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REMOVABLE SINGULARITIES OF CONTINUOUS HARMONIC FUNCTIONS IN R^m

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1.

Let E be a compact set of m-dimensional Euclidean space R^m. If $h(x)$ is harmonic and bounded in a neighbourhood of E, and if the $(m-2)$-capacity of E vanishes then $u(x)$ can be extended as a harmonic function to E. We say that E is removable for the class of bounded harmonic functions. On the other hand, if E is a smooth closed surface, that is a $(m-1)$-dimensional set, there exist harmonic functions with arbitrarily high smoothness, which cannot be extended to E. Our aim here is to prove a theorem which connects the above mentioned two results.

Let D be a domain bounded by a smooth outer surface Γ and a compact set E and denote by H_α the class of harmonic functions in D which satisfy a Hölder condition of order α, $0<\alpha<1$, in D:

$$|u(x)-u(x')| \leqq \text{Const.}\,|x-x'|^\alpha, \qquad x, x' \in D\,. \tag{1.1}$$

The set E is said to have β-dimensional measure zero, $0<\beta<m$, if E can be covered by open spheres of radii r_ν such that $\sum r_\nu{}^\beta$ is arbitrarily small. The following theorem holds.

THEOREM. *E is removable for the class H_α if and only if E has $(m-2+\alpha)$-dimensional measure zero.*

2.

We first assume the $(m-2+\alpha)$-dimensional measure does not vanish. It is then well-known (see Frostman [1]) that there is a distribution μ of unit mass on E such that

$$\mu(S) \leqq C r^{m-2+\alpha}$$

for all spheres S where r denotes the radius of S. We shall prove that

$$u(x) = \int_E \frac{d\mu(y)}{|x-y|^{m-2}}$$

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satisfies (1.1). We define $\mu(r,x)=\mu(\{y \mid |y-x|<r\})$ and find for x, $x' \in D$, $|x-x'|=\delta$,

$$u(x)-u(x') = \int_0^\infty r^{2-m}\, d\mu(r,x) - \int_0^\infty r^{2-m}\, d\mu(r,x')$$

$$= (m-2)\int_0^\infty \big(\mu(r,x)-\mu(r,x')\big) r^{1-m} dr$$

$$\leqq C_1 \int_0^{2\delta} r^{m-2+\alpha} r^{1-m}\, dr + (m-2)\int_{2\delta}^\infty \big(\mu(r,x)-\mu(r-\delta,x)\big) r^{1-m}\, dr$$

$$< C_2\delta^\alpha + (m-2)\int_\delta^\infty \big(\mu(r,x)(r^{1-m}-(r+\delta)^{1-m}\big)\, dr$$

$$< C_2\delta^\alpha + C_3 \int_\delta^\infty \frac{r^{m-2+\alpha}\,\delta}{r^m}\, dr = C_4\delta^\alpha .$$

Since x and x' can be interchanged we have proved (1.1).

3.

We now assume that the $(m-2+\alpha)$-dimensional measure of E vanishes and that $u(x)$ satisfies (1.1). Let $u_1(x)$ be the harmonic function inside Γ which is equal to $u(x)$ on Γ. If we define $v(x)=u(x)-u_1(x)$ then $v(x)=0$ on Γ and our assertion is that $v(x)\equiv 0$.

We can cover E by a finite number of closed spheres S_ν,

$$S_\nu:\ |x-x_\nu| \leqq r_\nu$$

such that

$$\sum r_\nu^{\,m-2+\alpha} \leqq \varepsilon .$$

We assume that ε has its smallest value when the number of spheres is $\leqq n$. In the proof we shall also use the expanded spheres

$$S_\nu(t):\ |x-x_\nu| \leqq r_\nu t, \qquad 1 \leqq t \leqq 3 .$$

For $t>1$ every point of E is strictly inside $\bigcup S_\nu(t)=\sum(t)$. The part of $\partial\sum(t)$ which is boundary of the unbounded component of the complement of $\sum(t)$ is denoted $\sigma(t)=\bigcup\sigma_\nu(t)$, where $\sigma_\nu(t)$ is $\sigma(t)\cap\partial S_\nu(t)$. Clearly $\sigma(t)$ does not meet E.

By Green's formula we have, $t>1$,

$$(3.1) \qquad \psi(t) = \int_{D-\Sigma(t)} |\operatorname{grad} v|^2\, dx = \int_{\sigma(t)} v \frac{\partial v}{\partial n}\, d\sigma = \tfrac{1}{2} \int_{\sigma(t)} \frac{\partial v^2}{\partial n}\, d\sigma .$$

If $v \not\equiv$ constant, $\psi(t)$ is $\geqq$ const. > 0 in $1 < t \leqq 3$, if ε is small enough. We rewrite (3.1) introducing the unit sphere U. Points on U are denoted ξ and its area element dA_ξ. The part of U for which $x_\nu + tr_\nu \xi \in \sigma_\nu(t)$ is called $a_\nu(t)$. Integrating (3.1) and using these notations we find

$$(3.2) \qquad -2 \int_2^3 \psi(t) t^{1-m}\, dt = \sum_{\nu=1}^{n} r_\nu^{\,m-2} \int_2^3 dt \int_{a_\nu(t)} \frac{\partial}{\partial t} v^2(x_\nu + r_\nu t \xi)\, dA_\xi .$$

In each term on the right of (3.2) we shall now interchange the order of integration. We must then study for ξ fixed for which values of t a certain ray $x_\nu + r_\nu t \xi$ belongs to $\sigma_\nu(t)$. We distinguish four cases, the first two of which are trivial.

