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SOME PROPERTIES OF PROXIMITY AND GENERALIZED UNIFORMITY

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The collection of all uniform structures on a set E and the collection of all proximity structures on E are organized to complete lattices by the order relations "finer—coarser". In the paper [2] E. M. Alfsen and the present author showed that these lattice operations are generally not compatible, i.e., the least upper bound (greatest lower bound) of a family of uniform structures need not be compatible with the least upper bound (greatest lower bound) of the family of corresponding proximity structures. Further it was shown that the collection of all *generalized uniform structures* on E—which were introduced in this paper—is also organized to a complete lattice by the order relation "finer—coarser", and that in this case the lattice operations are compatible with those on the collection of proximity structures.

In the present paper we utilize these results to establish the existence and compatibility of *initial* (resp. *final*) *generalized uniform structures* and *initial* (resp. *final*) proximity structures (cf. [4, § 2]).

Further we study the relation between *uniform convergence* with respect to a generalized uniform structure (defined in an obvious way) and the concept of *convergence in proximity* introduced by S. Leader in [5, p. 214]. We show that the latter coincides with uniform convergence with respect to the corresponding totally bounded uniform structure. Moreover we prove that for a generalized sequence with linearly ordered index set, uniform convergence and convergence in proximity coincide; a generalization of Leader's result for proper sequences (cf. [5, 214]).

1. Preliminaries.

We first recall that a *generalized uniform structure* on a set E is a collection $\mathcal{U}$ of sets $U \subset E \times E$, which satisfies the following requirements [2, p. 239]):

(G.U.1) $\Delta = \{(x,x) \mid x \in E\} \subset \bigcap_{U \in \mathcal{U}} U$.

(G.U.2) $U \in \mathcal{U},\ U \subset V \quad \Rightarrow \quad V \in \mathcal{U}$.

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(G.U.3) For every $U \in \mathscr{U}$ there exists a $V \in \mathscr{U}$ such that $V^{-1} = V \subset U$.

(G.U.4) For every $U \in \mathscr{U}$ there exists a $V \in \mathscr{U}$ such that $V^2 \subset U$.

(G.U.5) If $A_i \subset E$, $U_i \in \mathscr{U}$, are arbitrary for $i = 1, \ldots, n$, there exists a single $U \in \mathscr{U}$ such that $U(A_i) \subset U_i(A_i)$ for $i = 1, \ldots, n$.

The associated proximity structure $\mathscr{P}$ is defined in the customary way by the relation

$$(1.1) \qquad A \Subset B \ (\mathscr{P}) \ \Leftrightarrow \ \exists U \in \mathscr{U}\big(U(A) \subset B\big).$$

We list some properties of generalized uniform structures and proximity structures, which we shall need in the sequel (cf. [2, p. 239]).

An order relation ("finer—coarser") between generalized uniform structures on a set E is defined in the obvious way: "$\mathscr{U}_1$ coarser than $\mathscr{U}_2$" means $\mathscr{U}_1 \subset \mathscr{U}_2$. To every proximity structure $\mathscr{P}$ is associated a unique totally bounded structure $\mathscr{U}_\omega$, which is the coarsest generalized uniform structure compatible with $\mathscr{P}$ (cf. [1, p. 354], [2, p. 240]; the subscript ω always denotes the associated totally bounded structure).— A subset V of $E \times E$ is said to be *entourage-like* relatively to $\mathscr{P}$ if it admits a sequence $\{V_n\}_{n=1,2,\ldots}$ of symmetric subsets of $E \times E$ such that $V_1{}^2 \subset V$, $V_{n+1}{}^2 \subset V_n$, $A \Subset V_n(A)$ $(\mathscr{P})$, for all $A \subset S$ and $n = 1, 2, \ldots$. A generalized uniform structure $\mathscr{U}$ is called *total* if all *entourage*-like sets with respect to its associated proximity structure are entourages of $\mathscr{U}$. To every proximity structure $\mathscr{P}$ is associated a unique total structure $\mathscr{U}_\alpha$, which is the finest generalized uniform structure compatible with $\mathscr{P}$ (cf. [2, p. 241], [2, p. 242]; the subscript α always denotes the associated total structure). Further, a proximity structure $\mathscr{P}_1$ on E is said to be coarser than another proximity structure $\mathscr{P}_2$ on E provided that

$$A \Subset B \ (\mathscr{P}_1) \ \Rightarrow \ A \Subset B \ (\mathscr{P}_2).$$

This is equivalent to the conditions: The (unique) totally bounded uniform structure $\mathscr{U}_{\omega_1}$ associated with $\mathscr{P}_1$ is coarser than the corresponding structure $\mathscr{U}_{\omega_2}$ associated with $\mathscr{P}_2$, or: The (unique) total generalized uniform structure $\mathscr{U}_{\alpha_1}$ associated with $\mathscr{P}_1$ is coarser than the corresponding structure $\mathscr{U}_{\alpha_2}$ associated with $\mathscr{P}_2$. The collection of all proximity structures on E and the collection of all generalized uniform structures on E are both organized to complete lattices by these order relations. The *lattice supremum* and the *lattice infimum* will in both cases be denoted by the symbols $\vee$ and $\wedge$, respectively. The notations sup and inf will be used for the lattice operations on the collection of all *proper* uniform structures on E.

