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ON COMPLETE GRAPHS AND COMPLETE STARS CONTAINED AS SUBGRAPHS IN GRAPHS

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1. Introduction.

All graphs considered in this paper are finite and have no multiple edges and no loops. A *complete k-graph*, denoted by $\langle k \rangle$, is a graph having k vertices and $\frac{1}{2}k(k-1)$ edges, in particular a single vertex constitutes a $\langle 1 \rangle$. A *complete k,ϰ-star*, denoted by $\langle k,\varkappa \rangle$, is a $\langle k-1 \rangle$ together with $\varkappa$ further vertices each joined to every vertex of the $\langle k-1 \rangle$; k and $\varkappa$ are to be positive integers. A $\langle k \rangle$ is thus the same as a $\langle k,1 \rangle$, and a $\langle k,2 \rangle$ is the same as a $\langle k+1 \rangle$ with a single edge missing. Graphs will generally be denoted by Greek capitals. If Γ is a graph then n_Γ will denote the number of vertices of Γ and e_Γ the number of edges of Γ. The number of edges incident with a vertex is called the *valency* of the vertex in the graph.

P. Turán [2], [3] and K. Zarankiewicz [4] have found sufficient conditions for a $\langle k \rangle$ to be contained as a subgraph in a graph:

TURÁN'S THEOREM. *If* $n_\Gamma \geqq k \geqq 3$, *and if* $n_\Gamma = (k-1)t + r$, *where* $1 \leqq r \leqq k-1$, *and if*

$$e_\Gamma > \tfrac{1}{2}(n^2 - r^2)(k-2)/(k-1) + \tfrac{1}{2}r(r-1) ,$$

then $\Gamma \supseteqq \langle k \rangle$. *If equality holds then there exists a unique graph which satisfies the conditions of the theorem and does not contain a* $\langle k \rangle$ *as a subgraph.*

ZARANKIEWICZ'S THEOREM AS IMPROVED BY L. FINKIELSZTEJN. *If* $n_\Gamma \geqq k \geqq 3$ *and if each vertex has valency* $\geqq n_\Gamma(k-2)/(k-1)$ *and at least one vertex has valency* $> n_\Gamma(k-2)/(k-1)$, *then* $\Gamma \supseteqq \langle k \rangle$.

Zarankiewicz's theorem follows directly from Turán's [3].

The object of this paper is to improve Zarankiewicz's theorem—the new theorem is not implied by Turán's theorem—and to obtain conditions for $\langle k,\varkappa \rangle$'s to be contained as subgraphs in graphs.

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2. Stronger theorems of Zarankiewicz's type.

If Γ is a graph, then $\mathscr{V}(\Gamma)$ denotes the set of vertices of Γ. If $a \in \mathscr{V}(\Gamma)$ then $\mathscr{V}_a$ denotes the set of vertices to which a is joined, Γ_a denotes the subgraph of Γ spanned by these vertices, and $v(a,\Gamma)$ denotes the valency of a in Γ. If α is a real number, then $\mathscr{V}(\Gamma, \geqq \alpha)$ denotes the set of vertices having valency $\geqq \alpha$ in Γ. The sets $\mathscr{V}(\Gamma, > \alpha)$, $\mathscr{V}(\Gamma, \leqq \alpha)$, $\mathscr{V}(\Gamma, < \alpha)$ are defined analogously. Obviously $\mathscr{V}(\Gamma, > \alpha) \subseteq \mathscr{V}(\Gamma, \geqq \alpha)$ etc. The number $|\mathscr{V}(\Gamma, \geqq \alpha)|$ is denoted by $V(\Gamma, \geqq \alpha)$ etc. In this notation Zarankiewicz's theorem is as follows: If $n_\Gamma \geqq k \geqq 3$,

$$V\big(\Gamma, \geqq n_\Gamma(k-2)/(k-1)\big) = n_\Gamma ,$$

and

$$V\big(\Gamma, > n_\Gamma(k-2)/(k-1)\big) \geqq 1 ,$$

then $\Gamma \geqq \langle k \rangle$. The following stronger result holds:

