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ON SECONDARY COHOMOLOGY OPERATIONS

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1. Introduction.

The concept of cohomology operation has in recent years proved very fruitful. This is the case not only for ordinary cohomology but also for extraordinary cohomology. The present paper is a study of cohomology operations in the ordinary theory. This study is based on the presence of a cochain functor in this theory.

In sections 2 and 3 below we develop a general theory of cochain operations. A cochain operation of degree i is a natural transformation

$$\theta: \quad C \to C$$

of the cochain functor C into itself augmenting dimension by i.

These cochain operations are used in sections 4–6 to define secondary cohomology operations. The most elementary properties of these operations are also derived in sections 4–6.

The secondary operations defined here are closely related to the operations defined and used by J. F. Adams in [1]. In some sense we consider more operations here as we allow an additional unfactorized term b in the relations $\sum_\nu \alpha_\nu a_\nu + b = 0$ considered. This, in fact, means that our operations are in certain cases defined in a larger domain and have less indeterminacy than Adams' operations. This turns out to be of vital importance to the application given in a subsequent paper [6].

Besides the definition of secondary operations the main result in the present paper is an evaluation of secondary operations in low dimensions (theorem 4.6). This theorem contains the evaluation of the operation ψ on two dimensional classes done by Adams in [1].

In a second paper [6] the mod 2 cohomology of two-stage Postnikov systems with stable k-invariant will be computed. The algebra structure of this cohomology will also be given. Some of the details about the A-module structure, where A is the Steenrod algebra, are, however, still unknown to the author.

A possible third paper will treat the Cartan formula for secondary operations and give further applications.

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The results contained in the present paper were announced in Kristensen [5].

2. Preparations.

In the following we shall only be working over the groundfield Z_2. Coefficient groups are not mentioned any further in what follows.

Let $\pi = \{1, T\}$ be the symmetric group on two letters, and let W be the standard π-free resolution of Z_2. In each dimension $i \geqq 0$ W has two (Z_2-) generators e_i and Te_i, and

$$\begin{aligned} \partial(e_i) &= \partial(Te_i) = (1+T)e_{i-1} , \\ \varepsilon(e_0) &= \varepsilon(Te_0) = 1 . \end{aligned} \tag{1}$$

Let $\mathscr{K}$ denote the category of css-complexes, and let C_* denote the functor taking any css-complex K into its (non-normalized) chain complex $C_*(K)$ with coefficients in Z_2. It is well-known that there exists a natural π-equivariant chain transformation

$$\varphi' : \quad W \otimes C_* \to C_*^{(2)} \tag{2}$$

preserving augmentation. The action of π in $C_*^{(2)}$ is by permutation, in C^* it is trivial, and in $W \otimes C^*$ it is diagonal. The transformation (2) gives rise to a dual transformation

$$\varphi : \quad W \otimes_\pi C^{(2)} \to C , \tag{3}$$

where C denotes the normalized cochain functor of css-complexes.

As in [2] and [4] we make the following definition. Let $x \in C^n(K)$. Then, for any integer i we define

$$sq^i(x) = \varphi(e_{n-i} \otimes x^2 + e_{n-i+1} \otimes x\,\delta x) \in C^{n+i}(K) , \tag{4}$$

where $e_i = 0$ for $i < 0$. Then

$$\delta\, sq^i(x) = sq^i(\delta x) . \tag{5}$$

In (4) we have defined a natural transformation

$$sq^i : \quad C \to C \tag{6}$$

augmenting dimension by i. In general we make the following

DEFINITION 2.1. *A cochain operation of degree i is a natural transformation*

$$\theta : \quad C \to C \tag{7}$$

augmenting dimension by i.

We do not assume θ to be additive or to commute with coboundary. The concept of cochain operation will be examined in some detail in section 3.

Let A denote the Steenrod algebra, and let F be the free associative algebra with unit 1 generated by sq^i, $i=1,2,\ldots$. Let R denote the kernel of the obvious mapping $F \to A$. Then we have the exact sequence

$$0 \to R \to F \to A \to 0 . \tag{8}$$

It is well-known that R is generated by the Adem relations

$$sq^{2k-1-n}sq^n + \sum_{t=0}^{\frac{1}{2}(n-1)} \binom{n-1-t}{t} sq^{2k-1-t}sq^{k-n+t} \tag{9}$$

with $sq^0=1$.

In F we define the excess of an element as follows: For a monomial $sq^I = sq^{i_1}\ldots sq^{i_r}$, $I=(i_1,\ldots,i_r)$, put

$$e(sq^I) = \max_j (i_j - i_{j+1} - i_{j+2} - \ldots - i_r) . \tag{10}$$

For a sum $\sum_j m_j$ of monomials,

$$e(\textstyle\sum_j m_j) = \min_j(\{e(m_j)\}) . \tag{11}$$

Since any element in F can be uniquely written as a sum of monomials, the excess is well defined. According to Serre [7] an element a in the Steenrod algebra A is said to be of excess larger than or equal to n if a is zero on all cohomology classes of dimensions less than n. The subspace of A consisting of elements of excess larger than or equal to n, we denote by $E(n)$. The subspaces $E(*)$ define a decreasing filtration of A. It is not hard to see that if $\alpha \in F$ is of excess n, then its image in A is of excess larger than or equal to n. In fact, we have

LEMMA 2.2. *Let $\alpha \in F$ be of excess n, then $\alpha(u)=0$ if u is either a cochain of dimension $\leqq n-2$ or a cocycle of dimension $n-1$.*

PROOF. Since α is a sum of monomials each of which is of excess larger than or equal to n, it is clearly enough to prove the lemma for monomials. Let $\alpha = sq^I$, $I=(i_0,i_1,\ldots,i_r)$, and let $u \in C^m$. Since $e(\alpha)=n$, there is a j with

$$i_j - i_{j+1} - \ldots - i_r = n , \tag{12}$$

and hence we have, with $J=(i_0,\ldots,i_{j-1})$ and $K=(i_{j+1},\ldots,i_r)$,

$$\begin{aligned}\alpha(u) &= sq^J sq^{i_j} sq^K(u)\\ &= sq^J\big(\varphi(e_{m-n}\otimes(sq^K(u))^2 + e_{m-n+1}\otimes sq^K(u)\,sq^K(\delta u)\big) .\end{aligned}$$

If $m \leqq n-2$, then both e_{m-n} and e_{m-n+1} are zero since $m-n$ and $m-n+1$ are negative, and it follows that $\alpha(u)$ is zero. If $m \leqq n-1$ and $\delta u = 0$, $\alpha(u)$ is also seen to be zero.

This lemma will be used at a crucial point later in this paper.

LEMMA 2.3. *Let $a \in F$ and $x_i \in C^n$, $i = 1,2,\ldots,k$. Then there is a cochain operation $d = d(a; x_1,\ldots,x_y)$ depending on x_i with the property*

$$(13) \qquad a(\textstyle\sum x_i) = \sum_i a(x_i) + d(a; \delta x_1, \delta x_2, \ldots, \delta x_k) + \delta d(a; x_1,\ldots,x_k)\,.$$

For a equal to a monomial sq^I, $I = (i_0,\ldots,i_r)$, an explicit formula for d is given inductively by

$$(14) \qquad d(sq^i; x_1,\ldots,x_k) = \sum_{\nu<\mu} \varphi(e_{n-i+1} \otimes x_\nu x_\mu + e_{n-i+2} \otimes \delta x_\nu x_\mu)\,,$$

$$(15) \qquad d(sq^j sq^I; x_1,\ldots,x_k) = sq^j d(sq^I; x_1,\ldots,x_k) + \\ + d\big(sq^j; sq^I x_1,\ldots,sq^I x_k, d(sq^I; \delta x_1,\ldots,\delta x_k), \delta d(sq^I; x_1,\ldots,x_k)\big)\,,$$

and for a equal to a sum $\sum_\nu m_\nu$ of monomials

$$(16) \qquad d(a; x_1,\ldots,x_k) = \textstyle\sum_\nu d(m_\nu; x_1,\ldots,x_k)\,.$$

The proof consists in showing that the explicitly given d satisfies (13), and this is an easy exercise. Lemma 2.3 gives examples of cochain operations in several variables. We shall study this concept in section 3.

LEMMA 2.4. *For any $x_i \in C^n$ and any $a \in F$, we have*

$$d(a; 0,0,\ldots,0,x_i,0,\ldots,0) = 0\,.$$

This follows easily for monomials from (14) and (15) by induction on the length of the monomial. For arbitrary a it then follows from (16).

Let

$$(17) \qquad \varkappa\colon\ F \to F$$

be the derivation of degree -1 in F defined by $\varkappa(sq^i) = sq^{i-1}$ ($\varkappa$ induces a derivation in A; this can be seen from the Adem relations, but it can also be derived from lemma 2.5 and theorem 3.3). Then we easily prove

LEMMA 2.5. *For any cocycle $x \in C^n$ and any $a \in F$, we have*

$$d(a; x,x) \sim \varkappa(a)(x)\,.$$

LEMMA 2.6. *Let x and y be n-dimensional cocycles, and let $a \in F$ with $e(a) > n+1$, then $d(a; x,y) = 0$.*

PROOF. Clearly we need to consider only the case where a is a monomial. Let $a = sq^J sq^j sq^I$, where $j - \deg I > n+1$. The proof is by induction on the length of J. First, let the length be zero. Since $j > (n + \deg I - 1) + 1 = \deg(d(sq^I; x,y)) + 1$, we get

$$sq^j d(sq^I; x,y) = 0 ,$$

and since $(n + \deg I) - j + 1 < 0$, we get from (14)

$$d(sq^j; sq^I x, sq^I y, 0, \delta d(sq^I; x,y)) = 0 .$$

Now the conclusion follows from (15). Let the conclusion be true for $sq^J sq^j sq^I$. We shall then prove it for $a = sq^i sq^J sq^j sq^I$. Since $sq^J sq^j sq^I(x) = sq^J sq^j sq^I(y) = 0$, this follows immediately from (15).

