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ON THE COHOMOLOGY OF SPACES WITH TWO NON-VANISHING HOMOTOPY GROUPS

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1. Introduction.

The cohomology of Eilenberg–MacLane spaces $K(\pi, n)$ has been studied in great detail by Eilenberg–MacLane [7], Serre [14], and Cartan [2, 3]. A natural generalization of the problem of determining the cohomology of spaces of type $K(\pi, n)$ is the determination of the cohomology of spaces with two non vanishing homotopy groups. This problem has been considered by various authors, Cockroft [4, 5, 6], Hirsch [8, 9], and Kristensen [10, 11]. The computation carried out in the present paper includes a determination of the mod 2 cohomology of such spaces provided the homotopy groups are finite dimensional vectorspaces over Z_2 and the k-invariants are stable. These results were announced in Kristensen [11].

In section 2 we shortly review the concept of secondary cohomology operation, Adams [1], Kristensen [12], and in section 3 we study the behaviour of these operations in spectral sequences. Let $F \to E \to B$ be a fibration, and let $\bar{u} \in H^*(F)$ be a transgressive class. The problem considered in section 3 is to determine $d_r\{Qu(\bar{u})\}$, where Qu is a secondary operation defined on $\bar{u}$. The main results in this section are stated in theorems 3.2 and 3.3. Theorem 4.5 gives a clearer statement of theorem 3.2.

Section 4 does not have any interest in itself, but only in connection with the computations carried out in section 5. We shall call a finite product of Eilenberg–MacLane spaces with finitely generated homotopy groups a generalized Eilenberg–MacLane space. Let X be a generalized Eilenberg–MacLane space, and let us consider the fibration $\Omega X \to LX \to X$, where LX denotes the space of paths in X based at a point b. Let Y be a generalized Eilenberg–MacLane space; then a mapping $f\colon Y \to X$ induces a fibration $\Omega X \to E \to Y$ over Y. Any space E which can be obtained as total space in such a fibration we call a two-stage space. In section 5 we are concerned with the determination of the cohomology of certain two-stage spaces. To simplify the problem (and also to make

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the exposition clearer), we restrict ourselves to considering only two-stage spaces induced from mappings between generalized Eilenberg–MacLane spaces of a special type, namely, products of $K(Z_2,n)$'s for various integers n. The main result is stated in theorems 5.1 and 5.2. I see no great difficulty connected with generalizing this result to a theorem about arbitrary two-stage spaces replacing the condition about stable k-invariants by a condition on the mapping $f\colon Y \to X$ determining the two-stage space. It would also be desirable to obtain a description of the mod p cohomology of two-stage spaces in terms of secondary cohomology operations. Since the loop-space of any two-stage space has "stable k-invariants", it might also be possible to get some information in the general case. In his paper [13], C. R. F. Maunder has given an axiomatic description of cohomology operations of arbitrary high orders. A construction of these operations by means of cochain operations as it was done in Kristensen [12] for secondary operations could possibly lead to an evaluation of these operations in small dimensions. This problem is within reach for tertiary operations, but seems somewhat more involved in the general case. Also, such a description could possibly lead to a determination of the cohomology of spaces of the third or even higher stage.

I wish to thank Saunders MacLane for some very inspiring discussions on the subject.

2. Secondary operations.

In this section we shall review the definition of secondary cohomology operations as given in Adams [1] and Kristensen [12]. In what follows we will be working in the category of css-complexes. Coefficient groups will always be Z_2 (i.e. a field).

Let A denote the (mod 2) Steenrod algebra. The subspace of A consisting of all elements of excess larger than or equal to n is denoted by $E(n)$. The subspaces $E(*)$ define a decreasing filtration of A. We also put

$$A(m) = A/E(m+1)\,. \tag{2.1}$$

Let C_1 be a free A-module on generators c_1^ν, $\nu=1,2,\ldots,t$, of certain specified degrees. Let

$$C_0 = A(m_1)\oplus\ldots\oplus A(m_\tau) \tag{2.2}$$

with generators c_0^j, $j=1,2,\ldots,\tau$, and $m_j=\deg c_0^j$. Then C_0 is a left A-module. Let $n=\min_j\{\deg(c_0^j)\}$, and put

$$\deg(c_0^j) = n+\mu(j) = m_j\,. \tag{2.3}$$

Let a homogeneous mapping

$$d: \quad C_1 \to C_0 \tag{2.4}$$

be given by

$$d(c_1^{\nu}) = \sum_{j=1}^{\tau} a_j^{\nu} c_0^{j}, \qquad a_j^{\nu} \in A\big(n+\mu(j)\big) \,. \tag{2.5}$$

Then, of course $\deg(c_1^{\nu}) = \deg(a_j^{\nu}) + n + \mu(j)$. All degrees depend on n. We shall allow n to vary among all positive integers. The mapping (2.4) we shall also denote $d(n)\colon C_1(n) \to C_0(n)$. We have an isomorphism $\sigma\colon C_1(n+1) \to C_1(n)$ defined by $\sigma\big(c_1^{\nu}(n+1)\big) = c_1^{\nu}(n)$. This is a mapping of degree -1. Let

$$z = \sum_{\nu=1}^{t} \alpha_\nu c_1^{\nu} \in C_1(n) \tag{2.6}$$

be a d-cycle. This means that

$$\textstyle\sum_\nu \alpha_\nu a_j^{\nu} = 0 \in A\big(n+\mu(j)\big) \qquad \text{for all } j \,. \tag{2.7}$$

Sometimes we shall denote the image $\sigma^s(z) \in C_1(n-s)$, $s = \pm 1, \pm 2, \ldots$, by z.

Associated with the pair (d,z) there are some secondary cohomology operations (definition below). Let $Qu = Qu^{(d,z)}$ be associated with (d,z). Then for any css-complex K, Qu is defined on all homogeneous A-module mappings

$$\varepsilon\colon \quad C_0(n) \to H^*(K) \,, \tag{2.8}$$

or what is the same thing, on τ-tuples $\big(\varepsilon(c_0^{1}), \ldots, \varepsilon(c_0^{\tau})\big)$, with $\varepsilon d = 0$ provided $z \in \ker\big(d(n)\big)$. Furthermore,

$$Qu(\varepsilon) \in H^{n+i}(K)/\mathrm{Ind}\big(n,(d,z)\big) \,, \tag{2.9}$$

where $i = \deg(\alpha_\nu a_j^{\nu}) + \mu(j) - 1 = \deg(z) - 1 - n$, and where the indeterminacy subgroup is given by

$$\mathrm{Ind}\big(n,(d,z)\big) = \textstyle\sum_\nu \alpha_\nu H^{n+i-\deg(\alpha_\nu)}(K) \,. \tag{2.10}$$

Let m, $1 \leqq m \leqq \infty$, be such that $z \in \ker\big(d(n)\big)$ for all $n < m$. Then the operation Qu is additive for $n < m-1$. This is shown in [12] for $\tau = 1$, and the deviation from additivity for $n = m-1$ is also given there. The proof, however, immediately generalizes to several variables. In [12] it is also shown that the difference between any two operations associated with (d,z) for $n < m-1$ is a stable primary operation, and for $n = m-1$ a sum of products of stable primary operations. Let $\{z_r\}$ be a finite set of homogeneous elements in the kernel of d (see (2.4)) and let $z = \sum c_r z_r$, $c_r \in A$, be homogeneous. Let Qu^r be operations associated with z_r. Then there is an operation Qu associated with z, such that modulo the total indeterminacy

$$Qu(\varepsilon) = \sum c_r Qu^r(\varepsilon) \tag{2.11}$$

for each ε on which the operations are defined.

Finally we recall the definition of secondary operations given in [12]. Let $\pi = \{1, T\}$ be the symmetric group on two letters, and let W be the standard π-free resolution of Z_2. In each dimension $i \geqq 0$ W has two (Z_2-) generators e_i and Te_i,

$$\begin{aligned} \partial(e_i) &= \partial(Te_i) = (1+T)e_{i-1}, \\ \varepsilon(e_0) &= \varepsilon(Te_0) = 1. \end{aligned} \tag{2.12}$$

Let f: $K \to L$ be a css-map, and let $C^*(K,f)$ denote the (non-normalized) chain-complex of K with coefficients in Z_2 and increasingly filtered by chain-complexes of inverse images of skeletons in L. It is well-known that there exists a natural π-equivariant chain-transformation

$$\varphi'\colon\quad W \otimes C_* \to C_*^{(2)} \tag{2.13}$$

preserving filtration and augmentation. The action in $C_*^{(2)}$ is by permutation, in C_* it is trivial, and in $W \otimes C_*$ it is diagonal. The filtration in W is by dimension. The filtration of $W \otimes C_*$ and $C_*^{(2)}$ is the usual filtration of tensorproducts of filtered modules. Explicitly the filtration of $W \otimes C_*$ is given by $F_p(W \otimes C_*) = \sum W_i \otimes F_j(C_*)$, $i+j=p$.