THEOREM 1. *If* $k \geqq 3$,

$$V\big(\Gamma, \geqq n_\Gamma(k-2)/(k-1)\big) \geqq n_\Gamma(k-2)/(k-1) ,$$

and

$$V\big(\Gamma, > n_\Gamma(k-2)/(k-1)\big) \geqq 1 ,$$

then each vertex of $\mathscr{V}\big(\Gamma, > n_\Gamma(k-2)/(k-1)\big)$ *is joined to more than* $n_\Gamma(k-3)/(k-1)$ *vertices of* $\mathscr{V}\big(\Gamma, \geqq n_\Gamma(k-2)/(k-1)\big)$, *and if* Δ *is any* $\langle l \rangle$ *with* $1 \leqq l \leqq k-1$ *contained in* Γ *and such that*

$$\mathscr{V}(\Delta) \subseteq \mathscr{V}\big(\Gamma, \geqq n_\Gamma(k-2)/(k-1)\big)$$

and

$$\sum_{x \in \mathscr{V}(\Delta)} v(x,\Gamma) > l n_\Gamma(k-2)/(k-1) + \varkappa ,$$

where $\varkappa$ *is an integer* $\geqq 0$, *then* Γ *contains a* $\langle k-1 \rangle$ Λ *and a* $\langle k, \varkappa+1 \rangle$ Φ *such that*

$$\mathscr{V}(\Lambda) \subseteq \mathscr{V}\big(\Gamma, \geqq n_\Gamma(k-2)/(k-1)\big) \quad \text{and} \quad \Gamma \geqq \Phi \supset \Lambda \geqq \Delta .$$

NOTES. 1. The conditions of Theorem 1 imply, since the valency of every vertex is $\leqq n_\Gamma - 1$, that $n_\Gamma - 1 > n_\Gamma(k-2)/(k-1)$, that is, $n_\Gamma \geqq k$.

2. From $V\big(\Gamma, \geqq n_\Gamma(k-2)/(k-1)\big) \geqq n_\Gamma(k-2)/(k-1)$ it follows that if $n_\Gamma \leqq 2k-3$, then $\Gamma = \langle n_\Gamma \rangle$. For if $n_\Gamma \leqq 2k-3$, then $n_\Gamma(k-2)/(k-1) > n_\Gamma - 2$. Therefore at least $n_\Gamma - 1$ vertices have valency $n_\Gamma - 1$, consequently $\Gamma = \langle n_\Gamma \rangle$. So Theorem 1 is more significant for $n_\Gamma \geqq 2k-2$.

3. Theorem 1 implies that if a is any vertex the valency of which is $> n_\Gamma(k-2)/(k-1)$, then $\Gamma \geqq \langle k \rangle \in a$. (Take $\Delta = a$ $(l=1)$ and $\varkappa = 0$.)

PROOF OF THEOREM 1. Let a denote any vertex of

$$\mathscr{V}(\Gamma, > n_\Gamma(k-2)/(k-1)).$$

Then

$$\begin{aligned}|\mathscr{V}_a \cap \mathscr{V}(\Gamma, \geqq n_\Gamma(k-2)/(k-1))| &\geqq v(a,\Gamma) + V(\Gamma, \geqq n_\Gamma(k-2)/(k-1)) - n_\Gamma \\ &> 2n_\Gamma(k-2)/(k-1)) - n_\Gamma \\ &= n_\Gamma(k-3)/(k-1).\end{aligned}$$

The existence of a $\langle k-1\rangle$ Λ in Γ with the required properties will be proved by *reductio ad absurdum*: Suppose that no such $\langle k-1\rangle$ exists. Then $l \leqq k-2$. Let m denote the greatest integer with the property that Γ contains an $\langle m\rangle$ Θ such that $\mathscr{V}(\Theta) \subseteqq \mathscr{V}(\Gamma, \geqq n_\Gamma(k-2)/(k-1))$ and $\Theta \supseteqq \Delta$; obviously m exists and $l \leqq m \leqq k-2$.