(a) $x_\nu + r_\nu t\xi \notin \sigma_\nu(t)$, $2 \leqq t \leqq 3$. For such a ξ we get 0 as contribution to (3.2) from the νth term.

(b) $x_\nu + r_\nu t\xi \in \sigma_\nu(t)$, $2 \leqq t \leqq 3$. We can evaluate the t-integration and get the contribution

$$v^2(x_\nu + 3r_\nu \xi) - v^2(x_\nu + 2r_\nu \xi) = O(r_\nu^{\alpha}) .$$

(c) The remaining possibility is: $x_\nu + r_\nu t\xi \in \sigma_\nu(t)$, $\tau_i \leqq t \leqq \tau_i'$, $i = 0,1,2,\ldots,p$, $2 \leqq \tau_0 < \tau_0' < \tau_1 < \ldots < \tau_p' \leqq 3$. For every τ_i', $i < p$, there is an index $\mu \neq \nu$ so that $x_\nu + r_\nu \tau_i' \xi \in \sigma_\mu(\tau_i')$. We here have two essentially different cases.

(c1) $r_\mu \geqq r_\nu$. If we consider the two-dimensional plane containing x_ν, x_μ and $x = x_\nu + r_\nu \tau_i' \xi$, we see that $x' = x_\nu + r_\nu t \xi$, $t > \tau_i'$, must be interior to $S_\mu(t)$ and hence $x_\nu + r_\nu t\xi \notin \sigma_\nu(t)$, $t > \tau_i'$. (c1) can thus occur only if $i = p$.

(c2) We now assume $i < p$ and $r_\mu \leqq r_\nu$. We first observe that $x_\nu + r_\nu t\xi$, $2 \leqq t \leqq 3$, belongs to a certain sphere $S_\mu(t)$ in a t-interval and that its length is $\leqq 6 r_\mu r_\nu^{-1}$. We now consider an interval (τ_i', τ_{i+1}). The corresponding spheres are $S_\mu(t)$, $\mu = \mu_1, \ldots, \mu_k$. We can write, if $\varphi(t) = v^2(x_\nu + r_\nu t \xi)$,

$$(3.3) \qquad \varphi(\tau_{i+1}) - \varphi(\tau_i') = \sum_{j=1}^{k} \big(\varphi(s_{j+1}) - \varphi(s_j)\big) ,$$

where each pair s_j, s_{j+1} belongs to one $S_{\mu_l}(3)$. Hence

$$(3.4) \qquad |\varphi(\tau_{i+1}) - \varphi(\tau_i')| \leqq \sum_{j=1}^{k} C r_\nu^{\alpha} |s_{j+1} - s_j|^{\alpha} \leqq C 6^{\alpha} \sum_{j=1}^{k} r_{\mu_j}^{\alpha} .$$

We now evaluate the t-integral of the νth term in (3.2) and find

$$\sum_0^p \big(\varphi(\tau_i') - \varphi(\tau_i)\big) .$$

If we add the relations (3.3) for $i=0,1,\ldots,p-1$, and use (3.4) we get the estimate

$$O(r_\nu^{\alpha})+O(\textstyle\sum^1 r_\mu^{\alpha}) , \tag{3.5}$$

where $\sum^1$ indicates that the summation is extended over those μ for which $x_\nu+r_\nu t\xi$, $t\leqq 3$, meets $S_\mu(3)$.

We consider the estimate (3.5) for different points $\xi\in U$. The area of U for which $x_\nu+r_\nu t\xi\in S_\mu(3)$ for some t is $O(r_\mu^{m-1}r_\nu^{1-m})$. The total ν^{th} term in (3.2) is thus

$$O(r_\nu^{m-2+\alpha})+O(r_\nu^{-1}\textstyle\sum^2 r_\mu^{m-1+\alpha}) , \tag{3.6}$$

where $\sum^2$ indicates summation over those μ for which $S_\mu(3)\cap S_\nu(3)\neq\emptyset$ and $r_\mu\leqq r_\nu$. The last relations imply $S_\mu(1)\subset S_\nu(7)$. Since the covering by the spheres $S_\nu=S_\nu(1)$ was assumed to be minimal we have

$$\textstyle\sum^2 r_\mu^{m-2+\alpha} \leqq 7^{m-2+\alpha} r_\nu^{m-2+\alpha} .$$

If we use this and $r_\mu\leqq r_\nu$ in (3.6), we find that the ν^{th} term in (3.2) is $O(r_\nu^{m-2+\alpha})$ and so

$$\int_2^3 \psi(t)\, t^{1-m}\, dt \leqq \text{Const}\sum_1^n r_\nu^{m-2+\alpha} \leqq \text{Const}\cdot\varepsilon .$$

Hence $\psi(t)$ cannot be uniformly positive and so $v(x)\equiv$ constant, and then $v\equiv 0$, as was to be proved.

REFERENCE

1. O. Frostman, *Potentiel d'équilibre et capacité des ensembles,* Thèse, Medd. Lunds Univ. Mat. Sem. 3 (1935), Lund.

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