The transformation (2.13) gives rise to a dual transformation

$$\varphi\colon\quad W \otimes_\pi C^{(2)} \to C, \tag{2.14}$$

where C denotes the normalized cochain functor of css-complexes. In Kristensen [10] we studied the properties that φ inherits because φ' is filtration preserving. At a single point in this paper we shall make use of these properties. The transformation φ in (2.14) we shall assume to be fixed in what follows.

Let sq^i be the cochain operation (see [12]) defined on the n-dimensional cochain x by

$$sq^i(x) = \varphi(e_{n-i} \otimes x^2 + e_{n-i+1} \otimes x\delta x).$$

Let F be the free associative algebra generated by sq^i, $i > 0$, and let R be the ideal generated by the Adem relations. Then we have the exact sequence

$$0 \to R \to F \to A \to 0. \tag{2.15}$$

Since $z \in \ker d(n)$, $n < m$, there are elements $b_j \in E(m+\mu(j))$, such that in A

$$\textstyle\sum_\nu \alpha_\nu a_j^\nu + b_j = 0 \qquad \text{for all } j. \tag{2.16}$$

Let $\hat{\alpha}_\nu$, $\hat{a}_j^\nu$, and $\hat{b}_j$ be cochain operations in F mapping onto α_ν, a_j^ν, and b_j respectively, and with the excess of $\hat{b}_j$ larger than or equal to $m+\mu(j)$. By theorem 3.3 of [12] there are cochain operations θ_j with

$$(2.17) \qquad \Delta\theta_j = \textstyle\sum_\nu \hat{\alpha}_\nu \hat{a}_j^\nu + \hat{b}_j \qquad \text{for all } j\ .$$

Let $\varepsilon\colon C_0 \to H^*(K)$ be as in (2.8), and let u_j be a $(n+\mu(j))$-cocycle representing $\varepsilon(c_0{}^j)$. Then there are cochains w_ν with

$$(2.18) \qquad \delta w_\nu = \textstyle\sum_j \hat{a}_j^\nu(u_j) \qquad \text{for all } \nu\ .$$

A secondary cohomology operation associated with the pair (d,z) is then represented by the cocycle

$$(2.19) \qquad \textstyle\sum_j \theta_j(u_j) + \sum_\nu \hat{\alpha}_\nu(w_\nu) + \sum_\nu d\big(\hat{\alpha}_\nu;\, \hat{a}_1^\nu(u_1),\dots,\hat{a}_\tau^\nu(u_\tau)\big)\ ,$$

where the cochain operation d is defined in lemma 2.3 of [12].

3. Secondary operations and transgression.

It is a well-known fact that primary operations commute with transgression. In this section we shall examine to what extent this theorem can be generalized to secondary operations.

Let $f\colon E \to B$ be a css-mapping with fibre F_0 over a base point b. Then $C_*(E)$ is filtered by inverse images of skeletons in B. Therefore, associated with f there is a cohomology spectral sequence. The spectral sequences we shall consider below are all assumed, in the usual way, to have the property

$$(3.1) \qquad E(f)_2{}^{**} \cong H^*(B)\otimes H^*(F_0)\ .$$

LEMMA 3.1. *Let $x_j \in E_2{}^{0,\,n+\mu(j)}$ and $y_j \in E_2{}^{n+\mu(j)+1,\,0}$ in the spectral sequence $\{E_r, d_r\}$ associated with a* css*-mapping $f\colon E\to B$. Let x_j be transgressive, and let $d_{n+\mu(j)+1}\{x_j\} = \{y_j\}$. Let u_j be a cochain in E representing x_j such that $\delta u_j = v_j \in f^\#(C(B))$, where v_j is a representative for y_j. Let $k_j \in F$ be of degree $i-\mu(j)$. If $H^{n+i}(E)=0$, and if $\sum_j k_j(y_j)=0$ (considering k_j as a cohomology operation) in $H^{n+i}(B)$, then there exist cochains a and b with $b \in f^\#(C(B))$ such that*

$$\delta a = \textstyle\sum_j k_j(u_j) + b\ .$$

PROOF. Let b be a cochain in $f^\#(C(B))$ with $\delta b = -\sum_j k_j(v_j)$. Since $\delta(\sum_j k_j(u_j)+b)=0$, and since in E any $(n+i)$-dimensional cocycle is a coboundary, there is a cochain a with $\delta a = \sum_j k_j(u_j)+b$.

We use the setup and notation from section 2. In the choices made in the construction of secondary operations we shall, however, avoid one ambiguity, namely, in the choice of $\hat{b}_j$ in (2.17). Since $b_j \in E(m+\mu(j))$, we can choose (provided m is finite)

$$\begin{aligned} \hat{b}_j &= sq^{L(l(j),h(j))}(\hat{\beta}_j) + \hat{\zeta}_j , \\ h(j) &= m+\mu(j)+\deg(\hat{\beta}_j) , \\ L(l(j),h(j)) &= (2^{l(j)-1}h(j),\ldots,2h(j),h(j)) , \end{aligned} \tag{3.2}$$

where $\hat{\beta}_j$ is a sum of admissible monomials (possibly zero) of excess less than or equal to $m+\mu(j)$, and where the excess of ζ_j is larger than $m+\mu(j)$. In the following we shall only consider operations with this specific choice of cochain operations $\hat{b}_j$. It is then clear (cf. theorems 4.8 and 4.9 of [12]) that the difference between two operations associated with (d,z) coincides with a primary operation in the entire domain of definition. The statement of the following theorem is somewhat complicated. A clearer statement of the same theorem is given in theorem 4.5.

THEOREM 3.2. *Let $\{E_r,d_r\}$ be the spectral sequence associated with a* css-*map $f\colon E \to B$ satisfying* (3.1), *and let (d,z) be a pair as defined in section* 2. *Let*

$$\bar{u}_j \in E_2^{0,n+\mu(j)} \quad \textit{and} \quad \bar{v}_j \in E_2^{n+\mu(j)+1,0}, \qquad j = 1,\ldots,\tau .$$

Let u_j and v_j be cochains representing $\bar{u}_j$ and $\bar{v}_j$ respectively such that

$$\delta u_j = v_j \in f^{\#}(C(B)) .$$

Further, let w_ν and w_ν' be cochains such that

$$\delta w_\nu = \textstyle\sum_j \hat{a}_j^\nu(u_j) + w_\nu', \qquad \nu = 1,\ldots,t , \tag{3.3}$$

where $w_\nu' \in f^{\#}(C(B))$. Let us consider the class $qu(\bar{u}_1,\ldots,\bar{u}_\tau) \in E_2^{0,n+i}$ of the cochain

$$\begin{aligned} c = \textstyle\sum_j \theta_j(u_j) + \sum_\nu \hat{\alpha}_\nu(w_\nu) + \sum_\nu d(\hat{\alpha}_\nu;\, \hat{a}_1^\nu(u_1),\ldots,\hat{a}_\tau^\nu(u_\tau)) + \\ + \textstyle\sum_\nu d(\hat{\alpha}_\nu;\, w_\nu',\delta w_\nu) . \end{aligned} \tag{3.4}$$

This class clearly maps into a representative of $Qu(\bar{u}_1,\ldots,\bar{u}_\tau)$ in $H^{n+i}(F_0)$.

If $n<m-1$, then $qu(\bar{u}_1,\ldots,\bar{u}_\tau)$ is transgressive, and

$$d_{n+i+1}\{qu(\bar{u}_1,\ldots,\bar{u}_\tau)\} = \{qu(\bar{v}_1,\ldots,\bar{v}_\tau)\} , \tag{3.5}$$

where

$$\begin{aligned} qu(\bar{v}_1,\ldots,\bar{v}_\tau) = \{\textstyle\sum_j \theta_j(v_j) + \sum_\nu \hat{\alpha}_\nu(w_\nu') + \sum_\nu d(\hat{\alpha}_\nu;\, \hat{a}_1(v_1),\ldots,\hat{a}_\tau(v_\tau))\} \\ \in E_2^{n+i+1,0} . \end{aligned} \tag{3.6}$$