Let $\{\mathscr{U}_\beta\}_{\beta\in B}$ be a family of generalized uniform structures on E, with corresponding proximity structures $\{\mathscr{P}_\beta\}_{\beta\in B}$. Then $\bigvee_{\beta\in B}\mathscr{P}_\beta$ is the proximity structure associated with $\sup_{\beta\in B}\mathscr{U}_{\omega\beta}$, where $\mathscr{U}_{\omega\beta}$ denotes the totally bounded structure determined by $\mathscr{U}_\beta$. Further

$$\bigvee_{\beta\in B}\mathscr{U}_\beta = \bigcup_{\beta\in B}\mathscr{U}_\beta \cup \sup_{\beta\in B}\mathscr{U}_{\omega\beta}\,, \tag{1.2}$$

hence $\bigvee_{\beta\in B}\mathscr{P}_\beta$ is the proximity structure corresponding to $\bigvee_{\beta\in B}\mathscr{U}_\beta$. By duality it follows that $\bigwedge_{\beta\in B}\mathscr{P}_\beta$ is the proximity structure corresponding to $\bigwedge_{\beta\in B}\mathscr{U}_\beta$.

Now let f be a mapping of a set E into a set F, and let $\mathscr{U}$ be a generalized uniform structure on E. Set $g=(f\times f)$. The image of $\mathscr{U}$ by f is defined as the collection

$$g(\mathscr{U}) = \{V\subset F \mid \exists U\in\mathscr{U}(V\supset g(U))\}\,. \tag{1.3}$$

Similarly, if $\mathscr{V}$ is a generalized uniform structure on F, the inverse image of $\mathscr{V}$ by f is defined as the collection

$$g^{-1}(\mathscr{V}) = \{U\subset E \mid \exists V\in\mathscr{V}(g^{-1}(V)\subset U)\}\,. \tag{1.4}$$

Proposition 1. *Let f be a mapping of a set E into a set F. The inverse image $g^{-1}(\mathscr{V})$ by f of a generalized uniform structure $\mathscr{V}$ on F is a generalized uniform structure on E. The inverse image of a totally bounded structure is totally bounded, and if $\mathscr{V}_1$ and $\mathscr{V}_2$ are p-equivalent, (i.e. determine the same proximity structure) then $g^{-1}(\mathscr{V}_1)$ and $g^{-1}(\mathscr{V}_2)$ are p-equivalent.* (Cf. [2, p. 246].)

Proof. If $\mathscr{V}$ is a generalized uniform structure, then clearly $g^{-1}(\mathscr{V})$ satisfies the axioms (G.U.1)–(G.U.4). For $A_i\subset E$, $g^{-1}(V_i)\in g^{-1}(\mathscr{V})$, $i=1,\ldots,n$, there exists a $V\in\mathscr{V}$ such that

$$V(f(A_i)) \subset V_i(f(A_i)), \qquad i = 1,\ldots,n\,.$$

Now $(g^{-1}(V_i))(A_i)=f^{-1}(V_i(f(A_i)))$ and $(g^{-1}(V))(A_i)=f^{-1}(V(f(A_i)))$, hence

$$(g^{-1}(V))(A_i) \subset (g^{-1}(V_i))(A_i), \qquad i = 1,\ldots,n\,.$$

That is $g^{-1}(\mathscr{V})$ satisfies (G.U.5). Further, obviously, if $\mathscr{V}$ is totally bounded, so is $g^{-1}(\mathscr{V})$ since

$$g^{-1}\left(\bigcup_{i=1}^{n}(A_i\times A_i)\right) = \bigcup_{i=1}^{n}(f^{-1}(A_i)\times f^{-1}(A_i))$$

(cf. [2, p. 246]). And if $W(f(A))\subset V(f(A))$, then $(g^{-1}(W))(A)\subset(g^{-1}(V))(A)$, which implies

$$A \Subset B \ \ (g^{-1}(\mathscr{V})) \ \Rightarrow\ A \Subset B \ \ (g^{-1}(\mathscr{V}_\omega))\,.$$

This means $(g^{-1}(\mathscr{V}))_\omega \subset g^{-1}(\mathscr{V}_\omega)$. Evidently $(g^{-1}(\mathscr{V}))_\omega \supset g^{-1}(\mathscr{V}_\omega)$, so we conclude

$$(g^{-1}(\mathscr{V}))_\omega = g^{-1}(\mathscr{V}_\omega) . \tag{1.5}$$

This completes the proof.