Let $e(\Theta)$ denote the number of edges of Γ having one end in Θ and one end in $\Gamma - \Theta$. Clearly

$$\begin{aligned}e(\Theta) &= \sum_{x\in\mathscr{V}(\Theta)} (v(x,\Gamma) - (m-1)) \\ &= \sum_{x\in\mathscr{V}(\Delta)} v(x,\Gamma) + \sum_{x\in\mathscr{V}(\Theta)-\mathscr{V}(\Delta)} v(x,\Gamma) - m(m-1) \\ &> l n_\Gamma(k-2)/(k-1) + (m-l)n_\Gamma(k-2)/(k-1) - m(m-1) \\ &= n_\Gamma m(k-2)/(k-1) - m(m-1).\end{aligned}$$

Now suppose that in $\mathscr{V}(\Gamma) - \mathscr{V}(\Theta)$ there are n_0 vertices having valency $< n_\Gamma(k-2)/(k-1)$ in Γ and n_1 vertices having valency $\geqq n_\Gamma(k-2)/(k-1)$ in Γ. Each of the latter is joined to at most $m-1$ of the vertices of Θ, since otherwise the maximal property of m would be contradicted. Consequently

$$e(\Theta) \leqq mn_0 + (m-1)n_1 = (m-1)(n_0+n_1) + n_0.$$

Also $n_0 + n_1 = n_\Gamma - m$ and $n_0 \leqq n_\Gamma/(k-1)$, therefore

$$e(\Theta) \leqq (m-1)(n_\Gamma - m) + n_\Gamma/(k-1).$$

From

$$n_\Gamma m(k-2)/(k-1) - m(m-1) < e(\Theta) \leqq (m-1)(n_\Gamma - m) + n_\Gamma/(k-1)$$

it follows that $m(k-2) < (m-1)(k-1) + 1$, that is, $m > k-2$, whereas $m \leqq k-2$. This contradiction proves the existence of a $\langle k-1\rangle$ Λ in Γ with the properties mentioned.

Let $e(\Lambda)$ denote the number of edges of Γ having one end in Λ and one end in $\Gamma - \Lambda$. Clearly

$$\begin{aligned} e(\Lambda) &= \sum_{x \in \mathscr{V}(\Lambda)} \big(v(x,\Gamma) - (k-2)\big) \\ &= \sum_{x \in \mathscr{V}(\Delta)} v(x,\Gamma) + \sum_{x \in \mathscr{V}(\Lambda) - \mathscr{V}(\Delta)} v(x,\Gamma) - (k-1)(k-2) \\ &> l n_\Gamma (k-2)/(k-1) + \varkappa + (k-1-l) n_\Gamma (k-2)/(k-1) - (k-1)(k-2) \\ &= \big(n_\Gamma - (k-1)\big)(k-2) + \varkappa \,. \end{aligned}$$

Therefore at least $\varkappa+1$ of the $n_\Gamma-(k-1)$ vertices of $\Gamma-\Lambda$ are joined to all vertices of Λ. Consequently Γ contains a $\langle k, \varkappa+1\rangle$ Φ with the required properties as a subgraph.

REMARKS. Theorem 1 is primarily significant with $\varkappa=0$ as a criterion for the existence of one or more $\langle k\rangle$-s as subgraphs in a graph. In this respect the theorem is best possible:

(a) If $V\big(\Gamma, >n_\Gamma(k-2)/(k-1)\big)=0$, then Γ need not contain a $\langle k\rangle$ at all even if every vertex has valency $n_\Gamma(k-2)/(k-1)$. This is shown, for example, by a graph whose vertices are partitioned into $k-1$ mutually disjoint sets of τ vertices, $\tau \geqq 2$, any two vertices being joined by an edge if and only if they do not belong to the same set.

(b) If $V\big(\Gamma, \geqq n_\Gamma(k-2)/(k-1)\big) < n_\Gamma(k-2)/(k-1)$, then Γ need not contain a $\langle k\rangle$ at all even if $V\big(\Gamma, >n_\Gamma(k-2)/(k-1)\big) \geqq (n_\Gamma-1)(k-2)/(k-1)$, this is shown, for example, by a graph whose vertices are partitioned into $k-1$ mutually disjoint sets, $k-2$ of which contain τ vertices, $\tau \geqq 2$, and the remaining set $\tau+1$, any two vertices being joined by an edge if and only if they do not belong to the same set.