If $n=m-1$, $m<\infty$, let p be the least $l(j)$ (see (3.2)) *with $\beta_j \neq 0$, and let*

$$\hat{\beta}_j' = sq^{L(l(j)-p,h(j))}(\hat{\beta}_j) \tag{3.7}$$

such that

$$\hat{b}_j = sq^{L(p,\varkappa)}(\hat{\beta}_j') + \hat{\zeta}_j \tag{3.8}$$

with $\varkappa = 2^{-p}(n+i+2)$. *Then* $qu(\bar{u}_1,\ldots,\bar{u}_\tau)$ *persists till* E_s, $s=(1-2^{-p})(n+i+2)$, *and*

$$d_s\{qu(\bar{u}_1,\ldots,\bar{u}_\tau)\} = \{\sum_j \beta_j'(\bar{u}_j)\, \beta_j'(\bar{v}_j)^{2^p-1}\}\,, \tag{3.9}$$

where β_j' *denotes the cohomology operation corresponding to* $\hat{\beta}_j'$. *For* $p=1$ *we therefore have*

$$d_s\{\gamma_1\} = \{(\sum_j \beta_j'(\bar{u}_j))(\sum_j \beta_j'(\bar{v}_j))\}\,, \tag{3.10}$$

where

$$\gamma_1 = qu(\bar{u}_1,\ldots,\bar{u}_\tau) + \sum_{j<k} \beta_j'(\bar{u}_j)\cdot \beta_k'(\bar{u}_k)\,,$$

and for $p>1$

$$d_s\{\gamma_p\} = \{(\sum_j \beta_j'(\bar{u}_j))(\sum_j \beta_j'(\bar{v}_j))^{2^p-1}\} \tag{3.11}$$

with

$$\gamma_p = qu(\bar{u}_1,\ldots,\bar{u}_\tau)\,.$$

Furthermore, in the case $n=m-1$ *we have for* $p=1$ *that*

$$\xi_1 = (\gamma_1 + (\sum_j \beta_j'(\bar{u}_j))^2)(\sum_j \beta_j'(\bar{u}_j))(\sum_j \beta_j'(\bar{v}_j)) \tag{3.12}$$

is transgressive with

$$d_r\{\xi_1\} = \{qu^{z'}(\bar{v}_1,\ldots,\bar{v}_\tau)\}, \qquad r = \tfrac{3}{2}(n+i+2)-2\,, \tag{3.13}$$

and for $p>1$ *that*

$$\xi_p = \gamma_p(\sum_j \beta_j'(\bar{u}_j))(\sum_j \beta_j'(\bar{v}_j))^{2^p-1} \tag{3.14}$$

is transgressive with

$$d_r\{\xi_p\} = \{qu^{z'}(\bar{v}_1,\ldots,\bar{v}_\tau)\}, \qquad r = (1+2^{-p})(n+i+2)-2, \tag{3.15}$$

where $z' = \sum_\nu (Sq^{n+i+1}\alpha_\nu)c_1^{\,\nu}$, *and the cohomology class* $qu^{z'}(\bar{v}_1,\ldots,\bar{v}_\tau)$ *represents a secondary operation associated with* $(d(m),z')$.

Proof. Let us consider the cochain c in (3.4). An easy computation gives

$$\delta c = \sum_j \theta_j(v_j) + \sum_\nu \hat{\alpha}_\nu(w_\nu') + \sum_\nu d(\hat{\alpha}_\nu;\, \hat{a}_1^{\,\nu}(v_1),\ldots,\hat{a}_\tau^{\,\nu}(v_\tau)) + \sum_j \hat{b}_j(u_j)\,. \tag{3.16}$$

Then, in case $n<m-1$,

$$n+\mu(j) = \deg(u_j) < e(b_j)-1, \qquad \text{for all } j\,.$$

Hence $\sum_j \hat{b}_j(u_j)=0$. Since by (3.3)

$$\delta w_\nu' = \sum_j \hat{\alpha}_j^{\,\nu}(v_j)\,, \tag{3.17}$$

the formula (3.16) implies (3.5).

In case $n=m-1$, then

$$\textstyle\sum_j \hat{b}_j(u_j) = \sum_j \hat{b}_j(u_j) = \sum_j sq^{L(p,\varkappa)}(\hat{\beta}_j'(u_j)) = \sum_j \hat{\beta}_j'(u_j)\cdot\hat{\beta}_j'(v_j)^{2^p-1},$$

where $\varkappa=2^{-p}(n+i+2)$. Since in the expression (3.16) for δc all other terms are in $f^{\#}(C(B))$ (because $v_j, w_\nu' \in f^{\#}(C(B))$) this proves (3.9). The equation (3.10) follows easily from (3.9) since

$$\begin{aligned} d_s(\{\sum_{j<k} \beta_j'(\bar{u}_j)\beta_k'(\bar{u}_k)\}) &= \{\sum_{j\neq k} \beta_j'(\bar{u}_j)\beta_k'(\bar{v}_k)\} \\ &= \{(\textstyle\sum_j \beta_j'(\bar{u}_j))(\sum_j \beta_j'(\bar{v}_j))\} - \{\sum_j \beta_j'(\bar{u}_j)\beta_j'(\bar{v}_j)\}. \end{aligned}$$

For $p>1$ we must make sure that in E_s the class of

$$(\textstyle\sum_j \beta_j'(\bar{u}_j))(\sum_j \beta_j'(\bar{v}_j))^{2^p-1} - \sum_j \beta_j'(\bar{u}_j)\beta_j'(\bar{v}_j)^{2^p-1}$$

determines the zero element. An easy computation, however, gives this assertion. We prefer not to carry out this computation here.

To prove (3.13) and (3.15), let us consider the relations $sq^{n+i+1}sq^{L(p,\varkappa)}\hat{\beta}_j'$, $\varkappa=2^{-p}(n+i+2)$. Let ψ be a cochain operation such that

$$(3.18) \qquad \Delta\psi = sq^{n+i+1}sq^{\frac{1}{2}(n+i+2)}$$

with the properties stated in theorem 3.9 of Kristensen [12]. Then the cochain operation $\psi_j=\psi sq^{L(p-1,\varkappa)}\hat{\beta}_j'$ has the property

$$(3.19) \qquad \Delta\psi_j = sq^{n+i+1}sq^{L(p,\varkappa)}\hat{\beta}_j' \qquad \text{for all } j.$$

By (3.16) we have

$$\begin{aligned} (3.20) \qquad \delta(sq^{n+i+1}(c)) &= sq^{n+i+1}(y+\textstyle\sum_j \hat{b}_j(u_j)) \\ &= sq^{n+i+1}(y)+\textstyle\sum_j sq^{n+i+1}\hat{b}_j(u_j)+\delta d(sq^{n+i+1}; y,\delta c)+ \\ &\quad +\delta d(sq^{n+i+1}; \hat{b}_1(u_1),\ldots,\hat{b}_\tau(u_\tau)) \\ &\quad +d(sq^{n+i+1}; \hat{b}_1(v_1),\ldots,\hat{b}_\tau(v_\tau)), \end{aligned}$$

where

$$y = \textstyle\sum_j \theta_j(v_j)+\sum_\nu \hat{\alpha}_\nu(w')+\sum_\nu d(\hat{\alpha}_\nu; \hat{a}_1^\nu(v_1),\ldots,\hat{a}_\tau^\nu(v_\tau)).$$

Also

$$\delta \textstyle\sum \psi_j(u_j) = \sum \psi_j(v_j)+\sum sq^{n+i+1}\hat{b}_j(u_j).$$

Together with (3.20) this yields

$$(3.21) \qquad \delta z = \textstyle\sum_j \psi_j(v_j)+sq^{n+i+1}(y)+d(sq^{n+i+1}; \hat{b}_1(v_1),\ldots,\hat{b}_\tau(v_\tau)),$$

where

$$z = sq^{n+i+1}(c)+\textstyle\sum_j \psi_j(u_j)+d(sq^{n+i+1}; y,\delta c)+d(sq^{n+i+1}; \hat{b}_1(u_1),\ldots,\hat{b}_\tau(u_\tau)).$$

Now we shall draw some conclusions from (3.21). From the definition of the cochain operation d (lemma 2.3 of [12]) we notice that the filtration of the two last terms in z is larger than $s=(1-2^{-p})(n+i+2)$ which is the filtration of δc, i.e. $\delta c \in F_s$. By theorem 3.9 of [12] and the methods for computing filtrations given in [10] we see that the filtration of $\psi_j(u_j)$ is larger than s except for $p=1$, where $\psi_j(u_j)-\hat{\beta}_j'(u_j)^3\hat{\beta}_j'(v_j)$ is of filtration larger than s. Since $\delta z \in f^\#(C(B))$ and $sq^{n+i+1}(c)=c\,\delta c$, we conclude that the class of

$$(3.22)\qquad \begin{array}{ll} qu(\bar{u}_1,\ldots,\bar{u}_\tau)\big(\sum_j\beta_j'(\bar{u}_j)\beta_j'(\bar{v}_j)\big)-\sum_j\beta_j'(\bar{u}_j)^3\beta_j'(\bar{v}_j), & p=1\,, \\ qu(\bar{u}_1,\ldots,\bar{u}_\tau)\big(\sum_j\beta_j'(\bar{u}_j)\beta_j'(\bar{v}_j)^{2^p-1}\big)\,, & p>1\,, \end{array}$$

is transgressive, and that it hits the class of δz in $E_r^{2(n+i+1),0}$. An elementary argument shows that the class in (3.22) determines the same class as ζ_p in E_r. Therefore we have that ζ_p is transgressive with $d_r\{\zeta_p\}$ equal to the class af δz. All that is left then is to compare δz with a representative cocycle of $Qu^{z'}(\bar{v}_1,\ldots,\bar{v}_\tau)$.

Since the relations we are dealing with are $r_j'=\sum_\nu(sq^\lambda\hat{\alpha}_\nu)\hat{a}_j^\nu+sq^\lambda\zeta_j$, where $\lambda=n+i+1$, we must find cochain operations χ_j with $\Delta\chi_j=r_j'$. A convenient operation with this property is

$$(3.23)\qquad \chi_j = sq^\lambda\theta_j+d(sq^\lambda;\, \hat{\alpha}_1\hat{a}_j^1,\ldots,\hat{\alpha}_t\hat{a}_j^t,\hat{b}_j)+\psi_j\,,$$

where ψ_j is given in (3.19).