It follows that we may define the inverse image $f^{-1}(\mathscr{Q})$ of a proximity structure $\mathscr{Q}$ by means of an arbitrary generalized uniform structure $\mathscr{V}$ compatible with $\mathscr{Q}$. Now we recall that a mapping f of E into F is a *proximity mapping* with respect to the proximity structures $\mathscr{P}$ and $\mathscr{Q}$ if and only if

$$A \Subset B \quad (\mathscr{Q}) \quad \Rightarrow \quad f^{-1}(A) \Subset f^{-1}(B) \quad (\mathscr{P}) \tag{1.6}$$

(cf. e.g. [6, p. 550]). It is easily seen that this is equivalent to the requirement: $f^{-1}(\mathscr{Q})$ coarser than $\mathscr{P}$. Hence the following

COROLLARY. *A mapping of E into F is a proximity mapping with respect to the proximity structures $\mathscr{P}$ and $\mathscr{Q}$ if and only if it is uniformly continuous with respect to the corresponding totally bounded structures $\mathscr{U}_\omega$ and $\mathscr{V}_\omega$, equivalently if and only if it is uniformly continuous with respect to the corresponding total structures $\mathscr{U}_\alpha$ and $\mathscr{V}_\alpha$ (cf.* [2, p. 246]).

PROPOSTION 2. *Let f be a mapping of a set E into a set F. For every generalized uniform structure $\mathscr{U}$ on E there is on F a finest generalized uniform structure $g(\mathscr{U})^*$ contained in $g(\mathscr{U})$. If $\mathscr{U}_1$ and $\mathscr{U}_2$ are p-equivalent, then $g(\mathscr{U}_1)^*$ and $g(\mathscr{U}_2)^*$ are also p-equivalent.*

PROOF. Let $\{\mathscr{V}_\beta\}_{\beta\in\mathrm{B}}$ be the family of all generalized uniform structures on F such that $\mathscr{V}_\beta \subset g(\mathscr{U})$. Clearly $g^{-1}(\mathscr{V}_\beta) \subset \mathscr{U}$ for all $\beta \in \mathrm{B}$. As $g^{-1}(\mathscr{V}_{\omega\beta})$ is totally bounded (prop. 1), it follows that $g^{-1}(\mathscr{V}_{\omega\beta}) \subset \mathscr{U}_\omega$, hence $\mathscr{V}_{\omega\beta} \subset g(\mathscr{U}_\omega)$. Now let $V_i \in \mathscr{V}_{\omega\beta_i}$, $i=1,\ldots,n$. According to the previous result, $V_i \in g(\mathscr{U}_\omega)$, $i=1,\ldots,n$. Now $\mathscr{U}_\omega$ is a *proper* uniform structure, and consequently $g(\mathscr{U}_\omega)$ is closed under finite intersections. Hence

$$\sup_{\beta\in\mathrm{B}} \mathscr{V}_{\omega\beta} \subset g(\mathscr{U}_\omega) \subset g(\mathscr{U}) .$$

This implies $\bigvee_{\beta\in\mathrm{B}} \mathscr{V}_\beta \subset g(\mathscr{U})$, that is $g(\mathscr{U})^* = \bigvee_{\beta\in\mathrm{B}} \mathscr{V}_\beta$ has the required property. Now we have seen that a totally bounded structure is contained in $g(\mathscr{U})$ if and only if it is contained in $g(\mathscr{U}_\omega)$. Consequently $g(\mathscr{U}_1)^*$ and $g(\mathscr{U}_2)^*$ determine the same totally bounded structure if $\mathscr{U}_{\omega_1} = \mathscr{U}_{\omega_2}$, which completes the proof.

2. Initial and final structures.

We now introduce *initial* and *final* structures, in accordance with the usual terminology (cf. [4, § 2]).