The corollaries 3.5 and 3.6 in the next section contain a little more information about the cochains $d(a; x_1, \ldots, x_k)$.

3. Cochain operations.

Let $\mathcal{O}$ denote the set of all cochain operations in one variable as defined in 2.1. In $\mathcal{O}$ we define the structure of a graded associative algebra (over Z_2) as follows

$$(1) \qquad \begin{aligned} (\theta_1 + \theta_2)(u) &= \theta_1(u) + \theta_2(u), & \theta_1, \theta_2 \in \mathcal{O} \\ (\theta \cdot \psi)(u) &= \theta(\psi(u)), & \theta, \psi \in \mathcal{O} \end{aligned}$$

for all cochains u. It is then clear that $\theta_1 + \theta_2$ and $\theta\psi$ belong to $\mathcal{O}$ and that the degree of $\theta\psi$ is the sum of the degrees of θ and ψ. In $\mathcal{O}$ we define a differential Δ by

$$(2) \qquad \begin{aligned} \Delta\theta^i &= \delta\theta^i + (-1)^i \theta^i \delta \\ &= \delta\theta^i + \theta^i \delta \qquad \text{for } \theta^i \in \mathcal{O}^i . \end{aligned}$$

It follows easily that $\Delta\Delta = 0$. The differential Δ is clearly additive. The Δ-cycles in $\mathcal{O}$ we shall call primary cochain operations. They are denoted by $Z(\mathcal{O})$.

Each element $\theta \in Z(\mathcal{O})$ commutes with the coboundary and therefore maps cocycles into cocycles. In Lemma 3.2 below we shall show that the cohomology class of $\theta(u)$ depends only on the cohomology class of the cocycle u, so that θ defines a cohomology operation $\varepsilon(\theta)$ which is clearly stable. Hence

$$(3) \qquad \varepsilon(\theta)(\{u\}) = \{\theta(u)\} .$$

DEFINITION 3.1. *A cochain operation in m variables and of degree i is a natural transformation*

$$\theta\colon\ A \to C\,,$$

augmenting dimension by i, where A is the functor

$$A^n(K) = C^n(K)\oplus\ldots\oplus C^n(K), \qquad m \text{ summands}\,.$$

We do not assume θ to be additive or to commute with coboundary. The set of all cochain operations in m variables constitute a graded vectorspace denoted by $\mathcal{O}^{(m)}$. We define a differential in $\mathcal{O}^{(m)}$ by

$$(\Delta\psi)(x_1,\ldots,x_m) = \delta\psi(x_1,\ldots,x_m)+\psi(\delta x_1,\ldots,\delta x_m)\,. \tag{4}$$

The cycles with respect to Δ we denote by $Z(\mathcal{O}^{(m)})$. As in the case of one variable, each Δ-cycle $\theta \in Z(\mathcal{O}^{(m)})$ maps a set $(u_1,\ldots,u_m)$ of cocycles into a cocycle. In lemma 3.2 below we will show that the cohomology class of $\theta(u_1,\ldots,u_m)$ depends only on the set of cohomology classes $(\overline{u}_1,\ldots,\overline{u}_m)$ of $(u_1,\ldots,u_m)$ so that θ defines a cohomology operation $\varepsilon(\theta)$ by

$$\varepsilon(\theta)(\overline{u}_1,\ldots,\overline{u}_m) = \{\theta(u_1,\ldots,u_m)\}\,. \tag{5}$$

LEMMA 3.2. *In definition* (5) *the cohomology class $\varepsilon(\theta)(\overline{u}_1,\ldots,\overline{u}_m)$ is independent of the choice of representatives u_i of the cohomology classes $\overline{u}_i$.*

PROOF. Let $K(\pi,n)$ be the css-complex defined by Eilenberg and MacLane [3]. Let X be a css-complex, and let u_i and $u_i+\delta e_i$ be representatives of $\overline{u}_i \in H^n(X)$. Then there are unique mappings

$$f_{0,i}\colon\ X \to K(Z_2,n)_i\,, \tag{6}$$

$$f_{1,i}\colon\ X \to K(Z_2,n)_i\,, \tag{7}$$

such that $f_{0,i}(z_i^{(n)})=u_i$, $f_{1,i}(z_i^{(n)})=u_i+\delta e_i$, where $z_i^{(n)}$ is the basic cocycle in the i-th copy of $K(Z_2,n)$. The mappings $f_{0,i}$ and $f_{1,i}$ are homotopic. These mappings induce mappings

$$f_0,f_1\colon\ X \to \bigtimes_{i=1}^{m} K(Z_2,n)_i$$

by

$$\begin{aligned} f_0(\sigma_q) &= \big(f_{0,1}(\sigma_q),f_{0,2}(\sigma_q),\ldots,f_{0,m}(\sigma_q)\big)\,,\\ f_1(\sigma_q) &= \big(f_{1,1}(\sigma_q),f_{1,2}(\sigma_q),\ldots,f_{1,m}(\sigma_q)\big)\,. \end{aligned} \tag{8}$$

The mappings f_0 and f_1 are homotopic. The induced cochain transformations $f_0^{\#}$ and $f_1^{\#}$ therefore are cochain homotopic by a homotopy h. Let the image of $z_j^{(n)}$ under the maps induced by projection on the j-th factor $\times_i K(Z_2,n)_i \to K(Z_2,n)_j$ also be denoted by $z_j^{(n)}$. Then

$$(9) \qquad \begin{aligned} f_0{}^{\#}(z_j{}^{(n)}) &= u_j \\ f_1{}^{\#}(z_j{}^{(n)}) &= u_j+\delta e_j\,. \end{aligned}$$

Now we have

$$\begin{aligned} &\theta(u_1+\delta e_1,u_2+\delta e_2,\ldots,u_m+\delta e_m)-\theta(u_1,u_2,\ldots,u_m) \\ &\quad= \theta\big(f_1{}^{\#}(z_1{}^{(n)}),f_1{}^{\#}(z_2{}^{(n)}),\ldots,f_1{}^{\#}(z_m{}^{(n)})\big)-\theta\big(f_0{}^{\#}(z_1{}^{(n)}),f_0{}^{\#}(z_2{}^{(n)}),\ldots,f_0{}^{\#}(z_m{}^{(n)})\big) \\ &\quad= (f_1{}^{\#}-f_0{}^{\#})\,\theta(z_1{}^{(n)},z_2{}^{(n)},\ldots,z_m{}^{(n)}) \\ &\quad= (\delta h+h\delta)\,\theta(z_1{}^{(n)},z_2{}^{(n)},\ldots,z_m{}^{(n)}) \\ &\quad= \delta\big(h\theta(z_1{}^{(n)},z_2{}^{(n)},\ldots,z_m{}^{(n)})\big) \end{aligned}$$

since $h\delta\theta(z_1{}^{(n)},\ldots,z_m{}^{(n)})=h\,\theta(\delta z_1{}^{(n)},\ldots,\delta z_m{}^{(n)})=0$. This completes the proof.

Any element $(a_1,a_2,\ldots,a_m)$ of the direct sum of m copies of the Steenrod algebra $A^*\oplus A^*\oplus\ldots\oplus A^*$ is a stable cohomology operation in m variables, acting by

$$(a_1,a_2,\ldots,a_m)(\bar u_1,\bar u_2,\ldots,\bar u_m) = a_1(\bar u_1)+a_2(\bar u_2)+\ldots+a_m(\bar u_m)\,.$$

Since also any stable cohomology operation in m variables belongs to $A^*\oplus A^*\oplus\ldots\oplus A^*$, we obtain that $\varepsilon(\theta)\in A^*\oplus A^*\oplus\ldots\oplus A^*$ and that the sequence

$$(11) \qquad Z(\mathcal{O}^{(m)}) \to A^*\oplus A^*\oplus\ldots\oplus A^* \to 0$$

is exact. In case $n=1$, ε is an algebra homomorphism of $Z(\mathcal{O})$ onto A^*.

THEOREM 3.3. *The sequence*

$$(12) \qquad \mathcal{O}^{(m)} \xrightarrow{\Delta} Z(\mathcal{O}^{(m)}) \to A^*\oplus A^*\oplus\ldots\oplus A^* \to 0$$

is exact. The cohomology of $(\mathcal{O}^{(m)},\Delta)$ *is therefore* $A^*\oplus\ldots\oplus A^*$.

PROOF. We only have to prove the exactness at $Z(\mathcal{O}^{(m)})$. First, let $r=\Delta\theta$, and let $(u_1,\ldots,u_m)$ be a m-tuple of cocycles. Then

$$\begin{aligned} \varepsilon(r)\big(\{u_1\},\{u_2\},\ldots,\{u_m\}\big) &= \{r(u_1,u_2,\ldots,u_m)\} \\ &= \{\delta\theta(u_1,u_2,\ldots,u_m)+\theta(\delta u_1,\delta u_2,\ldots,\delta u_m)\} = 0\,. \end{aligned}$$

Since this is true for all m-tuples $(u_1,\ldots,u_m)$ and for all θ, we have $\varepsilon\Delta=0$. Next, let $r\in Z(\mathcal{O}^{(m)})$ with $\varepsilon(r)=0$. Then we must prove the existence of $\theta\in\mathcal{O}^{(m)}$ with $\Delta(\theta)=r$. Elements in the kernel of ε we shall call relations. In the proof we shall need the following lemma. It concerns partially defined cochain operations. These cochain operations are, for some dimension n, defined on all m-tuples of dimension less than n, on all m-tuples of cocycles of dimension n, and on $(0,0,\ldots,0)$ in higher dimensions. For short we shall say that the operation is defined in dimensions less than n.

