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# REMARKS ON THE EQUIVALENCE PRINCIPLE IN FLUCTUATION THEORY

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Let $c=(c_1,c_2,\ldots,c_n)$ be an arbitrary sequence of real numbers, and let $\mathscr{P}_c$ be the set of all $n!$ sequences that can be formed from $c$ by permutations. If $x=(x_1,\ldots,x_n)$ is in $\mathscr{P}_c$, let $s_0(x)=0$, and $s_i(x)=x_1+x_2+\ldots+x_i$ for $i=1,2,\ldots,n$. Set $L_n(x)$ equal to the first subscript $j$ for which $s_j(x)=\max\{s_i(x);\ 0\leqq i\leqq n\}$, and let $N_n(x)$ denote the number of partial sums $s_i(x)$ for which $s_i(x)>0$. Define $v_n(k;c)$, $w_n(k;c)$ to be the number of elements $x\in\mathscr{P}_c$ for which $N_n(x)=k$, $L_n(x)=k$, respectively.

A basic theorem in fluctuation theory states that $v_n(k;c)=w_n(k;c)$. This result (called the Equivalence Principle by Feller [4]) was first proved by Sparre Andersen [1] in 1953. A simpler proof was given by Feller [4] in 1959. Both authors used induction arguments. Although these arguments, particularly that of [4], are simple, they do not provide any hint as to the explicit natural one-to-one correspondence which exists between the set of paths for which $L_n$ is equal to $k$ and the set of paths for which $N_n$ is equal to $k$. A direct proof of this Equivalence Principle is given below by describing a natural one-to-one correspondence. [The authors appreciate the referee's pointing out that such a proof is attributed in [7] to Ian Richards. A similar method of proof is used by Sparre Andersen [8], to obtain a generalization of Theorem A, quoted below.] It is the purpose of this paper to show how this correspondence can be used to obtain a simple proof of a recent generalization of the Equivalence Principle due to Brandt [3].

Theorem A (Sparre Andersen). *For every sequence* $c=(c_1,c_2,\ldots,c_n)$,

$$v_n(k;c) = w_n(k;c), \qquad k = 0,1,\ldots,n\,. \tag{1}$$

Proof. This theorem will be proved by obtaining a one-to-one trans-

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formation, $\pi = \pi_n$, of $\mathscr{P}_c$ onto itself such that $N_n(x) = L_n(\pi(x))$ for all $x \in \mathscr{P}_c$. Choose $x \in \mathscr{P}_c$. If $N_n(x) = 0$, set $\pi(x) = x$. Then clearly $L_n(x) = 0 = N_n(x)$. Suppose $N_n(x) = k > 0$. Let $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \ldots < j_{n-k}$ denote, respectively, the subscripts of the positive and of the negative partial sums of $x$. That is, $s_{i_m} > 0$ for $m = 1, 2, \ldots, k$, while $s_{j_m} \leqq 0$ for $m = 1, 2, \ldots, n-k$. Then, define

$$\pi(x) = (x_{i_k}, x_{i_{k-1}}, \ldots, x_{i_1}, x_{j_1}, x_{j_2}, \ldots, x_{j_{n-k}}) .$$

It turns out that this definition insures that $\pi$ is one-to-one onto, and satisfies $N_n(x) = L_n(\pi(x))$. In order to prove this, we shall define the "natural" transformation, $\psi$ say, from $\mathscr{P}_c$ into itself, for which $L_n(x) = N_n(\psi(x))$. From its definition it will be clear that $\psi^{-1}$ exists and is equal to $\pi$, thereby proving that $\pi$ is one-to-one onto. The transformations $\pi$ and $\psi$ are illustrated in Fig. 1.

Let $x \in \mathscr{P}_c$ be such that $L_n(x) = k > 0$. Define ascending ladder points as the indices $k = m_0 > m_1 > \ldots > m_h = 0$ for which $s_{m_j} \geqq s_i$ for all $i \leqq m_j$. Also define descending ladder points as those indices $k = \mu_0 < \mu_1 < \ldots$ for which $s_{\mu_j} \geqq s_i$ when $i \geqq \mu_j$. From these sets of ascending and descending ladder points choose subsequences $k = m_{i_0} \geqq m_{i_1} > \ldots > m_{i_r}$ and $k = \mu_{j_0} < \mu_{j_1} < \mu_{j_2} < \ldots < \mu_{j_\varrho}$, defined recursively as follows: $i_1$ is the largest integer $v \geqq 0$ for which $s_{m_v} \geqq s_{\mu_1}, j_1$ is the largest integer $v \geqq 0$ for which $s_{\mu_v} > s_{m_{i_1+1}}, i_\alpha$ is the largest integer $v > i_{\alpha-1}$ for which $s_{m_v} \geqq s_{\mu_{j_{\alpha-1}+1}}$, while $j_\alpha$ is the largest integer $v > j_{\alpha-1}$ for which $s_{\mu_v} > s_{m_{i_\alpha+1}}$, $\alpha = 2, 3, \ldots$. Set