Theorem 1 is significant in the second place with $\varkappa=1$ as a criterion for the existence of $\langle k,2\rangle$-s as subgraphs in a graph, because a $\langle k,2\rangle$ is the same as a $\langle k+1\rangle$ with a single edge missing. I have proved elsewhere [1] that the conditions of Turán's theorem actually imply the existence not only of a $\langle k\rangle$ but of a $\langle k,2\rangle$ as a subgraph in the graph, even though the theorem is best possible. The conditions of Theorem 1 with $\varkappa=0$ do not always imply the existence of a $\langle k,2\rangle$ as a subgraph in the graph (see the remarks after the proof of Theorem 2). However, we can prove (cf. Note 2 after Theorem 1)

THEOREM 2. *If $n_\Gamma \geqq k+1 \geqq 4$, $V\big(\Gamma, \geqq n_\Gamma(k-2)/(k-1)\big) > n_\Gamma(k-2)/(k-1)$ and $V\big(\Gamma, >n_\Gamma(k-2)/(k-1)\big) \geqq 1$, then $\Gamma \supset \langle k,2\rangle$ except only if $k=3$, $n_\Gamma=4$ and Γ consists of a $\langle 3\rangle$ together with a fourth vertex joined to exactly one vertex of the $\langle 3\rangle$.*

The proof of Theorem 2 will require the following two simple results:

(1) *If Ψ is a graph and w is any vertex of Ψ, then if a vertex of Ψ_w has valency $\geqq 2$ in Ψ_w, it follows that $\Psi \geqq \langle 3,2\rangle$.*

For if, for example, $(x,y), (x,z) \in \Psi_w$, then w, x, y, z span a $\langle 3,2 \rangle$ or a $\langle 4 \rangle$ in Ψ.

(2) *If Ψ is a graph and (p,q) is an edge of Ψ and at least two vertices of Ψ are joined to both p and q, then $\Psi \geqq \langle 3,2 \rangle$.*

For if, for example, r and s are joined to both p and q, then p, q, r, s span a $\langle 3,2 \rangle$ or a $\langle 4 \rangle$ in Ψ.

PROOF OF THEOREM 2 FOR $k=3$ AND $n_\Gamma=4$. Let $a \in \mathscr{V}(\Gamma, >2)$, then $v(a,\Gamma)=3$. Further, $e_{\Gamma_a} \geqq 1$ because $V(\Gamma, \geqq 2) \geqq 3$. If $e_{\Gamma_a} \geqq 2$, then $\Gamma \geqq \langle 3,2 \rangle$ by (1). If $e_{\Gamma_a}=1$ then Γ consists of a $\langle 3 \rangle$ together with a vertex joined to exactly one vertex of the $\langle 3 \rangle$.

PROOF OF THEOREM 2 FOR $k=3$ AND ODD n_Γ by *reductio ad absurdum.* Suppose that the graph Γ has an odd number of vertices and satisfies the conditions of Theorem 2 with $k=3$, but $\Gamma \not\geqq \langle 3,2 \rangle$.

(3) *Γ contains no vertex of valency $> \frac{1}{2}n_\Gamma + \frac{1}{2}$.*

For otherwise Γ would contain a vertex a of valency $\geqq \frac{1}{2}n_\Gamma + \frac{3}{2}$ and so, by Theorem 1 with $k=3$, $\Delta = a$ and $\varkappa = 1$, $\Gamma \geqq \langle 3,2 \rangle$.

Therefore Γ contains at least $\frac{1}{2}n_\Gamma + \frac{1}{2}$ vertices of valency $\frac{1}{2}n_\Gamma + \frac{1}{2}$. Each of them is joined to at least one vertex of valency $\frac{1}{2}n_\Gamma + \frac{1}{2}$ by an edge. Let a and b denote two vertices of valency $\frac{1}{2}n_\Gamma + \frac{1}{2}$ joined by an edge. At least one vertex of Γ is joined to both a and b because $v(a,\Gamma) + v(b,\Gamma) = n_\Gamma + 1$, and only one vertex of Γ is joined to both a and b by (2). From this and (1) (2) it follows that