LEMMA 3.4. *Let $r \in Z(\mathcal{O}^{(m)})$ be a relation (i.e. $\varepsilon(r)=0$) of degree $i+1$, and let θ be a cochain operation defined in dimensions less than n such that*

$$\delta\theta(u_1,\ldots,u_m)+\theta(\delta u_1,\ldots,\delta u_m) = r(u_1,\ldots,u_m)\,, \tag{13}$$

whenever this makes sense (i.e. for $\dim u_i < n$, or u_i cocycles of dimension n). Then there exists a cochain operation θ' on m variables defined in dimensions less than $n+1$, such that

$$(\Delta\theta')(u_1,\ldots,u_m) = r(u_1,\ldots,u_m)\,,$$

whenever this makes sense, and such that

$$\theta'(u_1,u_2,\ldots,u_m) = \theta(u_1,u_2,\ldots,u_m), \quad \textit{for} \quad \dim u_j \leqq n-2\,, \tag{14}$$

$$\begin{aligned}\theta'(u_1,u_2,\ldots,u_m) = {} & \theta(u_1,u_2,\ldots,u_m) + \\ & +\textstyle\sum_k M_k'(u_1,u_2,\ldots,u_m) \quad \textit{for} \quad \dim u_j = n-1\,,\end{aligned} \tag{15}$$

and for u_j n-dimensional cocycles

$$\theta'(u_1,u_2,\ldots,u_m) = \theta(u_1,u_2,\ldots,u_m)+\textstyle\sum_k M_k(u_1,u_2,\ldots,u_m)\,, \tag{16}$$

where each M_k is an operation of degree i of the form

$$M_k(u_1,u_2,\ldots,u_m) = sq^{I(1)}(u_{j(1)})\, sq^{I(2)}(u_{j(2)}) \,\ldots\, sq^{I(r(k))}(u_{j(r(k))}) \tag{17}$$

with $I(j)=I(j,k)$ admissible and of excess $<n$, and $r(k)\geqq 2$ for each k. Each M_k' is derived from M_k by replacing all but one of the $sq^{I(j)}$'s by $sq^{I(j)}\delta$, e.g.

$$M_k'(u_1,u_2,\ldots,u_m) = sq^{I(1)}(u_{j(1)})\, sq^{I(2)}(\delta u_{j(2)}) \,\ldots\, sq^{I(r(k))}(\delta u_{j(r(k))}). \tag{18}$$

If $n>i$, M_k is zero for all k and θ' is an extension of θ.

PROOF. Let $L(\pi,n)$ be the css-complex introduced by Eilenberg–MacLane [3]. Let

$$p\colon\ L(\pi,n) \to K(\pi,n+1)$$

be the css-mapping induced by coboundary. Let us consider

$$L_n = \bigtimes_{i=1}^{m} L(Z_2,n)_i\,.$$

Then θ is not defined on the m-tuple $(c_1^{(n)},\ldots,c_m^{(n)})$ of basic cochains in L_n. As a first approximation to a definition of θ' on $(c_1^{(n)},\ldots,c_m^{(n)})$

let us choose an arbitrary cochain $\bar\theta(c_1^{(n)},\ldots,c_m^{(n)})$ in L_n with the property that, restricted to

$$K_n = \bigtimes_{i=1}^{m} K(Z_2,n)_i ,$$

it equals $\theta(z_1^{(n)},\ldots,z_m^{(n)})$. Putting

$$b = \delta\bar\theta(c_1^{(n)},\ldots,c_m^{(n)}) - r(c_1^{(n)},\ldots,c_m^{(n)}) , \tag{19}$$

we see that the restriction of b to K_n is zero and that

$$\delta b = r(\delta c_1^{(n)},\ldots,\delta c_m^{(n)}) . \tag{20}$$

Since r is a relation, there exists a cochain α in K_{n+1},

$$K_{n+1} = \bigtimes_{i=1}^{m} K(Z_2,n+1)_i ,$$

with

$$\delta\alpha = r(z_1^{(n+1)},\ldots,z_m^{(n+1)}) . \tag{21}$$

The projection $p\colon L_n \to K_{n+1}$ has the property $p^\#(z_j^{(n+1)}) = \delta(c_j^{(n)})$. Therefore

$$\delta p^\#(\alpha) = r(\delta c_1^{(n)},\ldots,\delta c_m^{(n)}) , \tag{22}$$

and by (20)

$$\delta\big(b - p^\#(\alpha)\big) = 0 . \tag{23}$$

Since L_n is acyclic, there exists a cochain e with

$$\delta e = b - p^\#(\alpha) . \tag{24}$$

If $i\colon K_n \to L_n$ denotes the inclusion, we have

$$\delta i^\#(e) = 0 . \tag{25}$$

The cohomology of

$$K_n = \bigtimes_{i=1}^{m} K(Z_2,n)_i$$

is isomorphic to the tensorproduct of m copies of the cohomology of $K(Z_2,n)$. From the computations by Serre [7] and (25) we therefore have

$$i^\#(e) \sim \textstyle\sum_{N,j} sq^N(z_j^{(n)}) + \sum_k M_k(z_1^{(n)},\ldots,z_m^{(n)}) , \tag{26}$$

where

$$M_k(z_1^{(n)},\ldots,z_m^{(n)}) = sq^{I(1)}(z_{j(1)}^{(n)})\, sq^{I(2)}(z_{j(2)}^{(n)}) \ldots sq^{I(r(k))}(z_{j(r(k))}^{(n)}) \tag{27}$$

with $I(j)$ admissible and of excess $< n$, and $r(k) \geqq 2$ for each k. If we replace α and e by $\alpha' = \alpha + \sum_{N,j} sq^N(z_j^{(n+1)})$ and $e' = e + \sum_{N,j} sq^N(c_j^{(n)})$, respectively, in the previous equations, then (21) to (25) still hold true while, for some cochain γ in L_n, (26) can be written

$$(28)\qquad i^{\#}(e') - \textstyle\sum_k M_k(z_1^{(n)},\ldots,z_m^{(n)}) = \delta i^{\#}(\gamma)\,.$$

Therefore, for some cochain β in L_n

$$(29)\qquad \delta\gamma = e' - \textstyle\sum_k M_k(c_1^{(n)},\ldots,c_m^{(n)}) + \beta\,.$$

Then

$$(30)\qquad i^{\#}(\beta) = 0\,.$$

From (19), (29), and (24) (with the primes) we get

$$(31)\qquad \begin{aligned}\delta[\bar\theta(c_1^{(n)},\ldots,c_m^{(n)}) + \beta + \textstyle\sum_k M_k(c_1^{(n)},\ldots,c_m^{(n)})]\\ = r(c_1^{(n)},\ldots,c_m^{(n)}) + p^{\#}(\alpha')\,.\end{aligned}$$

We define

$$(32)\qquad \begin{aligned}\theta'(z_1^{(n+1)},\ldots,z_m^{(n+1)}) &= \alpha'\\ \theta'(c_1^{(n)},\ldots,c_m^{(n)}) &= \bar\theta(c_1^{(n)},\ldots,c_m^{(n)}) + \beta + \textstyle\sum_k M_k(c_1^{(n)},\ldots,c_m^{(n)})\end{aligned}$$

and extend the definitions to all m-tuples of $(n+1)$-cocycles and n-cochains, respectively, by naturality. Because of (30) this definition makes (16) hold true. By (31) we get

$$(33)\qquad \delta\theta'(u_1,\ldots,u_m) + \theta'(\delta u_1,\ldots,\delta u_m) = r(u_1,\ldots,u_m)$$

for all m-tuples of n-cochains $(u_1,\ldots,u_m)$, and by (21)

$$(34)\qquad \delta\theta'(u_1,\ldots,u_m) + \theta'(\delta u_1,\ldots,\delta u_m) = r(u_1,\ldots,u_m)$$

for all m-tuples of $(n+1)$-cocycles.

Now, let $(u_1,\ldots,u_m)$ be a m-tuple of $(n-1)$ cochains. Then we must define $\theta'(u_1,\ldots,u_m)$ such that

$$(35)\qquad \delta\theta'(u_1,\ldots,u_m) = \theta'(\delta u_1,\ldots,\delta u_m) + r(u_1,\ldots,u_m)\,.$$

Since the δu_j's are cocycles, we apply (16) and see that the right hand side of (35) is equal to

$$(36)\qquad \begin{aligned}\theta(\delta u_1,\ldots,\delta u_m) + \textstyle\sum_k M_k(\delta u_1,\ldots,\delta u_m) + r(u_1,\ldots,u_m)\\ = \delta\big(\theta(u_1,\ldots,u_m) + \textstyle\sum_k M_k'(u_1,\ldots,u_m)\big)\,,\end{aligned}$$

where M_k' is as in (18). Putting

$$(37)\qquad \theta'(u_1,\ldots,u_m) = \theta(u_1,\ldots,u_m) + \textstyle\sum_k M_k'(u_1,\ldots,u_m)\,,$$

we see that both (35) and (15) are satisfied.

Let $(u_1,\ldots,u_m)$ be of dimension $(n-2)$. Then we must make a definition satisfying

$$\delta\theta'(u_1,\ldots,u_m) = \theta'(\delta u_1,\ldots,\delta u_m)+r(u_1,\ldots,u_m) \tag{38}$$
$$= \theta(\delta u_1,\ldots,\delta u_m)+r(u_1,\ldots,u_m)$$
$$= \delta\theta(u_1,\ldots,u_m)\ .$$

In this case and in dimensions less than $n-2$, we can therefore define $\theta'=\theta$. This completes the construction of θ'. If n is larger than i, we must have $\sum_k M_k=0$. If, namely, M_k occurred, then the degree of $M_k(u_1,\ldots,u_m)$ on one hand is equal to $n+i$, while on the other it equals $\sum_{j=1}^{r(k)}(n+\deg I(j))$. This implies

$$\sum_{i=1}^{r(k)}\deg\big(I(j)\big) < 0$$

which is a contradiction. Therefore it is clear that in this case θ' is an extension of θ. This completes the proof.