Let E be an arbitrary set, $\{F_\gamma\}_{\gamma\in\Gamma}$ a family of sets, and for each $\gamma\in\Gamma$ let f_γ be a mapping of E into F_γ. Let $\mathscr{V}_\gamma$ be a generalized uniform structure on F_γ, with associated proximity structure $\mathscr{Q}_\gamma$. A generalized uniform structure $\mathscr{U}$ on E is called an *initial generalized uniform structure* on E for the family $\{F_\gamma,\mathscr{V}_\gamma,f_\gamma\}_{\gamma\in\Gamma}$ if (and only if) for every set E', every generalized uniform structure $\mathscr{U}'$ on E', and every mapping h of E' into E the following condition is satisfied:

$$(2.1)\quad \begin{array}{l}\forall\gamma\in\Gamma\ (f_\gamma\circ h \text{ uniformly continuous with respect to } \mathscr{U}' \text{ and } \mathscr{V}_\gamma)\\ \quad\Leftrightarrow\ (h \text{ uniformly continuous with respect to } \mathscr{U}' \text{ and } \mathscr{U}).\end{array}$$

A proximity structure $\mathscr{P}$ on E is called an *initial proximity structure* on E for the family $\{F_\gamma,\mathscr{Q}_\gamma,f_\gamma\}_{\gamma\in\Gamma}$ if (and only if) for every set E', every proximity structure $\mathscr{P}'$ on E', and every mapping h of E' into E the following condition is satisfied:

$$(2.2)\quad \begin{array}{l}\forall\gamma\in\Gamma\ (f_\gamma\circ h \text{ proximity mapping with respect to } \mathscr{P}' \text{ and } \mathscr{Q}_\gamma)\\ \quad\Leftrightarrow\ (h \text{ proximity mapping with respect to } \mathscr{P}' \text{ and } \mathscr{P}).\end{array}$$

Similarly, let F be an arbitrary set, $\{E_\gamma\}_{\gamma\in\Gamma}$ a family of sets, and for each $\gamma\in\Gamma$ let f_γ be a mapping of E_γ into F. Let $\mathscr{U}_\gamma$ be a generalized uniform structure on E_γ, with associated proximity structure $\mathscr{P}_\gamma$. A generalized uniform structure $\mathscr{V}$ on F is called a *final generalized uniform structure* on F for the family $\{E_\gamma,\mathscr{U}_\gamma,f_\gamma\}_{\gamma\in\Gamma}$ if (and only if) for every set F', every generalized uniform structure $\mathscr{V}'$ on F', and every mapping h of F into F' the following condition is satisfied:

$$(2.3)\quad \begin{array}{l}\forall\gamma\in\Gamma\ (h\circ f_\gamma \text{ uniformly continuous with respect to } \mathscr{U}_\gamma \text{ and } \mathscr{V}')\\ \quad\Leftrightarrow\ (h \text{ uniformly continuous with respect to } \mathscr{V} \text{ and } \mathscr{V}').\end{array}$$

A proximity structure $\mathscr{Q}$ on F is called a *final proximity structure* on F for the family $\{E_\gamma,\mathscr{P}_\gamma,f_\gamma\}_{\gamma\in\Gamma}$ if (and only if) for every set F', every proximity structure $\mathscr{Q}'$ on F', and every mapping h of F into F' the following condition is satisfied:

$$(2.4)\quad \begin{array}{l}\forall\gamma\in\Gamma\ (h\circ f_\gamma \text{ proximity mapping with respect to } \mathscr{P}_\gamma \text{ and } \mathscr{Q}')\\ \quad\Leftrightarrow\ (h \text{ proximity mapping with respect to } \mathscr{Q} \text{ and } \mathscr{Q}').\end{array}$$

THEOREM 1. *Let E be an arbitrary set, and for each γ of an index set Γ let f_γ be a mapping of E into a set F_γ provided with a generalized uniform structure $\mathscr{V}_\gamma$ and corresponding proximity structure $\mathscr{Q}_\gamma$. Then the family $\{F_\gamma,\mathscr{V}_\gamma,f_\gamma\}_{\gamma\in\Gamma}$ determines a (unique) initial generalized uniform structure $\mathscr{U}$ on E, and the family $\{F_\gamma,\mathscr{Q}_\gamma,f_\gamma\}_{\gamma\in\Gamma}$ determines a (unique) initial proximity structure $\mathscr{P}$ on E. The latter coincides with the proximity structure associated with $\mathscr{U}$.*