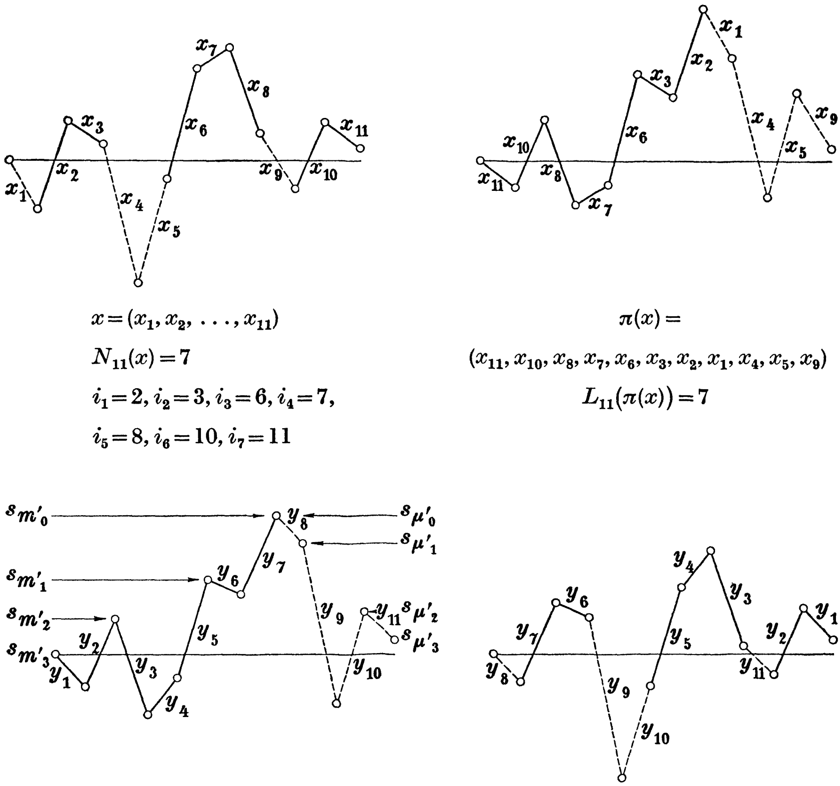
$$m'_\alpha = m_{i_\alpha}, \quad \mu'_\alpha = \mu_{j_\alpha}, \quad y^\alpha = (x_{m'_{\alpha-1}}, x_{m'_{\alpha-1}-1}, \ldots, x_{m'_\alpha+1})$$

and

$$z^\alpha = (x_{\mu'_{\alpha-1}+1}, x_{\mu'_{\alpha-1}+2}, \ldots, x_{\mu'_\alpha}) \quad \text{for} \quad \alpha = 1, 2, \ldots .$$

Let it be understood that $y^1$ is an empty sequence if $i_1 = 0$. Define the transformation $\psi : \mathscr{P}_c \to \mathscr{P}_c$ by $\psi(x) = x$ when $L_n(x) = 0$ and $\psi(x) = (y^1, z^1, y^2, z^2, \ldots)$ and otherwise, where $\psi(x)$ is to be considered as an $n$-tuple. It is a routine matter to check that $s_j(\psi(x)) > 0$ if and only if the $j$-th element of $\psi(x)$ is one of the elements $x_1, x_2, \ldots, x_k$, that is, one of the elements of an ascending $y^\alpha$ sequence. Consequently, $N_n(\psi(x)) = k = L_n(x)$. Furthermore, it is clear that $\pi(\psi(x)) = x$ for all $x \in \mathscr{P}_c$ thereby implying that $\pi$ and $\psi$ are one-to-one transformations of $\mathscr{P}_c$ onto itself. This completes the proof of the theorem.

In a recent paper, Brandt [3] derives a generalization of the Equivalence Principle. He defines $N_{n,\delta}(x)$ to be the number of partial sums greater than $\delta$, and $L_{n,\delta}(x)$ to be the index of the first partial sum greater than or equal to $\max_{0 \leqq i \leqq n}[s_i(x) - \delta]$ if $\delta \geqq 0$, and the index of the last partial sum greater than $\max_{0 \leqq i \leqq n}[s_i(x) + \delta]$ if $\delta < 0$. Define $v_n(k; c, \delta)$,



$x = (x_1, x_2, \ldots, x_{11})$
$N_{11}(x) = 7$
$i_1 = 2, i_2 = 3, i_3 = 6, i_4 = 7,$
$i_5 = 8, i_6 = 10, i_7 = 11$