Now we are able to complete the proof of theorem 3.3. We use an acyclic model argument. The acyclic models are the products

$$L_n = \bigtimes_{i=1}^{m} L(Z_2,n)\ .$$

Choose n larger than $i=\deg(r)-1$. Then we shall first construct a θ defined in dimensions less than n with $\Delta\theta=r$ whenever this makes sense. Since r is a relation, there exists a cochain α in K_n with $\delta\alpha=r(z_1{}^{(n)},\ldots,z_m{}^{(n)})$. Define

$$\theta(z_1{}^{(n)},\ldots,z_m{}^{(n)}) = \alpha\ . \tag{39}$$

Naturality then gives us θ in general in dimension n. In dimension $n-1$ we must define $\theta(c_1{}^{(n-1)},\ldots,c_m{}^{(n-1)})$ in L_{n-1} such that

$$\delta\theta(c_1{}^{(n-1)},\ldots,c_m{}^{(n-1)}) = \theta(\delta c_1{}^{(n-1)},\ldots,\delta c_m{}^{(n-1)})+r(c_1{}^{(n-1)},\ldots,c_m{}^{(n-1)}) \tag{40}$$
$$= p^{\#}(\alpha)+r(c_1{}^{(n-1)},\ldots,c_m{}^{(n-1)})\ ,$$

where p is the projection $L_{n-1}\to K_n$. The right hand side is a cocycle

$$\delta\big(p^{\#}(\alpha)+r(c_1{}^{(n-1)},\ldots,c_m{}^{(n-1)})\big) = p^{\#}(\delta\alpha)+\delta r(c_1{}^{(n-1)},\ldots,c_m{}^{(n-1)})$$
$$= p^{\#}\big(r(z_1{}^{(n)},\ldots,z_m{}^{(n)})\big)+\delta r(c_1{}^{(n-1)},\ldots,c_m{}^{(n-1)})$$
$$= r(p^{\#}z_1{}^{(n)},\ldots,p^{\#}z_m{}^{(n)})+\delta r(c_1{}^{(n-1)},\ldots,c_m{}^{(n-1)})$$
$$= 0\ .$$

Therefore we can choose a cochain $\theta(c_1{}^{(n-1)},\ldots,c_m{}^{(n-1)})$ in L_{n-1} making (40) hold true. Naturality defines θ on all m-tuples of $(n-1)$-cochains.

Continuing this argument a cochain operation θ is defined in the desired area with $\Delta\theta = r$. Since $n > i$, lemma 3.4 enables us to extend θ to all dimensions such that $\Delta\theta = r$. This completes the proof of theorem 3.3.

COROLLARY 3.5. *For any two elements $\alpha, a \in F$ there exists a cochain operation θ in two variables with*

$$(\Delta\theta)(x,y) = G(x,y) + \varkappa(\alpha)a(x) + \varkappa(\alpha)a(y) ,$$

where G is defined from the cochain operation d of lemma 2.3 by

$$G(x,y) = d(\alpha a; x,y) + d\big(x; a(x), a(y)\big) + d\big(\alpha; a(x+y), a(x) + a(y)\big) + \\ + \alpha d(a; x,y) + d\big(\alpha; \delta d(a; x,y), d(a; \delta x, \delta y)\big) ,$$

and $\varkappa$ is defined in (17) *of section* 2. *On cocycles therefore*

$$G(x,y) \sim \varkappa(\alpha)a(x) + \varkappa(\alpha)a(y) .$$

PROOF. First we note that $\Delta G = 0$. By theorem 3.3 there are cochain operations b, c and θ with $b, c \in F$, such that

$$(\Delta\theta)(x,y) = G(x,y) + b(x) + c(y) .$$

This means that for x and y cocycles, $G(x,y) + b(x) + c(y)$ is a coboundary. Further, if we put $y = 0$, then by lemmas 2.4 and 2.5 we get

$$b(x) \sim G(x,0) = d\big(\alpha; a(x), a(x)\big) \sim \varkappa(\alpha)\big(a(x)\big) .$$

A similar argument for c concludes the proof.

COROLLARY 3.6. *Let x and y be n-dimensional cocycles, and let $a = sq^J sq^j sq^I$, where I and J are sequences and j an integer with $j - \deg(I) = n + 1$. Then*

$$d(a; x,y) \sim sq^J\big(sq^I(x)\, sq^I(y)\big) .$$

PROOF. By dimensional reasons $sq^j sq^I x = 0$. Therefore, by corollary 3.5

$$d(sq^J sq^j sq^I; x,y) \sim sq^J\big(d(sq^j sq^I; x,y)\big)$$

and

$$\begin{aligned} d(sq^j sq^I; x,y) &\sim d(sq^j; sq^I x, sq^I y) + \\ &\quad + d\big(sq^j; sq^I(x+y), sq^I(x) + sq^I(y)\big) + sq^{j-1} sq^I(x) + \\ &\quad + sq^{j-1} sq^I(y) \\ &= sq^I(x)\, sq^I(y) + sq^I(x+y)\, \big(sq^I(x) + sq^I(y)\big) + \\ &\quad + sq^I(x)\, sq^I(x) + sq^I(y)\, sq^I(y) \\ &\sim sq^I(x)\, sq^I(y) . \end{aligned}$$

The equality is obtained by using the explicit definition of $d(sq^j; x_1, x_2)$. This completes the proof.

THEOREM 3.7. *Let $r \in R$, and let $e(r) \geqq n+1$. Then there exists a cochain operation θ with $\Delta\theta = r$ and with the values*

$$\begin{aligned}
\theta(u) &= 0 && \text{when } \dim(u) \leqq n-2\,, \\
\theta(u) &= \textstyle\sum_k sq^{I(1)}(u)\, sq^{I(2)}(\delta u) \ldots sq^{I(s(k))}(\delta u) && \text{when } \dim(u) = n-1\,, \\
\theta(u) &= \textstyle\sum_k sq^{I(1)}(u)\, sq^{I(2)}(u) \ldots sq^{I(s(k))}(u) && u \ n\text{-cocycle}
\end{aligned}$$

for certain monomials $sq^{I(j)}$ and $s(k) > 1$ for all k.

PROOF. Let θ' be a partially defined cochain operation with the value zero on all n-cocycles and all cochains of dimension less than n. Then $\Delta\theta' = r$ whenever this makes sense. A continued application of lemma 3.4 yields θ.

For special kinds of relations we can do better than theorem 3.7. Let sequences I be ordered lexicographically from the right, then we have

THEOREM 3.8. *Let there be given a relation r (homogeneous) from R of the form*

$$r = \sum_{s\in S} sq^{I(s)} sq^{j(s)} sq^{J(s)} + \sum_{t\in T} (sq^{2K(t)} sq^{j(t)} sq^{J(t)} + sq^{i(t)} sq^{K(t)\,J(t)}) + b\,,$$

where $e(b) \geqq n+2$; the sequence $I(s)$ contains at least one odd component, and

$$\begin{aligned}
j(s) &= n+1+\deg\big(J(s)\big)\,, \\
j(t) &= n+1+\deg\big(J(t)\big)\,, \\
i(t) &= n+1+\deg\big(K(t)J(t)\big)\,.
\end{aligned}$$

Then there exists a cochain operation θ with $\Delta\theta = r$, with $\theta(u) = 0$ for any cochain of dimension less than $n-1$, taking on $(n-1)$-cochains the value

$$\theta(u) = \sum_{\nu\in S\cup T} \ \sum_{\substack{A \prec B \\ A+B=I(\nu)}} sq^{AJ(\nu)}(u)\, sq^{BJ(\nu)}(\delta u)\,,$$

where $I(\nu) = 2K(\nu)$ for $\nu \in T$, and on n-cocycles the value

$$\theta(u) = \sum_{\nu\in S\cup T} \ \sum_{\substack{A \prec B \\ A+B=I(\nu)}} sq^{AJ(\nu)}(u)\, sq^{BJ(\nu)}(u)\,.$$

PROOF. First we shall show that the theorem follows if to each sequence I and each integer n there is a cochain operation $K_n{}^I$ defined on all n-cochains x and on all $(n+1)$-cocycles y and having the properties

$$
\begin{aligned}
& sq^I(x\,\delta x) = \sum_{A+B=I} sq^A(x)\, sq^B(\delta x) + \delta K_n{}^I(x) + K_n{}^I(\delta x) \\
(41) \qquad & sq^I(yy) = \sum_{A+B=I} sq^A(y)\, sq^B(y) + \delta K_n{}^I(y)\,, \\
& K_n{}^I(u) = 0 \quad \text{if } u \text{ is an } n\text{-cocycle.}
\end{aligned}
$$

Let θ' be the cochain operation defined in dimensions less than $n+1$ by the formulae

$$
\begin{aligned}
\theta'(u) &= 0, && u \text{ cochain, } \dim(u) < n-1\,, \\
\theta'(u) &= \sum_{\nu} \sum_{A \prec B} sq^{AJ(\nu)}(u)\, sq^{BJ(\nu)}(\delta u), && u\ (n-1)\text{-cochain}\,, \\
\theta'(u) &= \sum_{\nu} \sum_{A \prec B} sq^{AJ(\nu)}(u)\, sq^{BJ(\nu)}(u) + \\
&\quad + \sum_{\nu} \sum_{A \prec B} \varphi\big(e_1 \otimes sq^{AJ(\nu)}(\delta u)\, sq^{BJ(\nu)}(u)\big) + \\
&\quad + \sum_{\nu} K^{I(\nu)}_{m(\nu)}\big(sq^{J(\nu)}(u)\big), && u\ n\text{-cochain}\,,
\end{aligned}
$$

where $m(\nu) = n + \deg\big(J(\nu)\big)$, and for u an $(n+1)$-cocycle

$$
\begin{aligned}
\theta'(u) &= \sum_{\nu} \sum_{A \prec B} \varphi\big(e_1 \otimes sq^{AJ(\nu)}(u)\, sq^{BJ(\nu)}(u)\big) + \\
&\quad + \sum_{\nu} K^{I(\nu)}_{m(\nu)}\big(sq^{J(\nu)}(u)\big)\,.
\end{aligned}
$$

It is straightforward to verify that $\Delta\theta' = r$ whenever this makes sense. An application of lemma 3.4 now yields a θ with the properties stated in the theorem.