PROOF. The uniqueness follows from the general theory of initial structures [4, p. 27]. Let $g_\gamma = (f_\gamma \times f_\gamma)$. As $g_\gamma^{-1}(\mathscr{V}_\gamma)$ is a generalized uniform structure on E, the structure $\mathscr{U} = \bigvee_{\gamma \in \Gamma} g_\gamma^{-1}(\mathscr{V}_\gamma)$ is evidently the coarsest structure on E for which all f_γ are uniformly continuous. Now let E' be an arbitrary set with an arbitrary generalized uniform structure $\mathscr{U}'$. Let h be a mapping of E' into E, uniformly continuous with respect to $\mathscr{U}'$ and $\mathscr{U}$. Then clearly all $f_\gamma \circ h$ are uniformly continuous with respect to $\mathscr{U}'$ and $\mathscr{V}_\gamma$. Conversely, let all $f_\gamma \circ h$ be uniformly continuous with respect to $\mathscr{U}'$ and $\mathscr{V}_\gamma$. Then for each γ,

$$(f_\gamma \circ h \times f_\gamma \circ h)^{-1}(\mathscr{V}_\gamma) \subset \mathscr{U}' ,$$

thus $g_\gamma^{-1}(\mathscr{V}_\gamma) \subset (h \times h)(\mathscr{U}')$. Consequently $g_\gamma^{-1}(\mathscr{V}_\gamma) \subset (h \times h)(\mathscr{U}')^*$, hence

$$\mathscr{U} = \bigvee_{\gamma \in \Gamma} g_\gamma^{-1}(\mathscr{V}_\gamma) \subset (h \times h)(\mathscr{U}')^* \subset (h \times h)(\mathscr{U}') .$$

This implies $(h \times h)^{-1}(\mathscr{U}) \subset \mathscr{U}'$, and so h is uniformly continuous with respect to $\mathscr{U}'$ and $\mathscr{U}$.

In the same way it is shown that

$$\bigvee_{\gamma \in \Gamma} (g_\gamma^{-1}(\mathscr{V}_\gamma))_\omega = \sup_{\gamma \in \Gamma} (g^{-1}(\mathscr{V}_\gamma))_\omega$$

is the initial structure on E for the family $\{F_\gamma, \mathscr{V}_{\omega_\gamma}, f_\gamma\}_{\gamma \in \Gamma}$. From the characterization of proximity mappings in the corollary to proposition 1, it immediately follows that the proximity structure $\mathscr{P}$ determined by $\bigvee_{\gamma \in \Gamma} (g_\gamma^{-1}(\mathscr{V}_\gamma))_\omega$ actually is the initial proximity structure on E for the family $\{F_\gamma, \mathscr{Q}_\gamma, f_\gamma\}_{\gamma \in \Gamma}$. (This structure might also be defined directly in terms of proximity (see formula 2.6 of [1]), but we shall make no use of that fact.) $\mathscr{P}$ is also the proximity structure determined by $\mathscr{U} = \bigvee_{\gamma \in \Gamma} g_\gamma^{-1}(\mathscr{V}_\gamma)$ since $\mathscr{U}_\omega = \sup_{\gamma \in \Gamma} (g_\gamma^{-1}(\mathscr{V}_\gamma))_\omega$. This completes the proof.

If in particular $E = \prod_{\gamma \in \Gamma} F_\gamma$ and $f_\gamma = \mathrm{pr}_\gamma$ (the projection of E onto F_γ), the initial structures $\mathscr{U}$ and $\mathscr{P}$ may be called the (initial) *products* of the original generalized uniform and proximity structures, respectively. From theorem 1 we get immediately:

PROPOSITION 3. *Let $\{F_\gamma\}_{\gamma \in \Gamma}$ be a family of sets, each F_γ provided with a generalized uniform structure $\mathscr{V}_\gamma$ and corresponding proximity structure $\mathscr{Q}_\gamma$. Then an (initial) generalized uniform product structure and an (initial) proximity product structure may be defined on $\prod_{\gamma \in \Gamma} F_\gamma$, and the latter coincides with the proximity structure determined by the former.*

If all the structures $\mathscr{V}_\gamma$ of theorem 1 are *proper* uniform structures, there exists an initial *proper* uniform structure for the family $\{F_\gamma, \mathscr{V}_\gamma, f_\gamma\}_{\gamma \in \Gamma}$. The associated proximity structure is in general strictly finer than the initial proximity structure determined by the family $\{F_\gamma, \mathcal{Q}_\gamma, f_\gamma\}_{\gamma\in\Gamma}$. Consider for example a class of uniform structures $\{\mathcal{U}_\gamma\}_{\gamma\in\Gamma}$ of p-class $\mathcal{P}$ on a set E. It is easily seen that the *entourages* of $\mathcal{U}_\gamma$ coincide with the uniform neighbourhoods of the diagonal $\Delta\subset E\times E$ with respect to the corresponding *proper* uniform product structure $\mathcal{U}_\gamma\times\mathcal{U}_\gamma$. So all these products $\mathcal{U}_\gamma\times\mathcal{U}_\gamma$ are of different p-classes (cf. [1, p. 359]). But the *generalized* uniform product structures are of one and the same p-class, that of the (proper or generalized) product $\mathcal{U}_\omega\times\mathcal{U}_\omega$, of the totally bounded structure in the class.