(4) *Exactly one vertex, c say, is joined to both a and b, exactly half the vertices of $\Gamma - a - b - c$ are joined to a and not to b, the remaining half are joined to b and not to a, and c is joined only to a and to b.*

Since $V(\Gamma, \frac{1}{2}n_\Gamma + \frac{1}{2}) \geqq \frac{1}{2}n_\Gamma + \frac{1}{2} \geqq 3$, the notation can be chosen so that at least one of the vertices of $\mathscr{V}_a - b - c$, say d, has valency $\frac{1}{2}n_\Gamma + \frac{1}{2}$ in Γ. By (1) and (4)

(5) *d is joined to exactly one vertex of $\mathscr{V}_a - b - c$, say d', and to all vertices of $\mathscr{V}_b - c$.*

From (5), (2) and (4) it follows that

(6) *$e_{\Gamma_b} = 1$ and d' is joined to no vertex other than a and d.*

From (4), (5) and (6) it follows that $\{d'\} \cup \mathscr{V}_b - a \subseteqq \mathscr{V}(\Gamma, < \frac{1}{2}n_\Gamma)$, so $V(\Gamma, < \frac{1}{2}n_\Gamma) \geqq \frac{1}{2}n_\Gamma + \frac{1}{2}$ contrary to $V(\Gamma, \frac{1}{2}n_\Gamma + \frac{1}{2}) \geqq \frac{1}{2}n_\Gamma + \frac{1}{2}$. This proves Theorem 2 for $k=3$ and odd n_Γ.

PROOF OF THEOREM 2 FOR $k=3$ AND EVEN $n_\Gamma \geqq 6$ by *reductio ad absurdum.* Suppose that the graph Γ has an even number of vertices $\geqq 6$ and satisfies the conditions of Theorem 2 with $k=3$, but $\Gamma \not\geqq \langle 3,2 \rangle$.

(7) *Γ contains no vertex of valency $> \frac{1}{2}n_\Gamma + 1$.*

For otherwise Γ would contain a vertex a of valency $\geqq \frac{1}{2}n_\Gamma + 2$ and so, by Theorem 1 with $k=3$, $\Delta = a$ and $\varkappa = 1$, $\Gamma \supseteqq \langle 3,2\rangle$.

Γ contains at least one vertex of valency $\frac{1}{2}n_\Gamma + 1$, let a denote such a vertex.

(8) *No vertex of $\mathscr{V}_a$ has valency $> \frac{1}{2}n_\Gamma$ in Γ, and at least two vertices of $\mathscr{V}_a$ have valency $\frac{1}{2}n_\Gamma$ in Γ.*

For if $(a,a') \in \Gamma$ and $v(a',\Gamma) > \frac{1}{2}n_\Gamma$, then $v(a,\Gamma)+v(a',\Gamma) > n_\Gamma + 1$, so at least two vertices of Γ are joined to both a and a', consequently $\Gamma \supset \langle 3,2\rangle$ by (2). At least two vertices of $\mathscr{V}_a$ have valency $\frac{1}{2}n_\Gamma$ in Γ because $V(\Gamma, \geqq \frac{1}{2}n_\Gamma) \geqq \frac{1}{2}n_\Gamma + 1$ and $v(a,\Gamma) = \frac{1}{2}n_\Gamma + 1$.

Let b denote a vertex of $\mathscr{V}_a$ with valency $\frac{1}{2}n_\Gamma$ in Γ. At least one vertex of Γ is joined to both a and b because $v(a,\Gamma)+v(b,\Gamma) = n_\Gamma + 1$, and only one vertex of Γ is joined to both a and b by (2). From this and (1) and (2) it follows that

(9) *Exactly one vertex, c say, is joined to both a and b, exactly $\frac{1}{2}n_\Gamma - 1$ of the vertices of $\Gamma - a - b - c$ are joined to a and not to b, the remaining $\frac{1}{2}n_\Gamma - 2$ vertices of $\Gamma - a - b - c$ are joined to b and not to a, and c is joined only to a and to b.*

