)=F(u_3)=a$, B and A are as in Example (a). The element u_3 is of reduced degree 1 over B, because in order for an element x to be B-equivalent to u_3 it would have to satisfy the equation $F(x)=a$, and since this equation has three solutions in B, it can have no more than one additional solution in a given extension of B. On the other hand, C is obviously a separable extension of A.

EXAMPLE (c). A, B and C are as in Example (b), and $u=u_3$.

EXAMPLE (d). A and B are as in Example (a). It is easily seen that if C is an algebraically closed algebraic extension of A, then for every u in C the equation $F(x)=u$ has four distinct solutions. Thus every element that is in C but not in A has reduced degree at least three over A, and C is therefore a separable extension of A. From this it follows by 8.7 that A is perfect. On the other hand, if C is an algebraically closed algebraic extension of B, then the equation $F(x)=a$ has exactly one solution x in C that is not in B, and this element x is therefore of reduced degree 1 over B. Consequently B is not perfect.

EXAMPLE (e). A and C are as in Example (b), and H is the group of all even permutations of C that leave a fixed. The fixpoint set of H is in this case A, and $G(C/A)$ consists of all those permutations of C that leave a fixed. Therefore, there is no system B such that $A\leqq B\leqq C$ and $H=G(C/B)$.

11. Remarks on the notion of an algebraic element.

The notion of an algebraic element used here is rather restrictive, and one might ask whether there does not exist a more general concept for which results similar to the ones obtained here could be established. Theorem 11.1 below suggests, however, that if the present concept is

replaced by a more general one, then several of the principal results of the present paper are bound to fail. In particular, it shows that if the new concept has the property that the adjoining of an algebraic set of elements to a given system yields an algebraic extension, then it cannot be the case that every system has an algebraically closed algebraic extension.

THEOREM 11.1. *Suppose K is a class of systems that satisfies the condition* (I), *suppose A and B are systems in K with $A \leqq B$, and suppose u is an element in B that is not algebraic over A. If m is any cardinal number, then there exists in K an extension C of A such that there are at least m distinct elements in C that are A-equivalent to u.*

OUTLINE OF PROOF. (See footnote 1.) The method of proof is essentially the same as for Theorem 9.1, and there is no need to repeat all the details. The systems $A(U)$ and $A(U,u)$ that occur there are now replaced by $A(u)$ and A, respectively, and the set of all natural numbers, as the index set for the new operational symbols, is replaced by a set I of cardinality m. Except for this, the sentences which we add to Σ can be described in exactly the same manner as before, and one easily verifies that the conclusion of our theorem is equivalent to the assertion that the set of sentences, Σ_I, obtained in this manner, is consistent. We then consider, for each finite subset J of I, the set Σ_J of all those sentences in Σ_I that have the additional property that all the new operational symbols that occur in them are associated with members of J. The hypothesis that u is not algebraic over A implies that all these sets Σ_J are consistent, and this in turn implies the consistency of Σ_I.

COROLLARY 11.2. *Suppose K is a class of relational systems that satisfies the conditions* (I) *and* (II), *and suppose A and B are systems in K such that $A \leqq B$. Then the following conditions are equivalent*:

(i) *B is a normal extension of A.*

(ii) *For every extension C of B in K, every A-isomorphism of B into C maps B into itself.*

(iii) *For every extension C of B in K, every A-automorphism of C maps B onto itself.*

PROOF. By 5.5 the conditions (i)–(iii) are equivalent provided B is an algebraic extension of A. Inasmuch as normal extensions are by definition algebraic, it follows that (i) implies (ii). Since (ii) obviously implies (iii), it therefore suffices to show that if (iii) holds, then B is an algebraic extension of A.

If there is an element u in B that is not algebraic over A, and if m

is any cardinal, then it follows from 11.1 with the aid of (II) and (I_1) that there exists an extension C' of B in K with the property that there are at least m distinct elements v in C' that are A-equivalent to u. Taking m sufficiently large we infer that, for the corresponding extension C', some of these elements v are not in B. By 3.3, the A-isomorphism of $A(u)$ onto $A(v)$ can be extended to an A-automorphism of some extension C of C' that also belongs to K. Since v was so chosen that it does not belong to B, it follows that this automorphism does not map B into itself, and the statement (iii) therefore fails.

This result shows that if our hypothetical notion of a generalized algebraic element is accompanied by a suitable modification of the definition of a normal extension, and if it is true of these new concepts that every algebraic extension of a given system is a subsystem of a normal extension of that system, then it cannot be the case that normal extensions are characterized by the properties (ii) and (iii). We shall not try to make these observations more precise, but the above considerations suggest that the concept of an algebraic element which we have adopted is the least restrictive one for which the principal results of this paper are valid.

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