THEOREM 2. *Let F be an arbitrary set, and for each γ of an index set Γ let f_γ be a mapping into F of a set E_γ provided with a generalized uniform structure $\mathcal{U}_\gamma$ and corresponding proximity structure $\mathcal{P}_\gamma$. Then the family $\{E_\gamma, \mathcal{U}_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ determines a (unique) final generalized uniform structure $\mathcal{V}$ on F, and the family $\{E_\gamma, \mathcal{P}_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ determines a (unique) final proximity structure $\mathcal{Q}$ on F. The latter coincides with the proximity structure associated with $\mathcal{V}$.*

PROOF. The uniqueness follows from the general theory of final structures [4, p. 34]. According to proposition 2, for every $\gamma\in\Gamma$ there is a finest generalized uniform structure $\mathcal{V}_\gamma$ for which f_γ is uniformly continuous. Moreover, the corresponding proximity structure $\mathcal{Q}_\gamma$ is the finest proximity structure for which f_γ is a proximity mapping. It follows that $\mathcal{V}=\bigwedge_{\gamma\in\Gamma}\mathcal{V}_\gamma$ is the finest structure for which all f are uniformly continuous. Let F' be an arbitrary set with an arbitrary generalized uniform structure $\mathcal{V}'$. Let h be a mapping of F into F', uniformly continuous with respect to $\mathcal{V}$ and $\mathcal{V}'$. Then clearly all $h\circ f_\gamma$ are uniformly continuous with respect to $\mathcal{U}_\gamma$ and $\mathcal{V}'$. Conversely, let all $h\circ f_\gamma$ be uniformly continuous with respect to $\mathcal{U}_\gamma$ and $\mathcal{V}'$. For each γ,

$$(h\circ f_\gamma\times h\circ f_\gamma)^{-1}(\mathcal{V}')\subset\mathcal{U}_\gamma\,,$$

thus $(h\times h)^{-1}(\mathcal{V}')\subset g_\gamma(\mathcal{U}_\gamma)$. Hence

$$(h\times h)^{-1}(\mathcal{V}')\subset g_\gamma(\mathcal{U}_\gamma)^*=\mathcal{V}_\gamma$$

and consequently $(h\times h)^{-1}(\mathcal{V}')\subset\bigwedge_{\gamma\in\Gamma}\mathcal{V}_\gamma$. That is: h is uniformly continuous with respect to $\mathcal{V}$ and $\mathcal{V}'$.

In the same way it is shown that $\bigwedge_{\gamma\in\Gamma}\mathcal{V}_{\omega_\gamma}$ is a final generalized uniform structure for the family $\{E_\gamma, \mathcal{U}_{\omega_\gamma}, f_\gamma\}_{\gamma\in\Gamma}$. From the characterization of proximity mappings it follows that the proximity structure $\mathcal{Q}$ determined by $\bigwedge_{\gamma\in\Gamma}\mathcal{V}_{\omega_\gamma}$ actually is the final proximity structure for the family $\{E_\gamma, \mathcal{P}_\gamma, f_\gamma\}_{\gamma\in\Gamma}$. As $\mathcal{Q}$ also is the proximity structure associated with $\bigwedge_{\gamma\in\Gamma}\mathcal{V}_\gamma$, the proof is completed.

If all $\mathscr{U}_\gamma$ are proper uniform structures, we may easily define a final *proper* uniform structure for the family $\{E_\gamma, \mathscr{U}_\gamma, f_\gamma\}_{\gamma \in \Gamma}$. In general its associated proximity structure is strictly coarser than the corresponding final proximity structure. (Let for example all $E_\gamma = F$, all $f_\gamma =$ the identity mapping, and $\{\mathscr{U}_\gamma\}_{\gamma \in \Gamma}$ a class of uniform structures such that $\bigwedge_{\gamma \in \Gamma} \mathscr{U}_\gamma$ is of strictly finer p-class than $\inf_{\gamma \in \Gamma} \mathscr{U}_\gamma$ (cf. [2, p. 237])).