By (8) and (9) at least one of the vertices of $\mathscr{V}_a - b - c$, say d, has valency $\frac{1}{2}n_\Gamma$ in Γ. It follows from (1) and (9) that d is joined to exactly one of the vertices of $\mathscr{V}_a - b - c$, say d', and to all vertices of $\mathscr{V}_b - c$. From this, (2) and (9) it follows that

(10) $e_{\Gamma_b} = 1$, *and d' is joined to no vertex other than a and d.*

Consequently $\{d'\} \cup \mathscr{V}_b - a \subseteqq \mathscr{V}(\Gamma, \leqq \frac{1}{2}n_\Gamma - 1)$, so $V(\Gamma, \leqq \frac{1}{2}n_\Gamma - 1) \geqq \frac{1}{2}n_\Gamma$ contrary to $V(\Gamma, \geqq \frac{1}{2}n_\Gamma) \geqq \frac{1}{2}n_\Gamma + 1$. This proves Theorem 2 for $k=3$ and even $n_\Gamma \geqq 6$.

Proof of Theorem 2 for $k=4$ and $n_\Gamma = 5$. In this case $n_\Gamma = 2k-3$, so $\Gamma = \langle 5\rangle$ (see Note 2 after Theorem 1).

Proof of Theorem 2 for $k=4$ and $n_\Gamma = 6$. In this case $n_\Gamma(k-2)/(k-1) = 4$, so at least one vertex has valency 5 and at least five have valency $\geqq 4$. Let a denote a vertex having valency 5 in Γ. Then $V(\Gamma_a, \geqq 3) \geqq 4$, so by Theorem 2 with $k=3$ and $n_\Gamma = 5$, $\Gamma_a \supset \langle 3,2\rangle$. Adding a we have that $\Gamma \supset \langle 4,2\rangle$.

The rest of the proof of Theorem 2 will require the following two results:

(11) *Let Ψ denote a graph with the property that*

$$V(\Psi, \geqq n_\Psi(k'-2)/(k'-1)) > n_\Psi(k'-2)/(k'-1)\,,$$

where $n_\Psi \geqq k' \geqq 3$, and let b denote a vertex of valency $\geqq n_\Psi(k'-2)/(k'-1)$. If $x \in \mathscr{V}_b$ and $v(x,\Psi) = \nu + n_\Psi(k'-2)/(k'-1)$, then

$$v(x,\Psi_b) \geqq \nu + n_{\Psi_b}(k'-3)/(k'-2)\,.$$

For clearly

$$v(x,\Psi_b) \geqq v(x,\Psi)-(n_\Psi - n_{\Psi_b}) = \nu - n_\Psi/(k'-1)+n_{\Psi_b} .$$

Also

$$n_{\Psi_b} \geqq n_\Psi(k'-2)/(k'-1), \quad \text{so} \quad n_\Psi/(k'-1) \leqq n_{\Psi_b}/(k'-2) .$$

Consequently

$$v(x,\Psi_b) \geqq \nu + n_{\Psi_b}(k'-3)/(k'-2) .$$

(12) $V(\Psi_b, \geqq n_{\Psi_b}(k'-3)/(k'-2)) > n_{\Psi_b}(k'-3)/(k'-2)$.

For by (11) with $\nu=0$

$$\mathscr{V}(\Psi_b, \geqq n_{\Psi_b}(k'-3)/(k'-2)) \subseteqq \mathscr{V}_b \cap \mathscr{V}(\Psi, \geqq n_\Psi(k'-2)/(k'-1)) .$$

Now

$$\begin{aligned} |\mathscr{V}_b \cap \mathscr{V}(\Psi, \geqq n_\Psi(k'-2)/(k'-1))| &\geqq n_{\Psi_b} + V(\Psi, \geqq n_\Psi(k'-2)/(k'-1)) - n_\Psi \\ &> n_{\Psi_b} - n_\Psi/(k'-1) \\ &\geqq n_{\Psi_b}(k'-3)/(k'-2) . \end{aligned}$$

Proof of Theorem 2 for $k=4$ and $n_\Gamma \geqq 7$. Let a denote a vertex having valency $> n_\Gamma(k-2)/(k-1)$ in Γ, and let b denote a vertex of valency $\geqq n_\Gamma(k-2)/(k-1)$ joined to a. (By Theorem 1 a is joined to vertices of valency $\geqq n_\Gamma(k-2)/(k-1)$.) Clearly $n_\Gamma(k-2)/(k-1) > 4$, so

(13) $n_{\Gamma_b} \geqq 5$.