Now we only need to prove the existence of cochain operations $K_n{}^I$ satisfying (41). This we do by induction on the length of I. In section 7 of [4] a transformation

$$H\colon \quad W \otimes_\pi (C \otimes C)^2 \to C$$

was given with the property ((22) of section 7 in [4])

$$
(42) \qquad sq^k(xy) = \sum_{i+j=k} sq^i(x)\, sq^j(y) + \delta H(\eta) + H(\zeta)\,,
$$

where x is a p-cochain, and y a q-cocycle, and where

$$
\begin{aligned}
\eta &= e_{p+q-k} \otimes (x \otimes y)^2 + e_{p+q-k+1} \otimes (x \otimes y) \otimes (\delta x \otimes y)\,, \\
\zeta &= e_{p+q-k+1} \otimes (\delta x \otimes y)^2\,.
\end{aligned}
$$

If $I = (k)$ (that is length one), then (42) shows that if we put

$$K_n^{(k)}(x) = H\big(e_{2n+1-k}\otimes(x\otimes\delta x)^2 + e_{2n+2-k}\otimes(x\otimes\delta x)\otimes(\delta x\otimes\delta x)\big),$$
$$K_n^{(k)}(y) = H(e_{2n+2-k}\otimes y^4),$$

then (41) is satisfied.

Now, suppose that $K_n{}^I$ is defined such that (41) holds true. We shall then consider the sequence $J=(k)I$. Put

$$K_n{}^J(x) = sq^k K_n{}^I(x) + \sum_{A+B=I} H\big(\eta(A,B)\big) + \\ + d\big(sq^k;\, sq^{A_1}(x)\,sq^{B_1}(\delta x), \ldots, sq^{A_s}(x)\,sq^{B_s}(\delta x);\, \delta sq^k K_n{}^I(x), K_n{}^I(\delta x)\big),$$

where for $m=2n+1+\deg I-k$

$$\eta(A,B) = e_m\otimes\big(sq^A(x)\otimes sq^B(\delta x)\big)^2 + \\ + e_{m+1}\otimes\big(sq^A(x)\otimes sq^B(\delta x)\big)\otimes\big(sq^A(\delta x)\otimes sq^B(\delta x)\big),$$

and where $(A_1,B_1),\ldots,(A_s,B_s)$ is an ordering of the summands in $\sum sq^A x\, sq^B \delta x$,

$$K_n{}^J(y) = sq^k K_n{}^I(y) + \sum_{A+B=I} H\big(e_{m+1}\otimes\big(sq^A(y)\otimes sq^B(y)\big)^2\big) + \\ + d\big(sq^k;\, sq^{A_1}(y)\,sq^{B_1}(y), \ldots, sq^{A_s}(y)\,sq^{B_s}(y), \delta K_n{}^I(y)\big).$$

A straightforward computation shows that this $K_n{}^J$ satisfies (41). This completes the proof.

In a subsequent paper [6] we shall need a little more detailed information than is contained in theorem 3.8. This, however, is for a more special type of relation

THEOREM 3.9. *Let $r=sq^{2k+\varepsilon}sq^{n+1}+(1-\varepsilon)sq^{n+k+1}sq^k+b$ be a relation with $\varepsilon=0,1$ and $e(b)\geqq n+2$. Then there exists a cochain operation θ with $\Delta\theta=r$, and such that $\theta(u)=0$ when $\dim(u)\leqq n-2$, for $\dim u=n-1$,*

$$\theta(u) = \sum sq^\alpha u\; sq^\beta \delta u,$$

where the summation runs over all α,β with $\alpha<\beta$ and $\alpha+\beta=2k+\varepsilon$, and for $\dim(u)=n$ (same range of summation)

$$\theta(u) = \sum sq^\alpha u\; sq^\beta u + \sum\varphi(e_1\otimes sq^\alpha\delta u\; sq^\beta u) + \\ + H\big(e_{2(n-k)+1-\varepsilon}\otimes(u\otimes\delta u)^2 + e_{2(n-k)+2-\varepsilon}\otimes(u\otimes\delta u)\otimes(\delta u\otimes\delta u)\big) + \\ + \sum sq^{I_1}(u)\; sq^{I_2}(\delta u)\;\ldots\; sq^{I_s}(\delta u),$$

where $\deg(I_1)\leqq\deg(I_2)\leqq\ldots\leqq\deg(I_s)$, $s>1$, and where H is the homotopy of equation (42).

PROOF Let θ' be defined in dimensions less than $n+1$ as follows: $\theta'(u)=0$ on cochains of dimensions less than $n-1$, on $(n-1)$-cochains

$$\theta'(u) = \sum_{\alpha,\beta} sq^\alpha u\, sq^\beta \delta u, \qquad \alpha<\beta,\ \alpha+\beta = 2k+\varepsilon\,,$$

on n-cochains (same range of summation)

$$\theta'(u) = \sum_{\alpha,\beta} sq^\alpha u\, sq^\beta u + \sum_{\alpha,\beta}\varphi(e_1\otimes sq^\alpha \delta u\, sq^\beta u) + \\ + H\big(e_{2(n-k)+1-\varepsilon}\otimes(u\otimes\delta u)^2 + e_{2(n-k)+2-\varepsilon}\otimes(u\otimes\delta u)\otimes(\delta u\otimes\delta u)\big)\,,$$

and on $(n+1)$-cocycles

$$\theta'(u) = H\big(e_{2(n-k)+2-\varepsilon}\otimes(u\otimes u)^2\big) + \sum_{\alpha,\beta}\varphi(e_1\otimes sq^\alpha u\, sq^\beta u)\,.$$

An easy computation using (42) shows that $\Delta\theta' = r$ whenever this makes sense. An application of lemma 3.4 now yields the theorem.

4. Secondary cohomology operations.

Let

$$(1) \qquad r = \sum_\nu \alpha_\nu a_\nu + b \in R, \qquad \alpha_\nu, a_\nu; b \in F\,.$$

Then by theorem 3.3 there is a cochain operation θ with

$$(2) \qquad \Delta\theta = \sum \alpha_\nu a_\nu + b = r\,.$$

Let K be a css-complex, and let u be a cocycle in K of dimension less than $e(b)$. Suppose also that for all ν $a_\nu(u)$ is cohomologous to zero. Choose cochains v_ν in K such that

$$(3) \qquad \delta v_\nu = a_\nu(u), \qquad \text{all } \nu\,.$$

Consider the cochain

$$(4) \qquad qu^r(u)_{\theta,\{v_\nu\}} = qu^r(u) = \theta(u) + \sum_\nu \alpha_\nu(v_\nu)\,.$$

Actually, this cochain is a cocycle,

$$\delta qu^r(u) = \delta\big(\theta(u) + \sum_\nu \alpha_\nu(v_\nu)\big) = r(u) + \sum \alpha_\nu\big(a_\nu(u)\big) = b(u) = 0\,,$$

where $b(u)=0$ follows from lemma 2.2. Although it has not been made clear in the notation, $qu^r(u)$ depends on the choices of θ and v_ν. Any other cochain operation θ' satisfying (2) must be of the form $\theta'=\theta+\gamma$ with $\Delta\gamma=0$. Therefore, by (4)

$$(5) \qquad qu^r(u)_{\theta',\{v_\nu\}} - qu^r(u)_{\theta,\{v_\nu\}} = \gamma(u)\,.$$

Any other choice of cochains $\{v_\nu'\}$ satisfying (3) must be of the form $v_\nu' = v_\nu + x_\nu$ with $\delta x_\nu = 0$. We get

$$(6) \qquad qu^r(u)_{\theta,\{v_\nu+x_\nu\}} - qu^r(u)_{\theta,\{v_\nu\}} = \sum_\nu\big(\alpha_\nu(v_\nu+x_\nu) - \alpha_\nu(v_\nu)\big) \\ = \sum \alpha_\nu(x_\nu) + \delta \sum_\nu d(\alpha_\nu;\, v_\nu, x_\nu)\,.$$

We are also interested in the variation of $qu^r(u)$, when u is allowed to vary inside its cohomology class. First, however, let u and w be cocycles in K with $\dim(u) = \dim(w) < e(b)$. Besides (3), let

$$(7) \qquad \delta y_\nu = a_\nu(w), \qquad \text{all } \nu .$$

Then, for $z_\nu = v_\nu + y_\nu + d(a_\nu;\, u, w)$

$$(8) \qquad \delta z_\nu = a_\nu(u+w), \qquad \text{all } \nu .$$

Then,

$$(9) \quad \begin{aligned} \alpha_\nu(z_\nu) - \alpha_\nu(v_\nu) - \alpha_\nu(y_\nu) &\sim d\big(\alpha_\nu;\, a_\nu(u), a_\nu(w)\big) + \\ &\quad + d\big(\alpha_\nu;\, a_\nu(u+w), a_\nu(u)+a_\nu(w)\big) + \alpha_\nu d(a_\nu;\, u, w) \\ &= G_\nu(u,w) - d(\alpha_\nu a_\nu;\, u,w) , \end{aligned}$$

where G_ν is the cochain operation defined in corollary 3.5 for $\alpha = \alpha_\nu$ and $a = a_\nu$. Relative to (2), (3), (7), and (8) we therefore get

$$(10) \quad qu^r(u+w) - qu^r(u) - qu^r(w) \sim \psi(u,w) + \textstyle\sum_\nu G_\nu(u,w) - d(b;\, u,w) ,$$

where the cochain operation in two variables ψ is defined by

$$(11) \qquad \psi(x,y) = \theta(x+y) - \theta(x) - \theta(y) - d(r;\, x,y) .$$

It is easy to check that $\Delta\psi = 0$. Hence, by theorem 3.3 there are elements $f, g \in F$ and a cochain operator χ such that for all pairs (x,y) of cochains

$$(12) \qquad (\Delta\chi)(x,y) = \psi(x,y) + f(x) + g(y) .$$

If x is a cocycle, and y is zero, (12) shows that $f(x) \sim 0$. This means that f is a relation. Since also g is a relation, we see that $\psi(x,y) \sim 0$ for all pairs (x,y) of cocycles. Applying this and corollary 3.5 to (10), we get

$$(13) \quad \begin{aligned} qu^r(u+w) - qu^r(u) - qu^r(w) &\sim \textstyle\sum_\nu\big(\varkappa(\alpha_\nu)a_\nu(u) + \varkappa(\alpha_\nu)a_\nu(w)\big) - d(b;\, u,w) \\ &\sim d(b;\, u,w) . \end{aligned}$$