We define quotient structures of generalized uniform structures and of proximity structures in the customary way. From the results above we immediately get:

PROPOSITION 4. *Let R be an equivalence relation on the set E, and let E be provided with a generalized uniform structure $\mathscr{U}$ and corresponding proximity structure $\mathscr{P}$. Then there exists a generalized uniform quotient structure and a proximity quotient structure on E/R, and the latter coincides with the proximity structure derived from the former.*

3. Uniform convergence and convergences in proximity.

Let E be an arbitrary set, F a set provided with a generalized uniform structure $\mathscr{U}$ and corresponding proximity structure $\mathscr{P}$, and let $\{f_\alpha\}_{\alpha \in \mathsf{A}}$ be a generalized sequence of mappings of E into F. We define uniform convergence in the obvious way: $\{f_\alpha\}_{\alpha \in \mathsf{A}}$ converges uniformly (with respect to $\mathscr{U}$) to the mapping f if and only if for every $U \in \mathscr{U}$ there exists an $\alpha_0 \in \mathsf{A}$ such that $\alpha > \alpha_0$ (where $>$ is the order relation of A) implies $(f_\alpha(x), f(x)) \in U$ for all $x \in E$. We shall say that $\{f_\alpha\}_{\alpha \in \mathsf{A}}$ converges in proximity (with respect to $\mathscr{P}$) to f if and only if for every $G \subset E$ and every $H \subset F$, where $f(G) \Subset H(\mathscr{P})$, there exists an $\alpha_0 \in \mathsf{A}$ such that $\alpha > \alpha_0$ implies $f_\alpha(G) \subset H$; or equivalently: for every $G \subset E$ and every $H \subset F$, where $f(G) \Subset H(\mathscr{P})$, there exists an $\alpha_0 \in \mathsf{A}$ such that $\alpha > \alpha_0$ implies $f_\alpha(G) \Subset H(\mathscr{P})$ (cf. [5, p. 214]).

PROPOSITION 5. *Let $\mathscr{U}$ be a generalized uniform structure on F with corresponding proximity structure $\mathscr{P}$. If the generalized sequence $\{f_\alpha\}_{\alpha \in \mathsf{A}}$ of mappings of the set E into F converges uniformly to f with respect to $\mathscr{U}$, it converges in proximity to f with respect to $\mathscr{P}$.*

PROOF. Let $\{f_\alpha\}_{\alpha \in \mathsf{A}}$ converge uniformly to f with respect to $\mathscr{U}$. If $f(G) \Subset H(\mathscr{P})$, there is a $U \in \mathscr{U}$ such that $U(f(G)) \subset H$. Now choose $\alpha_0 \in \mathsf{A}$ such that $\alpha > \alpha_0$ implies $(f_\alpha(x), f(x)) \in U$ for all $x \in E$. For every $x \in G$ we thus have $f_\alpha(x) \in U(f(G))$, hence $f_\alpha(G) \subset H$.

In particular it follows that uniform convergence with respect to $\mathscr{U}_\omega$ implies convergence in proximity. We now state

THEOREM 3. *A generalized sequence $\{f_\alpha\}_{\alpha\in\mathsf{A}}$ of mappings of E into F converges in proximity to the mapping f with respect to the proximity structure $\mathscr{P}$ if and only if it converges uniformly with respect to the corresponding totally bounded structure $\mathscr{U}_\omega$.*

PROOF. It remains to prove that convergence in proximity implies uniform convergence with respect to $\mathscr{U}_\omega$. Now, suppose that $\{f_\alpha\}_{\alpha\in\mathsf{A}}$ is not uniformly convergent with respect to $\mathscr{U}_\omega$. Then there exist a $V\in\mathscr{U}_\omega$, a cofinal subset A_0 of A and for each $\alpha\in\mathsf{A}_0$ an $x_\alpha\in E$ such that

$$(f_\alpha(x_\alpha), f(x_\alpha)) \notin V .$$

We may choose a $W\in\mathscr{U}_\omega$ such that $W^2\subset V$ and $W=\bigcup_{i=1}^n(A_i\times A_i)$, where $\{A_i\}_{i=1,\ldots,n}$ is a p-covering of E ([1, p. 353]). Now for at least one $i\leqq n$, say i_0, there is a cofinal subset B of A_0 such that $f_\beta(x_\beta)\in A_{i_0}$ for $\beta\in\mathsf{B}$. Set $G=\{x_\beta \mid \beta\in\mathsf{B}\}$, and let $x_\gamma, x_\delta\in G$. Assume $(f_\gamma(x_\gamma), f(x_\delta))\in W$. As

$$(f_\delta(x_\delta), f_\gamma(x_\gamma)) \in A_{i_0}\times A_{i_0} \subset W ,$$

this implies $(f_\delta(x_\delta), f(x_\delta))\in W^2\subset V$, contrary to hypothesis. Consequently we have

$$(f_\gamma(x_\gamma), f(x_\delta)) \notin W .$$

Hence $f_\gamma(x_\gamma)\notin W(f(G))$, that is for all $\beta\in\mathsf{B}$ we have

$$f_\beta(G) \not\subset W(f(G)) .$$

But then $\{f_\beta\}_{\beta\in\mathsf{B}}$, and *a fortiori* $\{f_\alpha\}_{\alpha\in\mathsf{A}}$, does not converge to f in proximity. This completes the proof.