A cocycle representative of $qu^{z'}(\bar{v}_1,\ldots,\bar{v}_\tau)$ is

$$(3.24)\qquad \textstyle\sum_j\chi_j(v_j)+\sum_\nu sq^\lambda\hat{\alpha}_\nu(w_\nu')+\sum_\nu d\big(sq^\lambda\hat{\alpha}_\nu;\, \hat{a}_1^\nu(v_1),\ldots,\hat{a}_\tau^\nu(v_\tau)\big)\,.$$

Since

$$\begin{aligned}\textstyle\sum_\nu sq^\lambda\hat{\alpha}_\nu(w_\nu')+sq^\lambda(y) &\sim sq^\lambda\big(\textstyle\sum_\nu\hat{\alpha}_\nu(w_\nu')+y\big)+\\ &\quad +d\big(sq^\lambda;\, \hat{\alpha}_1(\textstyle\sum_j\hat{a}_j^1(v_j)),\ldots,\hat{\alpha}_\tau(\sum_j\hat{a}_j^t(v_j)),\, \sum_j\hat{b}_j(v_j)\big)\\ &\sim \textstyle\sum_j sq^\lambda\theta_j(v_j)+\sum_\nu sq^\lambda d\big(\hat{\alpha}_\nu;\, \hat{a}_1^\nu(v_1),\ldots,\hat{a}_\tau^\nu(v_\tau)\big)+\\ &\quad +d(sq^\lambda;\, r_1(v_1),\ldots,r_\tau(v_\tau),\eta_1,\ldots,\eta_\tau)+\\ &\quad +d\big(sq^\lambda;\, \hat{\alpha}_1(\textstyle\sum_j\hat{a}_j^1(v_j)),\ldots,\alpha_t(\sum_j\hat{a}_j^t(v_j)),\, \sum_j\hat{b}_j(v_j)\big)\,,\end{aligned}$$

where

$$\eta_\nu = \hat{\alpha}_\nu\big(\textstyle\sum_j\hat{a}_j^\nu(v_j)\big)-\sum_j\hat{\alpha}_\nu\hat{a}_j^\nu(v_j)\,,$$

and since (by an obvious generalization of corollary 3.5 of [12])

$$\begin{aligned} &d\big(sq^\lambda\hat{\alpha}_\nu;\, \hat{a}_1^\nu(v_1),\ldots,\hat{a}_\tau^\nu(v_\tau)\big)+sq^\lambda d\big(\hat{\alpha}_\nu;\, \hat{a}_1^\nu(v_1),\ldots,\hat{a}_\tau^\nu(v_\tau)\big)\\ &\qquad\sim d(sq^\lambda;\, \hat{\alpha}_\nu\hat{a}_1^\nu(v_1),\ldots,\hat{\alpha}_\nu\hat{a}_\tau^\nu(v_\tau))+d(sq^\lambda;\, \hat{\alpha}_\nu(\textstyle\sum_j\hat{a}_j^\nu(v_j)),\, \sum_j\hat{\alpha}_\nu\hat{a}_j(v_j))\,,\end{aligned}$$

we get that the difference between δz and (3.24) is cohomologous $\big($in $f^\#(C(B))\big)$ to

$$\begin{aligned}\textstyle\sum_j d(sq^\lambda;\, \hat\alpha_1\hat a_j^1(v_j),\ldots,\hat\alpha_t\hat a_j^t(v_j),\hat b_j(v_j)) + d(sq^\lambda;\, \hat b_1(v_1),\ldots,\hat b_\tau(v_\tau)) +\\ + d\big(sq^\lambda;\, \hat\alpha_1(\textstyle\sum_j \hat a_j^1(v_j)),\ldots,\alpha_t(\sum_j \hat a_j^t(v_j)),\, \sum_j \hat b_j(v_j)\big) +\\ + \textstyle\sum_\nu d\big(sq^\lambda;\, \hat\alpha_\nu\hat a_1^\nu(v_1),\ldots,\hat\alpha_\nu\hat a_\tau^\nu(v_\tau)\big) +\\ + d\big(sq^\lambda;\, r_1(v_1),\ldots,r_\tau(v_\tau),\eta_1,\ldots,\eta_\tau\big) +\\ + \textstyle\sum_\nu d\big(sq^\lambda;\, \hat\alpha_\nu(\sum_j \hat a_j^\nu(v_j)),\, \sum_j \hat\alpha_\nu \hat a_j^\nu(v_j)\big)\,.\end{aligned}$$

By an argument similar to the proof of corollary 3.5 of [12] it follows that this cochain is a coboundary. This completes the proof.

In case $H^*(E)$ is trivial, we shall give a slightly different statement of Theorem 3.2.

In F_0 we define a new operation $\overline{Qu}$ associated with a pair (d,z) as before. $\overline{Qu}$ is defined on τ-tuples $(\bar u_1,\ldots,\bar u_\tau)$, $\bar u_j \in H^{n+\mu(j)}(F_0)$, $n<m$, provided $\bar u_j$ is transgressive for all j, and provided there are $\bar v_j \in H^{n+\mu(j)+1}(B)$ with

$$(3.25)\qquad \begin{aligned} d_{n+\mu(j)+1}\{\bar u_j\} &= \{\bar v_j\}\,,\\ \textstyle\sum_j a_j(\bar v_j) &= 0\,. \end{aligned}$$

The indeterminacy is smaller than before. It is

$$(3.26)\qquad \overline{\mathrm{Ind}_n} = \textstyle\sum_\nu \alpha_\nu \bar H^{n+i-\deg(\alpha_\nu)}(F_0)\,,$$

where $\bar H^s(F_0)$ denotes the transgressive elements in $H^s(F_0)$. The definition is as follows: Let $\Delta\theta_j = r_j$, and let $u_j \in C(E)$ be such that $i^\#(u_j) \in C(F_0)$ represents $\bar u_j$, and $\delta u_j \in f^\#(C(B))$. Let $\delta w_\nu = \sum_j \hat a_j^\nu(u_j) + w_\nu'$ with $w_\nu' \in f^\#(C(B))$. From the assumptions it is clear that such cochains exist. Now we define $\overline{Qu}(\bar u_1,\ldots,\bar u_\tau)$ to be represented by

$$(3.27)\qquad i^\#\big(\textstyle\sum_j \theta_j(u_j) + \sum_\nu \hat\alpha_\nu(w_\nu) + \sum_\nu d(\hat\alpha_\nu;\, \hat a_1^\nu(u_1),\ldots,\hat a_\tau^\nu(u_\tau))\big)\,.$$

This cochain defines a well-determined class in $H^{n+i}(F_0)/\overline{\mathrm{Ind}}$.

THEOREM 3.3. *Let $\{E_r,d_r\}$ be the spectral sequence associated with a* css-*map $f\colon E \to B$ (see (3.1)), where $H^*(E)$ is trivial. Let (d,z) be a pair as previously defined, and let an associated operation $\overline{Qu}$ be defined on the τ-tuple $(\bar u_1,\ldots,\bar u_\tau)$, $\bar u_j \in H^{n+\mu(j)}(F_0)$.*

If $n<m-1$, then for any $\overline{qu}(\bar u_1,\ldots,\bar u_\tau)$ representing $\overline{Qu}(u_1,\ldots,\bar u_\tau)$ there is a $qu(\bar v_1,\ldots,\bar v_\tau)$ representing $Qu(\bar v_1,\ldots,\bar v_\tau)$ for some operation Qu associated with (d,z) such that

$$d_{n+i+1}\{\overline{qu}(\bar u_1,\ldots,\bar u_\tau)\} = \{qu(\bar v_1,\ldots,\bar v_\tau)\}\,,$$

where $\bar v_j$ is as in (3.25).

If $n=m-1$, let p be the least $l(j)$ (see (3.2)) with $\beta_j \neq 0$, and let $\beta_j' = Sq^{L(l(j)-p,\,h(j))}(\beta_j)$ such that

$$b_j = Sq^{L(p,\varkappa)}(\beta_j') + \zeta_j$$

with $\varkappa = 2^{-p}(n+i+2)$. *Then any* $\overline{qu}(\bar{u}_1,\ldots,\bar{u}_\tau)$ *persists till* E_s, $s = (1-2^{-p})(n+i+2)$, *and*

$$d_s\{\overline{qu}(\bar{u}_1,\ldots,\bar{u}_\tau)\} = \{\Sigma_j \beta_j'(\bar{u}_j)\beta_j'(\bar{v}_j)^{2^p-1}\}\,.$$

Therefore we have

$$d_s\{\bar{\gamma}_p\} = \{(\Sigma_j \beta_j'(\bar{u}_j))(\Sigma_j \beta_j'(\bar{v}_j))^{2^p-1}\}\,,$$

where

$$\bar{\gamma}_1 = \overline{qu}(\bar{u}_1,\ldots,\bar{u}_\tau) + \Sigma_j \beta_j'(\bar{u}_j)\cdot\beta_k'(\bar{u}_k)\,,$$

summation over pairs (j,k) *with* $j<k$, *and*

$$\bar{\gamma}_p = \overline{qu}(\bar{u}_1,\ldots,\bar{u}_\tau), \qquad \text{for } p > 1\,.$$

In the case $n=m-1$ *we also have that the class* $\bar{\xi}_j$, *where*

$$\bar{\xi}_1 = \big(\bar{\gamma}_1 + (\Sigma_j \beta_j'(\bar{u}_j))^2\big)(\Sigma_j \beta_j'(\bar{u}_j))(\Sigma_j \beta_j'(\bar{v}_j))\,,$$
$$\bar{\xi}_p = \bar{\gamma}_p(\Sigma_j \beta_j'(\bar{u}_j))(\Sigma_j \beta_j'(\bar{v}_j))^{2^p-1}, \qquad p > 1\,,$$

is transgressive with

$$d_r\{\bar{\xi}_p\} = qu^{z'}(\bar{v}_1,\ldots,\bar{v}_\tau), \qquad r = (1+2^{-p})(n+i+2)-2\,,$$

for some representative $qu^{z'}(\bar{v}_1,\ldots,\bar{v}_\tau)$ *of a secondary operation associated with* $(d(m),z')$, $z' = \Sigma_\nu (Sq^{n+i+1}\alpha_\nu)c_1^{\nu}$.