Especially, if $w = \delta x$, then we can put $y_\nu = a_\nu(x)$

$$(14) \qquad \delta\big(a_\nu(x)\big) = a_\nu(\delta x) = a_\nu(w) ,$$

and since $b(x) = 0$ (because $\dim x < e(b) - 1$),

$$(15) \qquad qu^r(\delta x) = \theta(\delta x) + \textstyle\sum_\nu \alpha_\nu a_\nu(x) = \theta(\delta x) + r(x) = \delta\theta(x) .$$

Therefore, by (13), corollary 3.6, and lemma 2.6

$$(16) \qquad qu^r(u+\delta x) - qu^r(u) \sim d(b;\, u, \delta x) \sim 0 .$$

This implies that we can define a coboundary cohomology operation

$$Qu^r\colon\ D(n,r,K) \to H^{n+i}(K)/\mathrm{Ind}(n,r,K) \tag{17}$$

associated with the relation $r=\sum_\nu \alpha_\nu a_\nu + b$ of degree $i+1$. The operation is defined in all dimensions n less than $e(b)$ on the subgroup $D(n,r,K)=D(n)$ of $H^n(K)$ consisting of all classes $\bar{u}$ satisfying for all ν

$$\varepsilon(a_\nu)(\bar{u}) = 0\ . \tag{18}$$

The indeterminacy of Qu^r is

$$\mathrm{Ind}(n,r,K) = \textstyle\sum_\nu \varepsilon(\alpha_\nu) H^{n-1+\deg(\alpha_\nu)}(K)\ . \tag{19}$$

Definition 4.1. *Let $\bar{u} \in H^n(K)$, $n < e(b)$, be a class satisfying* (18). *Let u be a cocycle representing $\bar{u}$, and let v_ν be cochains satisfying* (3). *Let θ be a cochain operation satisfying* (2). *Then*

$$Qu^r(\bar{u}) = \{\theta(u) + \textstyle\sum \alpha_\nu(v_\nu)\} \in H^{n+i}(K)/\mathrm{Ind}(n)\ . \tag{20}$$

This definition is independent of the choices of v_ν and u. This is seen from (6) and (16). From (5) follows

Theorem 4.2. *The difference between any two secondary operations associated with r is a primary operation.*

Theorem 4.3. *Let Qu^r be associated with $r=\sum_\nu \alpha_\nu a_\nu + b$. Then, if Qu^r is defined on the n-dimensional classes $\bar{u}$ and $\bar{v}$,*

$$\begin{aligned} Qu^r(\bar{u}+\bar{v}) &= Qu^r(\bar{u}) + Qu^r(\bar{v}) && \textit{if } n < e(b)-1\ ,\\ Qu^r(\bar{u}+\bar{v}) &= Qu^r(\bar{u}) + Qu^r(\bar{v}) + \{d(b;\, u,v)\} && \textit{if } n = e(b)-1\ , \end{aligned}$$

where u and v are arbitrary cocycles representing $\bar{u}$ and $\bar{v}$. In the case $n=e(b)-1$ the deviation from additivity can be computed from corollary 3.6.

This theorem is an immediate consequence of (13) and lemma 2.6.

Theorem 4.4. *The operation Qu^r, $r=\sum_\nu \alpha_\nu a_\nu + b \in R$, is natural. This means that for any* css-*mapping $f\colon K \to L$, the diagram*

$$\begin{array}{ccc} D(n,L) & \xrightarrow{Qu^r} & H^{n+i}(L)/\mathrm{Ind}(n,L) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ D(n,K) & \xrightarrow{Qu^r} & H^{n+i}(K)/\mathrm{Ind}(n,K) \end{array} \tag{21}$$

is commutative.

Proof. That $f^*(D(n,L)) \subseteq D(n,K)$ and $f^*(\mathrm{Ind}(n,L)) \subseteq \mathrm{Ind}(n,K)$ follows from the definitions (18) and (19). The mapping $f^*\colon H^*(L) \to H^*(K)$ therefore induces the vertical mappings in the diagram (21). The commutivity of (21) follows immediately from the naturality of the cochain operations θ, α_ν, a_ν used in the definition of Qu^r.

Let L be a subcomplex of K. Then, because of naturality any cochain operation θ will induce a mapping

$$\theta\colon \quad C(K,L) \to C(K,L)\ ,$$

natural with respect to mappings of pairs. It is not hard to see that everything we have said so far in this section carries over word by word to the relative case. In particular, the operations Qu^r are also defined in the relative cohomology group.

THEOREM 4.5. *Let $\delta^*\colon H^n(L) \to H^{n+1}(K,L)$ denote the coboundary operator from the cohomology sequence, and let $\bar{u} \in D(n,r,L)$. Then, if $n+1 < e(b)$, the diagram*

$$\begin{array}{ccc} D(n,L) & \xrightarrow{\quad Qu^r \quad} & H^{n+i}(L)/\mathrm{Ind}\,(n,L) \\ \downarrow{\scriptstyle \delta^*} & & \downarrow{\scriptstyle \delta^*} \\ D(n+1,(K,L)) & \xrightarrow{Qu^r} & H^{n+i+1}(K,L)/\mathrm{Ind}\big(n+1,(K,L)\big) \end{array} \tag{22}$$

is commutative.

PROOF. Since δ^* commutes with primary operations, it is clear that δ^* induces the vertical mappings in (22). Let $\bar{u} \in D(n,L)$ be represented by the cocycle u. We choose $v \in C^n(K)$ such that $i^\#(v) = u$, where $i\colon L \to K$ is the inclusion. Then, if $j^\#\colon C(K,L) \to C(K)$ denotes the inclusion, there is a cocycle $y \in C^{n+1}(K,L)$ with $j^\#(y) = \delta v$. This cocycle y is a representative of $\delta^*(\bar{u})$. Let v_ν be a cochain in $C(K)$ with $\delta i^*(v_\nu) = a_\nu(u)$ for all ν. Then there are cochains $w_\nu \in C(K,L)$ such that $j^\#(w_\nu) = a_\nu(v) - \delta v_\nu$. It follows that $\delta w_\nu = a_\nu(y)$. Let θ be a cochain operation with $\Delta(\theta) = r$. Then, by definition the cocycles $\theta(y) + \sum_\nu \alpha_\nu(w_\nu)$ and $\theta(u) + \sum_\nu \alpha_\nu\big(i^\#(v_\nu)\big)$ represent $Qu^r(\delta^*\bar{u})$ and $Qu^r(\bar{u})$ respectively. Since

$$\begin{aligned} i^\#\Big(\theta(v) + \textstyle\sum \alpha_\nu(v_\nu) + \sum_\nu d\big(\alpha_\nu; j^\#(w_\nu), \delta(v_\nu)\big)\Big) &= \theta(u) + \textstyle\sum_\nu \alpha_\nu\big(i^\#(v_\nu)\big)\ , \\ \delta\Big(\theta(v) + \textstyle\sum_\nu \alpha_\nu(v_\nu) + \sum_\nu d\big(\alpha_\nu; j^\#(w_\nu), \delta(v_\nu)\big)\Big) &= j^\#\big(\theta(y) + \textstyle\sum_\nu \alpha_\nu(w_\nu)\big)\ , \end{aligned}$$

it follows that $\theta(y) + \sum_\nu \alpha_\nu(w_\nu)$ is also a representative of $\delta^* Qu^r(\bar{u})$. This concludes the proof.

Operations defined in all dimensions and commuting with δ^* are usually called stable operations. Theorem 4.5 therefore states that Qu^r is a stable operation in case $b = 0$. We conclude this section by evaluating certain operations Qu^r in certain dimensions. Specifically we have

THEOREM 4.6. *Let there be given a relation $r=\sum_\nu \alpha_\nu a_\nu + \sum_\mu \beta_\mu b_\mu + b$ in R with $\alpha_\nu a_\nu$ a monomial for all ν and $e(\sum_\mu \beta_\mu b_\mu + b) \geqq n+2$. Further, we assume that $\sum \alpha_\nu a_\nu$ can be written in the form*

$$\sum \alpha_\nu a_\nu = \sum_{s\in S} sq^{I(s)} sq^{j(s)} sq^{J(s)} + \sum_{t\in T} (sq^{2K(t)} sq^{j(t)} sq^{J(t)} + sq^{i(t)} sq^{K(t)\,J(t)}) ,$$

where the sequence $I(s)$ contains at least one odd component, and

$$j(\gamma) = n+1+\deg(J(\gamma)), \quad \gamma \in S \cup T, \quad i(t) = n+1+\deg(K(t)J(t)) .$$

Then there is a secondary operation Qu^r associated with r taking the value

$$Qu^r(\bar{u}) = 0$$

on classes $\bar{u}$ of dimensions less than n, and in dimension n the value

$$Qu^r(\bar{u}) = \left\{ \sum_{\gamma\in S\cup T} \sum_{\substack{A \prec B \\ A+B=I(\gamma)}} Sq^{AJ(\gamma)}(\bar{u})\, Sq^{BJ(\gamma)}(\bar{u}) \right\} ,$$

where $I(\gamma)=2K(\gamma)$ for $\gamma \in T$.