By (11) with $k'=4$

(14) $v(a,\Gamma_b) > \frac{1}{2} n_{\Gamma_b}$.

By (12) with $k'=4$

(15) $V(\Gamma_b, \geqq \frac{1}{2} n_{\Gamma_b}) > \frac{1}{2} n_{\Gamma_b}$.

By (13), (14), (15) and Theorem 2 with $k=3$, we get $\Gamma_b \supseteqq \langle 3,2\rangle$. Adding b we have that $\Gamma \supset \langle 4,2\rangle$.

Proof of Theorem 2 for $k \geqq 5$ by induction over k. Suppose that $k=k' \geqq 5$ and that Theorem 2 is true if $k=k'-1$. Let Γ be a graph which satisfies the conditions of Theorem 2 with $k=k'$. Let a denote a vertex having valency $> n_\Gamma(k'-2)/(k'-1)$ in Γ, and let b denote a vertex of valency $\geqq n_\Gamma(k'-2)/(k'-1)$ joined to a (by Theorem 1, a is joined to vertices of valency $\geqq n_\Gamma(k'-2)/(k'-1)$).

(16) $n_{\Gamma_b} \geqq k'$.

For

$$n_{\Gamma_b} \geqq n_\Gamma(k'-2)/(k'-1) \geqq (k'+1)(k'-2)/(k'-1) = k'-2/(k'-1) ,$$

and $k' \geqq 5$.

(17) $v(a,\Gamma_b) > n_{\Gamma_b}(k'-3)/(k'-2)$

by (11), and

(18) $V(\Gamma_b, \geqq n_{\Gamma_b}(k'-3)/(k'-2)) > n_{\Gamma_b}(k'-3)/(k'-2)$
by (12). By (16), (17) and (18) Γ_b satisfies the conditions of Theorem 2 with $k=k'-1 \geqq 4$, therefore $\Gamma_b \supset \langle k'-1, 2\rangle$ by the induction hypothesis. Adding b we have that $\Gamma \supset \langle k', 2\rangle$.

Thus, Theorem 2 is true for $k=k' \geqq 5$ if it is true for $k=k'-1$. The theorem has been proved for $k=3$ and for $k=4$, so it is true generally.

REMARKS. Let the graph Ω be defined as follows:

$$\mathscr{V}(\Omega) = \{a_1, \ldots, a_{\gamma-1}, b_1, \ldots, b_{\gamma+1}\},$$

where $\gamma \geqq 2$, and

$$\mathscr{E}(\Omega) = \{(a_i, b_j), (a_1, b_{\gamma+1}), (b_\gamma, b_{\gamma+1})\}, \quad i = 1, \ldots, \gamma-1, \ j = 1, \ldots, \gamma,$$

$v(a_1, \Omega) = \frac{1}{2}n_\Omega + 1$, and $V(\Omega, \geqq \frac{1}{2}n_\Omega) = \frac{1}{2}n_\Omega$, but $\Omega \not\supseteq \langle 3, 2\rangle$. This example shows that in Theorem 1 with $\varkappa = 1$ the existence of an $\langle l\rangle$ Δ such that

$$\sum_{x \in \mathscr{V}(\Delta)} v(x, \Gamma) > l n_\Gamma (k-2)/(k-1) + 1$$

must be stipulated, if only

$$\sum_{x \in \mathscr{V}(\Delta)} v(x, \Gamma) = l n_\Gamma (k-2)/(k-1) + 1$$

holds (for one or more $\langle l\rangle$-s) then Γ need not contain a $\langle k, 2\rangle$ as a subgraph at all (in our example Ω take $\Delta = a_1$). The above example also shows that in Theorem 2

$$V(\Gamma, \geqq n_\Gamma(k-2)/(k-1)) > n_\Gamma(k-2)/(k-1)$$

must be stipulated; if $=$ holds instead of $>$, then Γ need not contain a $\langle k, 2\rangle$ as a subgraph.

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