This theorem follows easily from the definition of $\overline{Qu}$ and Theorem 3.2.

4. Lemmas about the Steenrod algebra.

In the following we shall often make use of certain vector spaces derived from the Steenrod algebra. Therefore it is convenient right away to introduce special notation for these vector spaces. As before $E(n+1)$ is the subspace of A consisting of all elements of excess larger than n and $A(n) = A/E(n+1)$. Let $d = d(n)\colon C_1(n) \to C_0(n)$ be as defined in (2.4) and (2.5) with $m_j = n+\mu(j)$. This means that

$$C_0(n) = A\big(n+\mu(1)\big)\oplus\ldots\oplus A\big(n+\mu(\tau)\big)$$

with generators $c_0^j(n) = c_0^j$. The degrees (positive) of these generators and of the generators $c_1^\nu(n) = c_1^\nu$ of $C_1(n)$ are

$$\text{(4.1)} \qquad \begin{aligned} \deg\big(c_0^j(n)\big) &= n+\mu(j)\,, \\ \deg\big(c_1^\nu(n)\big) &= \deg(a_j^\nu) + n + \mu(j) \qquad \text{for any } j\,. \end{aligned}$$

We put

$$\text{(4.2)} \qquad C_0(n,d) = C_0/im(d)\,,$$

(4.3) $$C_1(n,d) = K/K \cap \big(E(\deg c_1^{1})\oplus\ldots\oplus E(\deg c_1^{t})\big)\,,$$

where

(4.4) $$K = K(n) = \ker\big(d(n)\big)\,.$$

We have that C_0, C_1, $C_0(n,d)$, and $C_1(n,d)$ are all in an obvious fashion left A-modules, and d a mapping of A-modules. Let us consider mappings

(4.5) $$\begin{aligned}\bar{S}_n &= \bar{S}\colon\ \ C_1(n) \to C_1(n)\,,\\ S_n &= S\colon\ \ C_0(n) \to C_0(n)\,,\end{aligned}$$

defined on homogeneous components by

(4.6) $$\begin{aligned}\bar{S}(a) &= Sq^i\cdot a \qquad \text{for } a \in (C_1)_i\,,\\ S(b) &= Sq^i\cdot b \qquad \text{for } b \in (C_0)_i\,.\end{aligned}$$

The mapping S induces a mapping

(4.7) $$S\colon\ \ C_0(n,d) \to C_0(n,d)\,.$$

The mapping $S\colon C_0 \to C_0$ (4.5) is monic. To see this let $b=\sum_j(\sum_i \alpha_i^j)c_0^j \in \ker(S)$, where α_i^j is an admissible monomial of excess less than or equal to $n+\mu(j)$. Then $S(\alpha_i^j c_0^j) = (Sq^a\alpha_i^j)c_0^j$, where $a=n+\mu(j)+\deg(\alpha_i^j)$. We see that $Sq^a\alpha_i^j$ is admissible and of excess $n+\mu(j)$. Therefore, $S(b)=0$ implies $b=0$.

We define an increasing filtration F_* in $C_0(n,d)$ by

(4.8) $$\begin{aligned}F_j\big(C_0(n,d)\big) &= \ker S^j\,,\\ F_\infty\big(C_0(n,d)\big) &= C_0(n,d)\,,\end{aligned}$$

where S^j denotes the j-th iteration of S. There is a mapping $\sigma\colon C_i(n+1) \to C_i(n)$, $i=0,1$, of degree -1 defined by $\sigma\big(c_i^k(n+1)\big) = c_i^k(n)$. Then, clearly

(4.9) $$\begin{array}{ccc} 0 & & \\ \downarrow & & \\ C_1(n+1) & \xrightarrow{\ d\ } & C_0(n+1) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ C_1(n) & \xrightarrow{\ d\ } & C_0(n) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

is commutative and has exact vertical rows. There are induced mappings

(4.10) $$\begin{aligned}\sigma\colon\ \ C_1(n+1,d) &\to C_1(n,d)\,,\\ \sigma\colon\ \ C_0(n+1,d) &\to C_0(n,d)\,.\end{aligned}$$

The sequences

$$(4.11)\qquad \begin{array}{l} 0 \to C_0(n+1) \xrightarrow{S} C_0(n+1) \xrightarrow{\sigma} C_0(n) \to 0\,, \\ C_0(n+1,d) \xrightarrow{S} C_0(n+1,d) \xrightarrow{\sigma} C_0(n,d) \to 0 \end{array}$$

are exact. In both cases it is obvious that the composition σS is zero. Let

$$a = \sum_{j,k} \alpha_{j,k} c_0{}^j(n+1) \in \ker(\sigma) \in C_0(n+1)_i$$

with all $\alpha_{j,k}$ admissible monomials of excess less than or equal to $n+1+\mu(j)$. Since

$$\sigma(a) = \sum_{j,k} \alpha_{j,k} c_0{}^j(n) = 0\,,$$

it follows that $\alpha_{j,k}$ must be of excess $n+1+\mu(j)$. Therefore, any $\alpha_{j,k}$ can be written $Sq^s\beta_{j,k}$, where $\beta_{j,k}$ is admissible and

$$2s = n+1+\mu(j)+\deg\alpha_{j,k} = i\,.$$

For i odd, a must therefore be zero and is consequently in the image of S. For I even, we have

$$a = Sq^s \sum_{j,k} \beta_{j,k} c_0{}^j(n+1) = S\big(\sum_{j,k} \beta_{j,k} c_0{}^j(n+1)\big)\,,$$

because

$$\deg\big(\beta_{j,k} c_0{}^j(n+1)\big) = \deg\alpha_{j,k} - s + n + 1 + \mu(j) = i - s = s.$$

This proves exactness of the first sequence. The exactness of the second sequence now follows from the commutative diagram

$$\begin{array}{ccccc} & & C_1(n+1) & \xrightarrow[\cong]{\sigma} & C_1(n) \\ & & \downarrow d & & \downarrow d \\ C_0(n+1) & \xrightarrow{S} & C_0(n+1) & \xrightarrow{\sigma} & C_0(n)\,. \end{array}$$

For each $j = 1,2,\ldots,\infty$ choose a set of elements $\{x_i{}^j\}$, $x_i{}^j \in F_jC_0(n+1,d)$ such that for any s the set $\{\sigma(x_i{}^j) \mid j \leqq s, \text{all } i\}$ is a basis for $\sigma\big(F_sC_0(n+1,d)\big)$. Then

LEMMA 4.1. *The set of elements*

$$\{\sigma(x_i{}^j) \mid \textit{all } i,\ 1 \leqq j \leqq \infty\}$$

is a basis for $C_0(n,d)$.

Proofs will be given later in this section.

LEMMA 4.2. *The set of elements*

$$\{S^t x_i{}^j \mid \textit{all } i,\ 1 \leqq j \leqq \infty,\ 0 \leqq t < j\}$$

is a basis for $C_0(n+1,d)$.

Let us choose representatives for $x_i{}^j$ in $C_0(n+1)$. It is convenient also to denote these by $x_i{}^j$. Let $y_i{}^j \in C_1(n+1)$ be such that

(4.12) $$d(y_i^j) = S^j(x_i^j), \qquad 1 \leqq j < \infty ,$$

and let $\{z_s\}$, $z_s \in K(n+1) \subseteqq C_1(n+1)$, be a set of elements such that their images in

(4.13) $$D_1(n+1,d) = K(n+1)/K(n+1) \cap \Big(E\big(\deg(c_1^1(n))\big) \oplus \ldots \oplus E\big(\deg c_1^t(n)\big)\Big)$$

give a basis for this vector space. We denote the projection $K(n+1) \to D_1(n+1,d)$ by π. Since $\sigma S = 0$ and d commutes with σ (4.9), it follows from (4.12) that $\sigma(y_i^j) \in \ker(d(n))$, and therefore determines an element $w_i^j \in C_1(n,d)$, $1 \leqq j < \infty$. Then we have

LEMMA 4.3. *The set* $\{w_i^j, \sigma(z_s)\}$ *is a basis for* $C_1(n,d)$.