PROOF. To construct Qu^r let us choose the cochain operation θ with $\Delta\theta = r$ given by theorem 3.8. Let u be a cocycle representing $\bar{u}$, $(\dim(\bar{u}) \leqq n)$, and let

$$\delta v_\nu = a_\nu(u), \quad \delta w_\mu = b_\mu(u), \qquad \text{all } \nu \text{ and } \mu ,$$

then $Qu^r(\bar{u})$ is represented by the cocycle

$$\theta(u) + \sum_\nu \alpha_\nu(v_\nu) + \sum_\mu \beta_\mu(w_\mu) . \tag{23}$$

Since $e(\alpha_\nu a_\nu) \geqq n+1$, it follows that either $e(a_\nu) \geqq n+1$ or $e(\alpha_\nu) \geqq n+1+\deg(a_\nu)$. In the first case $a_\nu(u)=0$, and hence v_ν a cocycle. In the second case $\alpha_\nu(v_\nu)=0$. Since the same can be said about $\beta_\mu b_\mu$, we see that $\sum_\nu \alpha_\nu(v_\nu) + \sum_\mu \beta_\mu(w_\mu)$ is a cocycle determining a cohomology class in the indeterminacy subgroup of Qu^r. The only term in (23) of interest therefore is $\theta(u)$. By theorem 3.8, however, $\theta(u)$ is equal to zero if the dimension of $\bar{u}$ is less than n, and equal to

$$\sum_{\gamma\in S\cup T} \sum_{\substack{A \prec B \\ A+B=I(\gamma)}} sq^{AJ(\gamma)}(u)\, sq^{BJ(\gamma)}(u)$$

if $\bar{u}$ is n-dimensional. This completes the proof.

As an example let us consider the relation $r = sq^2 sq^3 + sq^4 sq^1 + sq^1 sq^4$. This relation is of the form considered in theorem 4.6. Accordingly there is an operation Qu^r with the values $Qu^r(\bar{u})=0$ for $\bar{u} \in D(1)$, and

$$Qu^r(\bar{u}) = \left\{ \sum_{\substack{\alpha \prec \beta \\ \alpha+\beta=2}} Sq^\alpha \bar{u}\, Sq^\beta \bar{u} \right\} = \{\bar{u}^3\} \tag{24}$$

for $\bar{u} \in D(2)$. In section 6 we shall see that the operation Qu^r coincides with ψ used by Adams in the paper [1]. The computation (24) was also carried out in that paper.

THEOREM 4.7. *Let $r=\sum_\nu \alpha_\nu a_\nu + b$ and $s=\sum_\nu \alpha_\nu a_\nu + b'$ be elements in R, and let $e(b)$ and $e(b')$ both be larger than an integer m. Then there are operations Qu^r and Qu^s which coincide in all dimensions smaller than m. In dimension m there is a sum of products of primary operations on $\bar{u}$ such that*

$$Qu^r(\bar{u}) - Qu^s(\bar{u}) = \{\textstyle\sum_k sq^{I(1,k)}(\bar{u}) \dots Sq^{I(s(k),k)}(\bar{u})\} .$$

PROOF. It is clear that $\varrho = b-b' \in R$ and that $e(\varrho) > m$. Let us choose a cochain operation ψ with $\Delta\psi = \varrho$, with the value $\psi(u)=0$ on cocycles in dimensions less than m, and the value

$$\psi(u) = \textstyle\sum_k sq^{I(1)}(u) \dots sq^{I(s(k))}(u)$$

on m-cocycles. This can be done by theorem 3.7. Let $\Delta\theta = r$. Then $\Delta(\theta-\psi)=s$. We shall use θ and $\theta-\psi$ to construct Qu^r and Qu^s respectively. That $Qu^r - Qu^s$ is as stated in the theorem is obvious.

By theorem 4.7 the secondary operation associated with the relation $r=\sum_\nu \alpha_\nu a_\nu + b$ does not essentially depend on the choice of b provided we stay inside the class $\varepsilon(b)$. Therefore, in the following it may sometimes be convenient to choose b to be a sum of terms of the form $sq^K(\sum_\nu m_\nu)$, where each m_ν is an admissible monomial, $K=L(j,k)=(2^{j-1}k, 2^{j-2}k, \dots, k)$ for some integers j and k, $sq^K m_\nu$ is admissible, and

$$e\big(sq^K(\textstyle\sum_\nu m_\nu)\big) > e(\textstyle\sum_\nu m_\nu) .$$

It is clear that in the class $\varepsilon(b)$ there is exactly one element of this form. If b is written in this form, we shall say it is admissible.

THEOREM 4.8. *Let $r=\sum_\nu \alpha_\nu a_\nu + b$ and $s=\sum_\nu \alpha_\nu b_\nu + b$ be relations with $a_\nu - b_\nu \in R$ for all ν. Then there are operations Qu^r and Qu^s with $Qu^r = Qu^s$.*

PROOF. Let ψ_ν be cochain operations with $\Delta\psi_\nu = a_\nu - b_\nu$. Let u be a n-cocycle representing a class in $D(n,r)=D(n,s)$. Let

$$\delta v_\nu = a_\nu(u), \qquad \text{all } \nu . \tag{25}$$

Then, for all ν

$$\delta\big(v_\nu - \psi_\nu(u)\big) = b_\nu(u) . \tag{26}$$

Let $\Delta\theta = r$, then an easy computation shows that if we define χ by

$$\chi(x) = \theta(x) + \textstyle\sum_\nu \alpha_\nu \psi_\nu(x) + \sum_\nu d\big(\alpha_\nu;\, (a_\nu - b_\nu)(x),\, b_\nu(x)\big)\,,$$

then $\Delta\chi = s$. Relative to θ, χ, (25), and (26) we have

$$\begin{aligned} qu^r(u) - qu^s(u) &= \theta(u) - \chi(u) + \textstyle\sum \alpha_\nu(v_\nu) - \sum \alpha_\nu(v_\nu - \psi_\nu u)) \\ &= \delta \textstyle\sum d\big(\alpha_\nu;\, \psi_\nu(u), v_\nu - \psi_\nu(u)\big) \sim 0\,. \end{aligned}$$

This completes the proof.

THEOREM 4.9. *Let $r = \sum_\nu \alpha_\nu a_\nu + b$ and $s = \sum_\nu \beta_\nu a_\nu + b$ be relations with $\alpha_\nu - \beta_\nu \in R$ for all ν. Then there are operations Qu^r and Qu^s with $Qu^r = Qu^s$.*

PROOF. Let ψ_ν be cochain operations with $\Delta\psi_\nu = \alpha_\nu - \beta_\nu$, and let $\Delta\theta = r$. Then, putting $\Phi = \theta + \sum_\nu \psi_\nu a_\nu$, we have $\Delta\Phi = s$. Let u be a n-cocycle representing a class in $D(n,r) = D(n,s)$, and let

$$\delta v_\nu = a_\nu(u), \qquad \text{all } \nu\,. \tag{27}$$

Relative to θ, Φ, and (27) we have

$$qu^r(u) - qu^s(u) = \theta(u) - \Phi(u) + \textstyle\sum_\nu \alpha_\nu(v_\nu) - \sum_\nu \beta_\nu(v_\nu) = \delta \sum_\nu \psi_\nu(v_\nu) \sim 0\,.$$

This completes the proof.

Let $r = \sum_\nu \alpha_\nu a_\nu + b$, $1 \leqq \nu \leqq t$, be a relation in the Steenrod algebra with α_ν, a_ν, and $b \in A$, then the above theorems show that we can associate a secondary operation Qu^r defined in dimensions less than $m + 1 \leqq e(b)$. The theorems show how much any two operations associated with r differ from each other. It is convenient to use the algebraic setup introduced by Adams [1]. Let C_1 be a free A-module on t generators c_1^ν, $\nu = 1,2,\ldots,t$. Let C_0 be the A-module, $A(m) = A/E(m+1)$ on one generator c_0 (for definition of $E(m+1)$ see section 2). The dimensions of the generators are given to be $\dim c_0 = n \leqq m$ and $\dim c_1^\nu = n + \deg(a_\nu)$. A mapping

$$d\colon\ C_1 \to C_0 \tag{28}$$

is given by $dc_1^\nu = a_\nu c_0$. Let $z = \sum_\nu \alpha_\nu c_1^\nu$, then

$$d(z) = \textstyle\sum_\nu \alpha_\nu a_\nu c_0 = rc_0 = 0\,.$$

An operation associated with r will also be called associated with the pair (d,z). In this algebraic setup it is easy to see how to generalize secondary operations from one to several variables.