REMARK. The concept *uniform convergence* of a generalized sequence *with respect to a proximity structure* $\mathscr{P}$ introduced by Leader in [5, p. 214] is nothing but uniform convergence with respect to the corresponding total structure $\mathscr{U}_\alpha$ (cf. [2, p. 241]).

It follows immediately from theorem 3 that convergence in proximity implies uniform convergence in the case that $\mathscr{P}$ admits only *one* uniform structure. (In this case $\mathscr{U}_\alpha=\mathscr{U}_\omega$ (cf. [2, p. 241]).

Leader has shown that if $\{f_\alpha\}_{\alpha\in\mathsf{A}}$ is a proper sequence, uniform convergence and convergence in proximity coincide. We shall now prove a generalization of this result.

THEOREM 4. *Let $\{f_\alpha\}_{\alpha\in\mathsf{A}}$ be a generalized sequence of mappings of the set E into the set F, and assume that the ordering of the index set A is linear. Then if $\{f_\alpha\}_{\alpha\in\mathsf{A}}$ converge in proximity to the mapping f with respect to a proximity structure $\mathscr{P}$ on F, it also converges uniformly to f with respect to an arbitrary generalized uniform structure compatible with $\mathscr{P}$.*

PROOF. Assume that $\{f_\alpha\}_{\alpha\in A}$ is *not* uniformly convergent with respect to $\mathscr{U}$. Then there exists a $U\in\mathscr{U}$ and a cofinal subset B of A such that for each $\beta\in B$ there is a $x_\beta\in E$ with $(f_\beta(x_\beta), f(x_\beta))\notin U$. Choose a symmetric $W\in\mathscr{U}$ such that $W^4\subset U$. We now make use of the following lemma, the proof of which is found in [3, p. 254]:

LEMMA. *Let U and W be subsets of a Cartesian product $F\times F$, and assume that W is symmetric and contains the diagonal Δ, and that $W^4\subset U$. Moreover, let $(x_\beta, y_\beta)_{\beta\in B}$ be some generalized sequence with a linearly ordered index set B, and assume that $(x_\beta, y_\beta)\notin U$ for $\beta\in B$. Then there exists a cofinal subset Γ of B such that $(x_\gamma, y_\delta)\notin W$ whenever γ and δ both belong to Γ.*

From this lemma it follows that there exists a cofinal subset Γ of B such that $(f_\gamma(x_\gamma), f(x_\delta))\notin W$ for $\gamma,\delta\in\Gamma$. Now let $G=\{x_\gamma \mid \gamma\in\Gamma\}$. Then

$$f_\gamma(x_\gamma)\notin W(f(G))$$

for all $\gamma\in\Gamma$. This means

$$f_\gamma(G)\not\subset W(f(G))$$

for all $\gamma\in\Gamma$. Hence $\{f_\gamma\}_{\gamma\in\Gamma}$, and *a fortiori* $\{f_\alpha\}_{\alpha\in A}$, does not converge in proximity to f with respect to the proximity structure $\mathscr{P}$ associated with $\mathscr{U}$. This completes the proof.

REMARK. Uniform convergence with respect to a generalized uniform structure is generally not a topological convergence. But given a structure $\mathscr{U}$ on F, there always is a finest topology on the set of mappings of E into F for which all uniformly convergent generalized sequences are convergent (namely the least upper bound of the uniform convergence topologies of all proper uniform structures, or all pseudometrizable structures, coarser than $\mathscr{U}$).

REFERENCES

1. E. M. Alfsen and J. E. Fenstad, *On the equivalence between proximity structures and totally bounded uniform structures*. Math. Scand. 7 (1959), 353–360.
2. E. M. Alfsen and Olav Njåstad, *Proximity and generalized uniformity*. Fund. Math. 52 (1963), 235–252.
3. E. M. Alfsen and Olav Njåstad, *Totality of uniform structures with a linearly ordered base*. Fund. Math. 52 (1963), 253–256.
4. N. Bourbaki, *Théorie des ensembles*, Ch. IV (Act. Sci. et Ind. 1258) Paris, 1957.
5. S. Leader, *Completion of proximity spaces by local clusters*, Fund. Math. 48 (1960), 201–216.
6. Yu. M. Smirnov, *On proximity spaces*, Mat. Sbornik N. S. 31 (1952), 543–574 (Russian).

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