Let

(4.14) $$S'_{n+1} = S': \quad C_1(n+1) \to C_1(n+1)$$

be a mapping defined by $S'(a) = Sq^{i-1} \cdot a$ for $a \in (C_1)_i$. We get an induced mapping

(4.15) $$S': \quad C_1(n+1,d) \to C_1(n+1,d) .$$

The element $S'(y_i^j)$ is in the kernel of d: $C_1(n+1) \to C_0(n+1)$, since for $k = 2^{j-1} \deg(x_i^j)$

$$d\big(S'(y_i^j)\big) = Sq^{2k-1} Sq^k\big(S^{j-1}(x_i^j)\big) = 0 ,$$

according to the Adem relation $Sq^{2k-1} Sq^k = 0$. It follows for $h > 0$ that $(S')^h(y_i^j)$ determines an element in $C_1(n+1,d)$. This element we also denote $(S')^h(y_i^j)$.

LEMMA. *The set of elements*

$$\{(S')^h(y_i^j),\ h > 0;\ (S')^t(z_s),\ t \geqq 0\}$$

is a basis for $C_1(n+1,d)$.

PROOF OF LEMMA 4.1. From the choice of x_i^j it is clear that the $\sigma(x_i^j)$'s are independent. Since σ: $C_0(n+1,d) \to C_0(n,d)$ is onto (see (4.11)), they also span $C_0(n,d)$.

PROOF OF LEMMA 4.2. First, let us show that the elements in $\{S^t x_i^j \mid 0 \leqq t < j\}$ are independent. Let

$$\sum \lambda_i^{j,t} S^t x_i^j = 0, \qquad \lambda_i^{j,t} \in Z_2 ,$$

be a (finite) relation in $C_0(n+1,d)$. Since $\sigma S = 0$, we get

(4.16) $$\sum \lambda_i^{j,0} \sigma(x_i^j) = 0 \in C_0(n,d) .$$

The set $\{\sigma(x_i^j)\}$ is independent so that we can conclude $\lambda_i^{j,0}=0$. The equation (4.16) then becomes

$$S(\sum \lambda_i^{j,t} S^{t-1} x_i^j) = 0, \qquad 1 \leqq t < j .$$

Assume that $\lambda_i^{j,t}=0$ for $t<p$, then

$$S^p(\sum \lambda_i^{j,t} S^{t-p} x_i^j) = 0, \qquad p \leqq t < j .$$

Hence

$$\sum \lambda_i^{j,t} S^{t-p} x_i^j \in F_p C_0(n+1,d) .$$

This implies

$$\sum \lambda_i^{j,p} \sigma(x_i^j) \in \sigma\big(F_p C_0(n+1,d)\big) .$$

Therefore each $\lambda_i^{j,p} x_i^j \in F_p C_0(n+1,d)$. From the definition of the filtration F_* we conclude $\lambda_i^{j,p} S^p x_i^j = 0 \in C_0(n+1,d)$. Since $p<j$, it follows that $\lambda_i^{j,p}=0$. This proves that the elements $\{S^t x_i^j \mid t<j\}$ are independent. To show that they span $C_0(n+1,d)$ we argue by induction on degree in $C_0(n+1,d)$. We assume that the statement is true in dimensions less than p. Let $a \in C_0(n+1,d)_p$. By Lemma 4.1 we get that $\sigma(a)$ can be written as a sum $\sum \sigma(x_i^j)$ where the summation is over some subset of the index set. Then the element $b=a-\sum x_i^j$ has the property $\sigma(b)=0$. It is enough to show that b can be written as a sum of elements from $\{S^t x_i^j \mid t<j\}$. By the exactness of (4.11) there is an element $c \in C(n+1,d)$ with $S(c)=b$. By the induction hypothesis c is a sum of elements from $\{S^t x_i^j \mid t<j\}$. Since $S^j x_i^j=0$, the same is true for $S(c)$. This completes the proof.

Proof of Lemma 4.3. Let us put

$$C_1'(n) = C_1(n)/E\big(\deg(c_1^1(n))\big)\oplus\ldots\oplus E\big(\deg(c_1^t(n))\big) .$$

Then the sequence

$$(4.17) \qquad 0 \to C_1'(n+1) \xrightarrow{S'} C_1'(n+1) \xrightarrow{\sigma} C_1'(n) \to 0$$

is exact. The mappings in (4.17) are induced by S' (see (4.14)) and σ. The exactness of (4.17) follows from (4.11) by shifting the dimensions in C_1' down by one.

Let

$$(4.18) \qquad \sum \lambda_i^j w_i^j + \sum \lambda_s \sigma(z_s) = 0$$

be a relation in $C_1(n,d)$. This relation implies that

$$(4.19) \qquad \beta = \sum \lambda_i^j y_i^j + \sum \lambda_s z_s \in C_1'(n+1)$$

is in the kernel of σ. First, let us consider the coefficients λ_i^j. Since the dimension of y_i^j is even, they are clearly zero in case β is odd dimensional. The exactness of (4.17) implies that β is in the image of S'. Since the image of S' is of odd dimension, we get in case β is of even dimension

$$\sum \lambda_i^j y_i^j + \sum \lambda_s z_s \in E\big(\deg(c_1^1(n+1))\big) \oplus \ldots \oplus E\big(\deg(c_1^t(n+1))\big) ,$$

considering y_i^j and z_s to be in $C_1(n+1)$. By an exact sequence similar to (4.11) we get

$$(4.20) \qquad \sum \lambda_i^j y_i^j + \sum \lambda_s z_s = Sq^p \alpha + \gamma, \qquad p = \deg(\alpha) ,$$

for some $\alpha \in C_1(n+1)$ and

$$\gamma \in E(\deg c_1^1(n+1)+1) \oplus \ldots \oplus E(\deg c_1^t(n+1)+1) .$$

The formula for $d(n+1)$ shows that $d(\gamma)=0$. Applying d to (4.20) and using (4.12), we therefore get

$$(4.21) \qquad \sum \lambda_i^j S^j x_i^j = S\big(d(\alpha)\big) \in C_0(n+1)$$

or

$$\sum \lambda_i^j S^{j-1} x_i^j - d(\alpha) = 0 .$$

Lemma 4.2 now implies that $\lambda_i^j = 0$.

Now we are left with a relation of the form $\sum \lambda_s \sigma(z_s) = 0 \in C_1(n,d)$. By (4.17) this gives the relation $\sum \lambda_s \pi(z_s) = 0$ in $D_1(n+1,d)$ (see (4.13)). This implies $\lambda_s = 0$ and proves the independence of $\{w_i^j, \sigma(z_s)\}$.

Secondly, we must show that $\{w_i^j, \sigma(z_s)\}$ span $C_1(n,d)$. Let $b \in C_1(n,d)$. There is an element $a \in C_1(n+1)$ such that the class of $\sigma(a)$ in $C_1(n,d)$ is b. Since $\sigma(da) = d\big(\sigma(a)\big) = 0$, there is an element α in $C_0(n+1)$ with $S(\alpha) = da$. This equation tells us that the class of α in $C_0(n+1,d)$ is in $F_1 C_0(n+1,d)$. Therefore, by Lemma 4.2

$$\alpha = \sum S^{j-1} x_i^j + d\gamma ,$$

where the summation is over some subset of the index set. Let us consider the element

$$(4.22) \qquad \beta = a - \sum y_i^j - \bar{S}\gamma \in C_1(n+1), \qquad \text{(for definition of } \bar{S} \text{ see (4.5))} .$$

Since $d\beta = 0$, β determines an element $\{\beta\}$ in $D_1(n+1,d)$. By the choice of the element z_s we get

$$(4.23) \qquad \beta = \sum z_s + c ,$$

where $c \in E\big(\deg(c_1^1(n+1)) - 1\big) \oplus \ldots \oplus E\big(\deg(c_1^t(n+1)) - 1\big)$. From (4.22) and (4.23) we get

$$b = \sum w_i^j + \sum \sigma(z_s) .$$

This completes the proof.

PROOF OF LEMMA 4.4. First we must show that the elements in the set $\{(S')^h(y_i^j), (S')^t(z_s); h > 0, t \geqq 0\}$ are independent. Let

$$\sum \lambda_i^{h,j}(S')^h(y_i^j) + \sum \lambda_s^t(S')^t(z_s) = 0$$

be a relation in $C_1(n+1,d)$. Suppose that $\lambda_i^{h,j} = \lambda_s^t = 0$ for $h,t < q$. Then

$$(S')^q\big(\sum \lambda_i^{h,j}(S')^{h-q}(y_i^j) + \sum \lambda_s^t(S')^{t-q}(z_s)\big) = 0 .$$

Therefore, by (4.17)

$$\sum \lambda_i^{h,j}(S')^{h-q}(y_i^j) + \sum \lambda_s^t(S')^{t-q}(z_s) = 0 . \tag{4.24}$$

Applying σ to (4.24), we get

$$\sum \lambda_i^{q,j} w_i^j + \sum \lambda_s^q \sigma(z_s) = 0 \in C_1(n,d) .$$

By Lemma 4.3, $\lambda_i^{q,j} = 0$ and $\lambda_i^q = 0$. This proves the independence.