$\pi(x) =$
$(x_{11}, x_{10}, x_8, x_7, x_6, x_3, x_2, x_1, x_4, x_5, x_9)$
$L_{11}(\pi(x)) = 7$

$y = (y_1, y_2, \ldots, y_{11})$
$L_{11}(y) = 7$
$m'_0 = 7, m'_1 = 5, m'_2 = 2, m'_3 = 0,$
$\mu'_0 = 7, \mu'_1 = 8, \mu'_2 = 10, \mu'_3 = 11$

$\psi(y) =$
$(y_8, y_7, y_6, y_9, y_{10}, y_5, y_4, y_3, y_{11}, y_2, y_1)$
$N_{11}(\psi(y)) = 7$

Fig. 1.

$w_n(k; c,\delta)$ to be the number of sequences $x \in \mathscr{P}_c$ for which $N_{n,\delta}(x) = k$, $L_{n,\delta}(x) = k$, respectively. In [3], Brandt proves the following:

THEOREM B (Brandt). *For every sequence $c = (c_1, \ldots, c_n)$ and all real $\delta$*

$$v_n(k; c,\delta) = w_n(k; c,\delta), \qquad k = 0, 1, \ldots, n\,. \tag{2}$$

PROOF. For $\delta = 0$, (2) is exactly (1). It therefore suffices to consider $\delta \neq 0$. However, this case may be derived as a direct application of the mappings $\pi$ and $\psi$ defined above in the proof of Theorem A. This is seen as follows. Assume first that $\delta > 0$. For each $x \in \mathscr{P}_c$ for which

$N_{n,\delta}(x)=k$, form the new $(n+1)$-tuple $x'=(-\delta,x_1,x_2,\ldots,x_n)$. Observe that $N_{n+1}(x')=k$. By applying the transformation $\pi$, considered as being defined for $(n+1)$-tuples, to $x'$, one obtains a sequence $\pi(x')=(b_1,b_2,\ldots,b_{n+1})$ for which $L_{n+1}(\pi(x'))=k$. Observe that $b_{k+1}=-\delta$, and that by deleting this element, the modified sequence

$$b = (b_1,\ldots,b_k,b_{k+2},\ldots,b_{n+1}) \in \mathscr{P}_c$$

and $L_{n,\delta}(b)=k$. On the other hand, if

$$x = (x_1,x_2,\ldots,x_n) \in \mathscr{P}_c$$

is a sequence for which $L_{n,\delta}(x)=k$, form the new $(n+1)$-tuple

$$x'' = (x_1,x_2,\ldots,x_k,-\delta,x_{k+1},\ldots,x_n)\ ,$$

for which $L_{n+1}(x'')=k$. Now apply $\psi$, considered as being defined for $(n+1)$-tuples, to $x''$ to obtain a sequence $\psi(x'')=(d_1,d_2,\ldots,d_{n+1})$ for which $N_{n+1}(\psi(x''))=k$. By the definition of $\psi$, it may be checked that $d_1=-\delta$ and that the modified sequence

$$d = (d_2,d_3,\ldots,d_{n+1}) \in \mathscr{P}_c$$

satisfies $N_{n,\delta}(d)=k$. This proves (2) for $\delta>0$. Now assume that $\delta<0$. If $x\in\mathscr{P}_c$ is such that $N_{n,\delta}(x)=k$, then observe that for this case, $N_{n+1}(x')=k+1$. Hence, with $x'$ and $b$ as before, it follows that $L_{n+1}(\pi(x'))=k+1$ and $L_{n,\delta}(b)=k$. On the other hand, if $x\in\mathscr{P}_c$ is such that $L_{n,\delta}(x)=k$, then $L_{n+1}(x'')=k+1$ and $N_{n+1}(\psi(x''))=k+1$. Thus with $d$ as before, $N_{n,\delta}(d)=k$. Consequently, in both cases, the mappings described above, which take $x$ into $b$ and $x$ into $d$ respectively, suffice to prove (2).

REMARKS. (i) It is possible to obtain results for the location of the *minimum* partial sum, analogous to those obtained above for $L_n$. For if

$$x = (x_1,x_2,\ldots,x_n) \in \mathscr{P}_c$$

is such that $L_n(x)=k$, then for $y=(x_n,x_{n-1},\ldots,x_1)$, the index of the last minimum partial sum among $\{s_i(y);\,0\leqq i\leqq n\}$, is $n-k$.