As before, let C_1 be a free A-module on generators c_1^ν, $\nu = 1,2,\ldots,t$. Let

$$C_0 = A(m_1) \oplus A(m_2) \oplus \ldots \oplus A(m_\tau) \tag{29}$$

with generators c_0^j, $j=1,2,\ldots,\tau$. The dimensions of the generators of C_0 are given to be

(30) $$\dim c_0^j = n+\mu(j) \leqq m_j \, .$$

Then C_0 is a left A-module. As in (28) let there be given a mapping $d\colon C_1 \to C_0$ by

(31) $$d(c_1^\nu) = \sum_{j=1}^{\tau} a_j^\nu c_0^j \, ,$$

where the right hand side is supposed to be homogeneous. To make d homogeneous we put

(32) $$\dim c_1^\nu = \deg a_j^\nu + n + \mu(j) \, .$$

Let

$$z = \sum_{\nu=1}^{t} \alpha_\nu c_1^\nu \in C_1$$

be a d-cycle. This means that

(33) $$\textstyle\sum_\nu \alpha_\nu a_j^\nu = 0 \in A(m_j) \qquad \text{for all } j \, .$$

Above the symbol n is supposed to be a parameter taking all values such that $n+\mu(j) \leqq m_j$ is true for all j. Let K be a css-complex, and let $\varepsilon\colon C_0 \to H^*(K)$ be a homogeneous mapping. We shall define a secondary operation associated with (d,z) defined on ε, or what is the same thing, on the set $\big(\varepsilon(c_0^1),\ldots,\varepsilon(c_0^\tau)\big)$ of classes in $H^*(K)$ if and only if $\varepsilon d = 0$. We do this as follows. Let $b_j \in E(m_j+1)$ such that in A

(34) $$\textstyle\sum_\nu \alpha_\nu a_j^\nu + b_j = 0 \, .$$

Here α_ν, a_j^ν, and b_j are all in A, but let us choose representatives of these elements in F. The representative of b_j we choose so that it is of excess larger than m_j. We shall denote these representatives by the same symbols. The relations obtained are denoted

(35) $$r_j = \textstyle\sum_\nu \alpha_\nu a_j^\nu + b_j, \qquad \alpha_\nu, a_j^\nu, b_j \in F, \qquad e(b_j) > m_j \, .$$

Let u_j be a $\big(n+\mu(j)\big)$-cocycle representing $\varepsilon(c_0^j)$, and let

(36) $$\delta v_\nu = \textstyle\sum_j a_j^\nu(u_j) \, .$$

Let $\varDelta\theta_j = r_j$, then the cochain

$$qu^{(d,z)}(u_1,\ldots,u_\tau) = \textstyle\sum_j \theta_j(u_j) + \sum_\nu \alpha_\nu(v_\nu) + \sum_\nu d\big(\alpha_\nu;\, a_1^\nu(u_1),\ldots,a_\tau^\nu(u_\tau)\big)$$

is actually a cocycle and represents our operation

(37) $$Qu^{(d,z)}(\varepsilon) = Qu^{(d,z)}(u_1,\ldots,u_\tau) \in H^{n+i}(K)/\mathrm{Ind}\big(n,(d,z),K\big) \, ,$$

where $i = \deg r_j + \mu(j) - 1$, and

$$\operatorname{Ind}(n,(d,z),K) = \textstyle\sum_\nu \alpha_\nu H^{n+i-\deg \alpha_\nu}(K) . \tag{38}$$

The theorems in this section and their proofs are easily seen to generalize to the case of several variables.

5. Properties of secondary operations.

The properties of secondary operations treated in this section are the very elementary ones concerning manipulation of the relations associated with the operations. Properties of more advanced character, e.g. the behaviour of secondary operations in spectral sequences associated with fibrations, are postponed to a subsequent paper (mentioned in the introduction). The theorems in this section are stated in a very simple form. They might, however, just as well be put into the form given to these theorems by Adams in [1]. To indicate the method of proof, we shall here give a proof of one of the theorems.

THEOREM 5.1. *In any of the following three cases*

(i) $\quad r = (\alpha\beta)a + \sum_\nu \alpha_\nu a_\nu + b, \qquad s = (\alpha)(\beta a) + \sum \alpha_\nu a_\nu + b ,$

(ii) $\quad r = \alpha(a+a') + \sum_\nu \alpha_\nu a_\nu + b, \qquad s = \alpha a + \alpha a' + \sum \alpha_\nu a_\nu + b ,$

(iii) $\quad r = (\alpha+\alpha')a + \sum_\nu \alpha_\nu a_\nu + b, \qquad s = \alpha a + \alpha' a + \sum \alpha_\nu a_\nu + b ,$

there are operations Qu^r, Qu^s, such that for classes $\bar{u}$ on which both operations are defined

$$Qu^r(\bar{u}) = Qu^s(\bar{u})$$

modulo the total indeterminancy involved.

THEOREM 5.2. *Let $r = \sum_\nu \alpha_\nu a_\nu + b$, $s = \sum_\nu \alpha_\nu (a_\nu c) + bc$ be relations, and let Qu^r be associated with r. Then there exists an operation Qu^s, such that for any $\bar{u}$ on which Qu^s is defined*

$$Qu^r(c\bar{u}) = Qu^s(\bar{u})$$

modulo the total indeterminacy of Qu^s.

THEOREM 5.3. *Let $r_t = \sum_\nu \alpha_\nu{}^t a_\nu{}^t + b^t$ be a finite set of relations, and let*

$$r = \textstyle\sum_{t,\nu} (c^t \alpha_\nu{}^t) a_\nu{}^t + \sum_t (c^t b^t) .$$

Let Qu^{r_t} be operations associated with r_t. Then there is an operation Qu^r such that for any $\bar{u}$ on which Qu^{r_t} is defined for all t

$$\textstyle\sum_t c^t Qu^{r_t}(\bar{u}) = Qu^r(\bar{u})$$

modulo the total indeterminacy involved.

PROOF OF THEOREM 5.3. Let θ_t be a cochain operation associated with r_t. Then

$$\theta = \sum_t\bigl(c^t\theta_t + d(c^t;\ \alpha_1{}^t a_1{}^t, \alpha_2{}^t a_2{}^t, \ldots, b^t) + d(c^t;\ \theta_t\delta, \delta\theta_t)\bigr)$$

is associated with r. Let $u \in C^n$ be a cocycle representing $\bar{u}$, and let $\delta e_\nu{}^t = a_\nu{}^t(u)$ for all ν and t. Then

$$\begin{aligned}\sum_t c^t q u^{r_t}(u) - q u^r(u) \\ &= \sum_t c^t\bigl(\theta_t(u) + \sum_\nu \alpha_\nu{}^t(e_\nu{}^t)\bigr) - \bigl(\sum_t c^t\theta_t(u) + \sum_t d(c^t;\ \alpha_1{}^t a_1{}^t(u), \ldots, b^t(u)) + \\ &\qquad + \sum_{t,\nu} c^t \alpha_\nu{}^t(e_\nu{}^t)\bigr) .\end{aligned}$$

Since $b^t(u) = 0$, and

$$c^t\theta_t(u) \sim c^t\bigl(\theta_t(u) + \sum_\nu \alpha_\nu{}^t(e_\nu{}^t)\bigr) + c^t\bigl(\sum_\nu' \alpha_\nu{}^t(e_\nu{}^t)\bigr) ,$$

we get

$$\begin{aligned}\sum c^t q u^{r_t}(u) - q u^r(u) \\ &\sim \sum c^t\bigl(\sum_\nu \alpha_\nu{}^t(e_\nu{}^t)\bigr) - \sum d\bigl(c^t;\ \alpha_1{}^t a_1{}^t(u), \ldots, b^t(u)\bigr) - \sum_{t,\nu} c^t\alpha_\nu{}^t(e_\nu{}^t)\bigr) \sim 0 .\end{aligned}$$

This completes the proof.

6. Comparison with operations of Adams.

In [1] Adams considered operations Φ associated with relations $r = \sum \alpha_\nu a_\nu$ with no unfactorized term. These operations are characterized by the following axioms:

AXIOM 1. For any $\bar{u} \in H^n(X)$, n arbitrary, $\Phi(\bar{u})$ is defined if and only if

$$a_\nu(\bar{u}) = 0 \qquad \text{for all } \nu .$$

AXIOM 2. If $\Phi(\bar{u})$ is defined, then

$$\Phi(\bar{u}) \in H^{n+i}(X)/\mathrm{Ind}\,(r,n) ,$$

where $i = \deg(r) - 1$, and $\mathrm{Ind}\,(r,n) = \sum_\nu \alpha_\nu H^{n-1+\deg(a_\nu)}(X)$.

AXIOM 3. The operation Φ is natural.

AXIOM 4. Let (X,A) be a pair of topological spaces, and let $\bar{v} \in H^n(X,A)$ be a class such that Φ is defined on $j^*(\bar{v}) \in H^n(X)$. Let $\bar{w}_\nu \in H^*(A)$ be classes such that

$$\delta^*(\bar{w}_\nu) = a_\nu(\bar{v}) \qquad \text{for all } \nu .$$

Then, in $H^{n+i}(A)/i^*(\mathrm{Ind}\,(r,n))$ we have

$$i^*\Phi(j^*(\bar{v})) = \{\sum_\nu \alpha_\nu(\bar{w}_\nu)\} .$$

AXIOM 5. The operation Φ commutes with suspension σ: $\tilde{H}^j(X) \to \tilde{H}^{j+1}(SX)$.

THEOREM 6.1. *Any operation Qu^r associated with the relation $r=\sum_\nu \alpha_\nu a_\nu$ (with no unfactorized term) satisfies the axioms* 1–5 *of Adams.*

PROOF. Axioms 1–3 are obvious. To prove axiom 4, let $v \in C(X,A)$ represent $\bar{v}$ and choose cochains e_ν in $C(X)$ such that $\{i^\# e_\nu\}=\overline{w}_\nu$ and $\delta e_\nu = a_\nu(v)$. Let θ be a cochain operation with $\Delta\theta = r$. Then $i^* Qu^r(j^*(\bar{v}))$ is represented by

$$\begin{aligned} i^\#\big(\theta(v)+\textstyle\sum \alpha_\nu(e_\nu)\big) &= \theta\big(i^\#(v)\big)+\textstyle\sum \alpha_\nu(i^\# e_\nu) \\ &= \textstyle\sum \alpha_\nu(i^\# e_\nu)\ . \end{aligned}$$

This cochain also represents $\sum_\nu \alpha_\nu(\overline{w}_\nu)$. Axiom 5 follows immediately from commutation with the coboundary operator from the cohomology sequence (theorem 4.5). This completes the proof.

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