Next, we shall prove that $\{(S')^h(y_i^j), (S')^t(z_s)\}$ span $C_1(n+1,d)$. Assume this is true in dimensions less than q, and let $b \in C_1(n+1,d)$ be of dimension q. Let $a \in C_1(n+1)$ be a representative of b. By the definition of z_s we get $\pi(b) = \sum \pi(z_s) \in D_1(n+1,d)$ (see (4.13)). Let us consider $b - \sum z_s \in C_1'(n+1)$; then clearly $\sigma(b - \sum z_s) = 0$. By (4.17) there is a $\zeta \in C_1(n+1)$ such that

$$b - \sum z_s = S'\{\zeta\} \in C_1'(n+1) . \tag{4.25}$$

Then obviously $\sigma d S'\zeta = 0 \in C_0(n)$. This implies $S\sigma d\zeta = 0$. Since S is one-one, we get $\sigma d\zeta = 0$ or

$$d\zeta = S\alpha \tag{4.26}$$

for some $\alpha \in C_0(n+1)$. Hence $\{\alpha\} \in F_1C_0(n+1,d)$ so that by Lemma 4.2

$$\{\alpha\} = \sum S^{j-1}x_i^j .$$

Considered in $C_0(n+1)$ this gives

$$\alpha - \sum S^{j-1}x_i^j = d\gamma \tag{4.27}$$

for some $\gamma \in C_1(n+1)$. Now (4.27) implies

$$\zeta - \sum y_i^j - S\gamma \in \ker\big(d(n+1)\big) .$$

This element therefore determines an element in $C_1(n+1,d)$ of dimension less than q. By the induction hypothesis

$$\{\zeta - \sum y_i^j - S\gamma\} = \sum (S')^h y_i^j + \sum (S')^t(z_s) .$$

Since $S'S\gamma = 0$ (Adem relation), we get the following equation in $C_1'(n+1)$:

$$S'\{\zeta\} = \sum S'y_i^j + \sum (S')^{h+1}y_i^j + \sum (S')^{t+1}(z_s) .$$

Combining this with (4.25) we get the following equation in $C_1(n+1,d)$:

$$b = \sum S' y_i^j + \sum (S')^{h+1} y_i^j + \sum z_s + \sum (S')^{t+1}(z_s) .$$

This completes the proof.

We conclude this section by restating Theorem 3.2.

THEOREM 4.5. *Let* $\{E_r,d_r\}$ *be a spectral sequence associated with a* css-*map* $f\colon E \to B$ *(see* (3.1)*), and let* $z \in \ker(d(n))$*, where* $d(n)\colon C_1(n) \to C_0(n)$ *is a mapping as defined in* (4.1)*. Let the assumption be as in Theorem* 3.2*. Let* $\varepsilon_n\colon C_0(n) \to H^*(F)$ *and* $\varepsilon_{n+1}\colon C_0(n+1) \to H^*(B)$ *be defined by* $\varepsilon_n(c_0{}^j)=\bar{u}_j$ *and* $\varepsilon_{n+1}(c_0{}^j)=\bar{v}_j$*.*

If $\sigma^{-1}(z) \in \ker(d(n+1))$*, then* $qu^z(\varepsilon_n)$ *is transgressive, and*

$$d_s\{qu^z(\varepsilon_n)\} = \{qu^{\sigma^{-1}(z)}(\varepsilon_{n+1})\}, \qquad s = \deg(z) .$$

If $d(n+1)(\sigma^{-1}(z))=S^t x$*,* $x \in C_0(n+1)$*, then* $qu^z(\varepsilon_n)$ *persists till* E_r*,* $r=(2^t-1)\cdot\deg(x)$*, and*

$$d_r\{\gamma_t\} = \{\sigma(x)(\varepsilon_n)\big(x(\varepsilon_{n+1})\big)^{2^t-1}\} ,$$

where $x(\varepsilon_{n+1})=\varepsilon_{n+1}(x)$*, and where for* $t \geqq 2$*,* γ_t *equals* $qu^z(\varepsilon_n)$*,* γ_1 *equals* $qu^z(\varepsilon_n)$ *plus a product of primary operations on* ε_n*.*

Furthermore, in this case we have that ξ_t *is transgressive with*

$$d_q\{\xi_t\} = \{qu^{S'\sigma^{-1}(z)}(\varepsilon_{n+1})\}, \qquad q = (2^t+1)\deg x - 2 ,$$

where

$$\xi_1 = \big(\gamma_1 + (\sigma x)(\varepsilon_n)^2\big)(\sigma x)(\varepsilon_n)\cdot x(\varepsilon_{n+1}) ,$$
$$\xi_t = \gamma_t(\sigma x)(\varepsilon_n)x(\varepsilon_{n+1})^{2^t-1} .$$

5. Computations.

Let us in this section call a space a generalized Eilenberg–MacLane space if it is a finite cartesian product

(5.1) $$B = \times_j K(Z_2, n(j))$$

of Eilenberg–MacLane spaces with the non-vanishing homotopy group isomorphic to Z_2. In B we have one basic cohomology class $\bar{u}_j \in H^{n(j)}(B,Z_2)$ for each factor in the product (5.1).

From generalized Eilenberg–MacLane spaces we construct two-stage spaces. Let B and K be generalized Eilenberg–MacLane spaces, and let $f\colon B \to K$ be a mapping. The homotopy type of this mapping is determined by the set $\{k_\nu\}$ of cohomology classes

(5.2) $$k_\nu = f^*(\bar{b}_\nu) \in H^{n(\nu)}(B) ,$$

where $\bar{b}_\nu \in H^{n(\nu)}(K)$ is a basic class in K. Since K is base space in the fibering $\Omega K \to LK \to K$, where ΩK and LK denote the loop-space and the space of paths in K (based at a certain base-point), the mapping $f\colon B \to K$ induces a fibration

$$\Omega K \to E \xrightarrow{p} B . \tag{5.3}$$

The total space E in this fibration we shall denote a two-stage space. The homotopy type of the space E is determined by the base space B and the set $\{k_\nu\}$ of cohomology classes in $H^*(B)$. These cohomology classes are the k-invariants of the space E. These k-invariants can be considered as cohomology operations. We say that the k-invariants are stable if these operations are stable (or rather can be extended to stable operations).

It is clear that $p^*(k_\nu)=0$ in $H^*(E)$. In case the k-invariants are stable, we have

$$k_\nu = \textstyle\sum_j a_j^\nu \bar{u}_j , \tag{5.4}$$

where a_j^ν is in the Steenrod algebra. Let us choose cochain operations $\hat{a}_j^\nu$ (see [12]) representing a_j^ν. Let $\bar{u}_j$, $j=1,\ldots,\tau$, be the basic classes in B, and let u_j be cycles representing $p^*(\bar{u}_j)$. Then there are cochains w_ν in $C(E)$ with

$$\delta w_\nu = \textstyle\sum_j \hat{a}_j^\nu(u_j) , \tag{5.5}$$

such that the restriction of w_ν to the fibre ΩK in the fibration $E \to B$ gives a cocycle representing a basic class in that space. If we are given cochains u_j, w_ν and cochain operations $\hat{a}_j^\nu$ with the properties stated above, we shall say that E is oriented. We shall not exploit this notion any further. We only need it as it is stated.

Associated with a two-stage space E with stable k-invariants there is a mapping

$$d\colon\ C_1 \to C_0 \tag{5.6}$$

of the sort considered in section 2. The generators c_1^ν of C_1 are in one to one correspondence with the basic classes $\bar{b}_\nu \in H^*(K)$ and $\deg(c_1^\nu)=\dim(\bar{b}_\nu)$. The generators c_0^j of C_0 are in one to one correspondence with the basic classes $\bar{u}_j \in H^*(B)$ and $\deg(c_0^j)=\dim(\bar{u}_j)$. Putting $\deg(c_0^j)=n+\mu(j)$, $\min_j\{\mu(j)\}=0$, we have

$$C_0 = A(n+\mu(1))\oplus\ldots\oplus A(n+\mu(\tau)) .$$

The mapping d is given by (cf. (5.5))

$$d(c_1^\nu) = \textstyle\sum_j a_j^\nu c_0^j . \tag{5.7}$$

There is also a mapping $\varepsilon_n=\varepsilon\colon C_0 \to H^*(E)$ given by $\varepsilon(c_0^j)=p^*(\bar{u}_j)$.

Let E be oriented with the orientation given by cochains u_j, w_ν, and cochain operations $\hat{a}_j^\nu$. Let Qu be a secondary operation associated with a d-cycle $z=\sum \alpha_\nu c_1^\nu \in C_1$. This operation is defined on ε, and in $Qu(\varepsilon)$ there is a specified class $qu(\varepsilon)$ given by the orientation. This class is represented by the cocycle

$$\sum_j \theta_j(u_j)+\sum_\nu \hat{\alpha}_\nu(w_\nu)+\sum_\nu d\big(\hat{\alpha}_\nu;\, \hat{a}_1^\nu(u_1),\ldots,\hat{a}_\tau^\nu(u_\tau)\big)\,, \tag{5.9}$$

where θ_j satisfies (2.17) and $\hat{\alpha}_\nu$ is a cochain operation representing α_ν. It is easy to see that $qu(\varepsilon)$ is independent of other choices than the ones done by orientation. The loop-space ΩE is also a two-stage space. It is determined by the mapping $\Omega f\colon \Omega B \to \Omega K$. The k-invariants of ΩE are therefore the same as the k-invariants of E (considered as stable operations). The spaces with these particular k-invariants therefore depend on a parameter which we for instance can choose to be the minimal dimension of the basic classes. We use the notation $E=E_n$, $n=\min\{\dim \bar{u}_j\}$. Then, of course, $\Omega E=E_{n-1}$. Now we are ready to state the main theorems of this section.

