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A NOTE ON INVARIANT UNIFORMITIES IN COSET SPACES

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The author can find no reference in the literature to the following statement:

PROPOSITION. *If G is a topological group and H is a compact subgroup of G, then there is a natural uniformity $\mathcal{U}$ on the coset space G/H which is invariant under the action of G.*

Invariant is taken in the sense that there exists a base $\mathcal{B}$ of entourages of $\mathcal{U}$ so that (Hx,Hy) belongs to an entourage B of $\mathcal{B}$ if and only if (Hxg,Hyg) belongs to B for any g in G, where Hz is a coset in G/H for $z \in G$.

The problem arose in an attempt to find conditions on a topological loop so that it have an invariant uniformity [2, Theorem 1]. L. Kristensen [4] has shown that G/H has an invariant metric if G/H is first countable (and H is compact). The purpose of this note is to point out that the generalization from invariant metric to invariant uniformity is possible (this is the above proposition) and to show how Kristensen's results may be obtained as a corollary.

Let G and H be as above, and let $\pi\colon G \to G/H$ be the natural mapping defined by $\pi(x) = Hx$. For each symmetric open neighborhood U of the identity e of G, define

$$U^{\sim} = \{(\pi(x),\pi(y))\colon \ x \in Uy\} .$$

Then the collection of all such $U^{\sim}$ and their finite intersections forms a base for a uniformity $\mathcal{U}$ of G/H, because the following conditions are satisfied:

a) the diagonal of $G/H \times G/H$ is contained in each $U^{\sim}$,
b) each $U^{\sim}$ is symmetric, because U is symmetric,

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c) for a given $U^{\sim}$, let $V^2 \subset U$. Then it can be seen that $V^{\sim} \circ V^{\sim} \subset U^{\sim}$,

d) for each $U^{\sim}$, $U^{\sim}(Hx) = HUx$, which is an open neighborhood of $Hx \in G/H$,

e) if $Hx \neq Hy$, compactness of H implies that there exists a symmetric open neighborhood U of e in G such that $UHx \cap UHy$ is empty [5, p. 51]. If $(Hx, Hy) \in U^{\sim}$, there exist h_1 and h_2 in H such that $h_1 x \in U h_2 y$. Consequently $(Hx, Hy) \notin U^{\sim}$.

From its definition $\mathscr{U}$ is invariant in the above sense. Therefore G/H has an invariant uniformity.

If it is assumed further that G/H has a countable fundamental system $\{W_i\}$ of neighborhoods of He, then it may also be assumed that the collection of all $\pi^{-1}(W_i) = V_i$ has the following properties [5, p. 28]: 1) each V_i is a symmetric neighborhood of G, 2) for any i, there is a j such that $V_j^2 \subset V_i$, and 3) for any i and j there is a k such that $V_k \subset V_i \cap V_j$. Then the collection of $\{V_i^{\sim}\}$ and their finite intersections forms a countable base for the uniformity $\mathscr{U}$ [3, p. 177], as a result of 1)–3) above and the following: if $U^{\sim} \in \mathscr{U}$, we will show that there is a $V_i^{\sim} \subset U^{\sim}$. There is an i such that $HV_i \subset HU$, since $\{W_i\}$ is a fundamental system in G/H. If $(Hx, Hy) \in V_i^{\sim}$, then $x \in V_i y$, $x \in HUy$, and finally $(Hx, Hy) \in U^{\sim}$.

Thus the existence of a countable base for the neighborhoods of $He \in G/H$ implies the existence of a countable base for $\mathscr{U}$ (note that it is not true in general that a uniform space with a countable base of neighborhoods at each point has a (separated) uniformity with a countable base of entourages).

The remarks of Kristensen [4, pp. 35–36] now follow from the above discussion together with classical theorems on metrizability. The space G/H is metrizable if and only if it is first countable, because a uniform space with a countable base of entourages is metrizable [3, p. 186]. Also if G/H is metrizable, the existence of an invariant uniformity implies the existence of an invariant metric [1]. Finally, as Kristensen points out, if G is locally compact, then the usual construction [3, p. 185] of the metric obtained from the uniformity allows one to pick the metric so that a subset of G/H is compact if and only if it is closed and bounded with respect to the metric.

Special cases, of course, of the foregoing discussion are the facts that every group has a right invariant uniformity and a first countable group has a right invariant metric. We also remark that if G is a topological group with the compact subgroup H, then there is a uniformity for G/H so that the elements of G are uniformly equicontinuous homeomorphisms of G/H.

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