(ii) A further generalization of the Equivalence Principle is possible as follows. In [5], we studied the joint behavior of the two quantities $N_n(x)$ and $M_n(x)\equiv N_n((x_n,x_{n-1},\ldots,x_1))$. It is natural to ask whether or not there exists a quantity $K_n(x)$ such that a one-to-one correspondence exists between paths for which $N_n=k$ and $M_n=j$, and paths for which

$L_n = k$ and $K_n = j$. That this is possible follows directly from the proof of Theorem A, as we indicate now. Let $x \in \mathscr{P}_c$ be such that $L_n(x) = k$. Obtain the ladder points

$$m_r' < m'_{r-1} < \ldots < m_1' \leqq m_0' = k = \mu_0' < \mu_1' < \ldots < \mu_\varrho'$$

and the corresponding subsequences $y^1, y^2, \ldots, y^r$ and $z^1, z^2, \ldots, z^\varrho$ as described in the proof of Theorem A. For $\delta \geqq 0$, let $K_{n,\delta}^{(i)}(x)$, $1 \leqq i \leqq \varrho$, be the number of partial sums of the path $z^{(i)}$ which do not exceed $s_{\mu'_{i-1}}(x) - s_{m'_i}(x) - \delta$. Set

$$K_{n,\delta}(x) = \sum_i K_{n,\delta}^{(i)}(x) .$$

For $\delta \leqq 0$, let $K_{n,\delta}^{(i)}(x)$, $1 \leqq i \leqq r$, be the number of partial sums of the path $y^{(i)}$ which do not exceed $s_{m'_{i-1}}(x) - s_{\mu'_{i-1}}(x) - \delta$, and set

$$K_{n,\delta}(x) = \sum_i K_{n,\delta}^{(i)}(x) + n - k .$$

In either case, let $K_n$ denote $K_{n,\delta}$ for the special case $\delta = s_n(x)$. One may then check that the mappings $\pi$ and $\psi$ of Theorem A exhibit the desired equivalence between $(N_n, M_n)$ and $(L_n, K_n)$. In fact, the above description leads to an equivalence between $(N_n, n - N_{n,\delta})$ and $(L_n, K_{n,\delta})$.

For a sequence $x = (c_1, c_2, \ldots, c_n)$, let $\mathscr{C}_c$ be the set of all $n$ sequences formed from $c$ by *cyclic* permutations. Let $v_n^*(k; c)$, $w_n^*(k; c)$ denote, respectively, the number of elements $x \in \mathscr{C}_c$ for which $N_n(x) = k$, $L_n(x) = k$. The following specialization of the Equivalence Principle to the case of cyclic permutations is implicit in the proof of Spitzer's Theorem 2.1, [6].

LEMMA. *If $c$ is such that for all $x \in \mathscr{C}_c$, $s_i(x) \neq 0$, $i = 1, 2, \ldots, n-1$, and $s_n(x) = 0$, then $v_n^*(k; c) = w_n^*(k; c) = 1$ for $k = 0, 1, \ldots, n-1$.*

An interesting application of this lemma is the following. For a path $x = (x_1, \ldots, x_n)$ define $L_n^*(x)$ to be the smallest index $j$ for which

$$\max_{0 \leqq i \leqq n} \max [s_i(x), s_n(x) - s_i(x)]$$

occurs at $i = j$. For a sequence $c = (c_1, c_2, \ldots, c_n)$, let $\mathscr{I}_c$ denote the set of all $n$ *inverted cyclic* permutations, namely, those permutations which take $(x_1, x_2, \ldots, x_n)$ into $(x_2, x_3, \ldots, x_n, -x_1)$. One then obtains the following

THEOREM. *If $c$ is such that for all $x \in \mathscr{I}_c$, $s_i(x) \neq s_j(x)$, $0 \leqq i < j \leqq n$, then for each $r = 0, 1, \ldots, n-1$, there exists exactly one $x \in \mathscr{I}_c$ satisfying $L_n^*(x) = r$.*

PROOF. As was done in Corollary 2.3 of [5], associate with any path $x$, the new path $x^* = (x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n)$. Then $s_{2n}(c^*) = 0$

and $c^*$ satisfies the hypotheses of Lemma 1. Consequently, each path $y \in \mathscr{C}_{c^*}$ yields a distinct value for $L_{2n}(y)$. However, it is clear that

$$\max_{0 \leqq i \leqq n} \max [s_i(x), s_n(x) - s_i(x)] = \max_{0 \leqq i < 2n} s_i(x^*) .$$

Therefore, since $s_n(x) - s_i(x) = s_{n+i}(x^*)$, it follows that if $L_{2n}(x^*) = j$, then $L_n{}^*(x) = j$ if $j < n$ and $L_n{}^*(x) = j - n$ if $j \geqq n$. The proof is then complete since among the first $n$ cyclic permutations of $c^*$, $L_{2n}$ cannot take on both of the values $i$ and $n+i$ for any $i$.

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