THEOREM 5.1. *Let E_n be a two-stage space with stable k-invariants of the form* (5.4), *and let E_n be oriented. Let the mapping $d\colon C_1 \to C_0$ be associated with E_n. If $e(a_j^\nu) \geqq 2+\mu(j)$ for all ν and j, where* $\dim(\bar{u}_j)=n+\mu(j)$, $\min_j\{\mu(j)\}=0$, *then*

$$\{a_i(\varepsilon), qu^{z(j)}(\varepsilon)\}\,,$$

forms a simple system of generators for $H^*(E)$. Here $\{a_i\}$, $a_i \in C_0$ and $\{z(j)\}$, $z(j) \in \ker(d)$, are arbitrary sets such that the images of a_i and $z(j)$ in $C_0(n,d)$ and $C_1(n,d)$ respectively (*notation as in section* 4) *constitute a basis for these vector spaces.*

The algebra structure of $H^*(E)$ is contained in the following theorem. We use the same notation as introduced in section 4.

THEOREM 5.2. *Let E_n be as in Theorem* 5.1. *Let $\{x_i^j\}$ be a set of elements in $C_0(n,d)$ as defined prior to Lemma* 4.1. *Let $\{y_i^j, z_s\}$ be a set of elements in $C_1(n,d)$ as defined prior to Lemma* 4.3. *Then $H^*(E)$ as an algebra is the tensor product of a polynomial algebra in $\{qu^{S' y(i,j)}(\varepsilon_n), qu^{z_s}(\varepsilon_n), x_i^\infty(\varepsilon_n)\}$ and truncated polynomial algebras $Z_2[x_i^j(\varepsilon_n), 2^j]$ of height 2^j, $j<\infty$.* Here $y(i,j)=y_i^j$.

PROOF. The proof of Theorems 5.1 and 5.2 is by induction on n. First we show that Theorem 5.1_n implies Theorem 5.2_{n+1}, and secondly that Theorem 5.2_{n+1} implies Theorem 5.1_{n+1}. The condition $e(a_j^\nu) \geqq$

$2+\mu(j)$ on the k-invariants serves the purpose of making the start of this induction obvious.

Theorem $5.1_n \Rightarrow$ Theorem 5.2_{n+1}. Let us consider the fibration

$$\Omega E_{n+1} \xrightarrow{i} LE_{n+1} \xrightarrow{\pi} E_{n+1} . \tag{5.10}$$

Let $\hat{a}_j^\nu$ be cochain operations in F ((2.15)) representing $a_j^\nu \in A$. Let E_{n+1} be oriented by cocycles v_j and cochains w_ν' with

$$\delta w_\nu' = \textstyle\sum_j \hat{a}_j^\nu(v_j) .$$

Let the images of v_j and w_ν' in $C(LE_{n+1})$ under $\pi^\#$ be denoted by v_j and w_ν' also. Then there are cochains u_j and w_ν in LE_{n+1} with

$$\begin{aligned} \delta u_j &= v_j , \\ \delta w_\nu &= \textstyle\sum_j \hat{a}_j^\nu(u_j) + w'_\nu \end{aligned} \tag{5.11}$$

such that $i^\#(u_j)$, $i^\#(w_\nu)$ gives an orientation of $E_n = \Omega E_{n+1}$.

Let us pick elements $x_i^j \in C_0(n+1,d)$ and y_i^j, $z_s \in C_1(n+1,d)$ as done in section 4. Lemmas 4.1 and 4.3 and our hypothesis give that $H^*(\Omega E_{n+1})$ as a vector space is isomorphic to the exterior algebra in

$$\{\sigma(x_i^j)(\varepsilon_n), qu^{w(i,j)}(\varepsilon_n), qu^{\sigma(z_s)}(\varepsilon_n)\} , \tag{5.12}$$

where $w(i,j) = w_i^j$. We proceed by studying the behaviour of the generators (5.12) in the spectral sequence associated with the fibration (5.10). By Theorem 4.5, $qu^{\sigma(z_s)}(\varepsilon_n)$ is transgressive and the transgression is the class of $qu^{z_s}(\varepsilon_{n+1})$. Also, $\gamma(w_i^j)$, $\gamma(w_i^j) = qu^{w(i,j)}(\varepsilon_n)$ for $j \geqq 2$, and $\gamma(w_i^j) = qu^{w(i,j)}(\varepsilon_n) +$ a product of primary operations on ε_n for $j=1$, persists till E_r, $r = (2^j-1)\deg(x_i^j)$, with

$$d_r\{\gamma(w_i^j)\} = \left\{\sigma(x_i^j)(\varepsilon_n)\cdot\left(x_i^j(\varepsilon_{n+1})\right)^{2^j-1}\right\} .$$

The element $\xi(w_i^j)$,

$$\begin{aligned} \xi(w_i^j) &= \left(\gamma(w_i^j) + (\sigma x_i^j)(\varepsilon_n)^2\right)(\sigma x_i^j)(\varepsilon_n)x_i^j(\varepsilon_{n+1}) && \text{for } j = 1 , \\ \xi(w_i^j) &= \gamma(w_i^j)(\sigma x_i^j)(\varepsilon_n)x_i^j(\varepsilon_{n+1}) && \text{for } i \geqq 2 , \end{aligned}$$

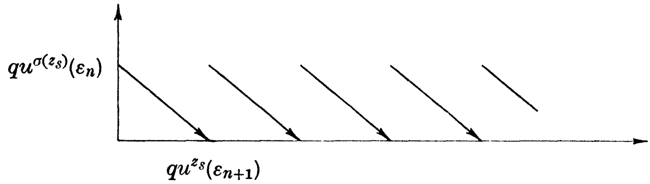
is transgressive, and

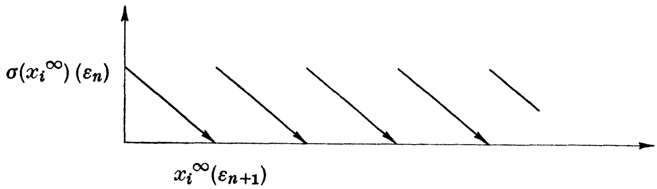
$$d_q\{\xi(w_i^j)\} = \{qu^{S'y(i,j)}(\varepsilon_{n+1})\} ,$$

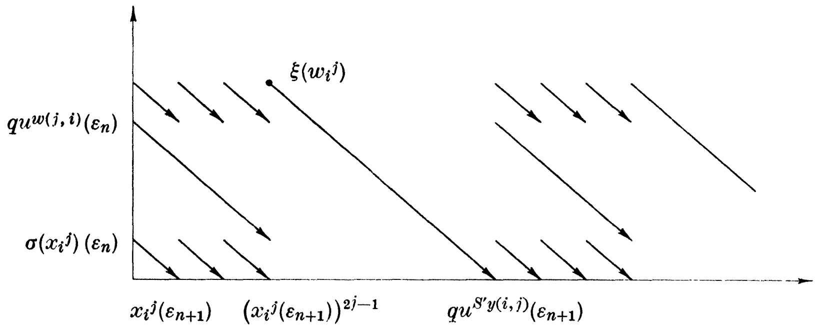
where $y(i,j) = y_i^j$ and $q = (2^j+1)\deg x_i^j - 2$.

Furthermore, we clearly have that $\sigma(x_i^j)(\varepsilon_n)$, $j = 1,\dots,\infty$, is transgressive and transgresses into the class of $x_i^j(\varepsilon_{n+1})$.

We review the situation in the following diagrams







The technique used in sections 9–12 of [10] can also be applied here. The spectral sequence of (5.10) splits up into a tensor product of very simple spectral sequences, one for each z_s, each x_i^{∞}, and each x_i^j, $j < \infty$, as the above diagrams indicate. Since $(x_i^j(\varepsilon_{n+1}))^{2^j} = 0$, it follows that the algebra structure of $H^*(E_{n+1})$ is as given by Theorem 5.2_{n+1}.

Theorem $5.2_{n+1} \Rightarrow$ Theorem 5.1_{n+1}. This is very easy. Theorem 5.2_{n+1} and Lemmas 4.2 and 4.4 give Theorem 5.1_{n+1} for special generators. The validity of Theorem 5.1_{n+1} for this special set of generators implies the theorem for any other set of generators.

Finally, we only need to start the induction. By the assumptions, however, the E_n is of the same homotopy type as $\Omega K \times B$ for small n. For these n the theorems follows from the computation by Serre [14]. This completes the proof.

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