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A NOTE ON INDUCTIVE LIMITS OF LINEAR SPACES

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1.

It is well known [4, p. 92] that a closed subspace of an (LF)-space (that is an inductive limit of a sequence of Fréchet-spaces) need not be an (LF)-space. Hence, if $E = \lim_{\to} E_\alpha$ is an inductive limit (in the sense of Bourbaki [2, Chap. II, p. 63]) and F is a linear subspace of E, it might happen that the induced topology in F is different from the topology of the inductive limit $F = \lim_{\to} E_\alpha \cap F$, where each $E_\alpha \cap F$ has the topology induced from E_α. We prove, however, in the present note the following. Let $^{-\alpha}$ denote closure in E_α, equip every $\overline{E_\alpha \cap F}^\alpha$ with the topology induced from E_α, and let $\lim_{\to} \overline{E_\alpha \cap F}^\alpha$ denote the inductive limit on the union E_0 of the spaces $\overline{E_\alpha \cap F}^\alpha$. Assume that E is the union of the spaces E_α. Then the topology of $\lim_{\to} E_\alpha \cap F$ coincides with the topology induced from $\lim_{\to} \overline{E_\alpha \cap F}^\alpha$. Specializing a corollary of this proposition, we obtain a result, due to Bourbaki [3, p. 55], concerning a construction of Radon measures on a locally compact space. As another application we show that the completion of a strict inductive limit of metric spaces is an (LF)-space.

PROPOSITION 1. *The topology $\mathscr{T}_F$ of $F = \lim_{\to} E_\alpha \cap F$ coincides with the topology induced from the topology $\mathscr{T}_0$ of $E_0 = \lim_{\to} \overline{E_\alpha \cap F}^\alpha$. Furthermore, $\lim_{\to} E_\alpha \cap F$ is a dense subspace of $\lim_{\to} \overline{E_\alpha \cap F}^\alpha$.*

PROOF. It is easy to verify that the topology induced from $\mathscr{T}_0$ is coarser than $\mathscr{T}_F$. In order to prove that $\mathscr{T}_F$ is coarser than the topology induced from $\mathscr{T}_0$, we first prove that if a linear functional f on F is continuous in the first mentioned topology, then f is also continuous in the latter. The restriction f_α of f to $E_\alpha \cap F$ is continuous, and f_α can therefore be extended in a unique way to a continuous linear functional $\bar{f}_\alpha$ on $\overline{E_\alpha \cap F}^\alpha$. Suppose that $E_\alpha \subset E_\beta$ and let $x \in \overline{E_\alpha \cap F}^\alpha$. Hence there exists a net $\{x_\gamma\} \subset E_\alpha \cap F$, such that $x_\gamma \to x$ in E_α. Since the topology induced from E_β is coarser than the topology of E_α, we also have that $x_\gamma \to x$ in E_β. Consequently,

$$\bar{f}_\beta(x) = \lim f_\beta(x_\gamma) = \lim f_\alpha(x_\gamma) = \bar{f}_\alpha(x) .$$

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Now there exists, whenever E_α and E_β are given, an E_γ such that $E_\alpha \cup E_\beta \subset E_\gamma$. We can therefore infer that the linear functional $\bar{f}$ is defined uniquely on E_0 by the equation

$$\bar{f}(x) = \bar{f}_\alpha(x), \qquad x \in \overline{E_\alpha \cap F}^\alpha .$$

Furthermore, since the restriction of $\bar{f}$ to $\overline{E_\alpha \cap F}^\alpha$ is $\bar{f}_\alpha$, the functional $\bar{f}$ is $\mathscr{T}_0$-continuous. Now the restriction of $\bar{f}$ to F is f. Hence f is continuous in the topology induced from $\mathscr{T}_0$. A consequence of this result is that if K is a $\mathscr{T}_F$-closed, convex subset of F then K is also closed in the topology induced from $\mathscr{T}_0$. Let U be a zero-neighbourhood in $\mathscr{T}_F$. We may and shall assume that U is symmetric, convex and $\mathscr{T}_F$-closed. By definition $U \cap E_\alpha \cap F$ is a zero-neighbourhood in $E_\alpha \cap F$. Hence it follows, by a simple result in general topology [1, p. 39], that $\overline{U \cap E_\alpha \cap F}^\alpha$ is a neighbourhood in $\overline{E_\alpha \cap F}^\alpha$. Letting bar denote $\mathscr{T}_0$-closure, we have

$$\overline{U \cap E_\alpha \cap F}^\alpha \subset \overline{U \cap E_\alpha \cap F} \subset \bar{U} .$$

Therefore the convex hull of the union of the sets $\overline{U \cap E_\alpha \cap F}^\alpha$ is contained in $\bar{U}$. This shows that $\bar{U}$ is a $\mathscr{T}_0$-neighbourhood. Since U is closed in the topology induced from $\mathscr{T}_0$, we infer that $U = \bar{U} \cap F$ is a neighbourhood in the induced topology. The second assertion in the proposition follows at once from the fact that $\overline{E_\alpha \cap F}^\alpha \subset \overline{E_\alpha \cap F} \subset \bar{F}$.

Corollary 1. *Suppose that the following condition is fulfilled.*

(B) *For every α there exists a β such that*

$$E_\alpha \subset \overline{E_\beta \cap F}^\beta .$$

Then the topology of $F = \lim_{\rightarrow} E_\alpha \cap F$ is the topology induced from $E = \lim_{\rightarrow} E_\alpha$, and F is a dense subspace of E. Furthermore, a linear functional f on F admits a continuous extension to $E = \lim_{\rightarrow} E_\alpha$ if and only if the restriction of f to $E_\alpha \cap F$ is continuous for each α.

Proof. It follows from the hypothesis and an elementary result for inductive limits [2, Chap. II, p. 62] that $E_0 = E$, and that $\lim_{\rightarrow} E_\alpha = \lim_{\rightarrow} \overline{E_\alpha \cap F}^\alpha$. This proves the first assertion, and the last one is an immediate consequence of the first one.

Condition (B) has been used by Bourbaki in his theory of integration [3, Chap. III, § 2 Definition 3]. The following gives another example.

PROPOSITION 2. *If* $E = \lim_{\rightarrow} E_n$ *is an* (LF)*-space, and*

$$\bigcup_{n=1}^{\infty} \overline{E_n \cap F}^n = E ,$$

then condition (B) *is satisfied.*

PROOF. This is an immediate consequence of [5, Théorème 1, p. 268].

2.

We assume in this section that E is a strict inductive limit of the sequence $\{E_n\}$, and that each E_n is a closed subspace of E_{n+1}. We claim no novelty for the following lemma, but being unable to give a ready reference, we give the proof.

LEMMA 1. *Suppose that* F *is a subspace of* $E = \lim_{\rightarrow} E_n$ *such that* F *is a metric topological vector space in a topology finer than the induced topology. Then* $F \subset E_n$ *for some* n.

PROOF. If the assertion is false, there exists a sequence $\{x_i\} \subset F$ and an increasing sequence $\{n_i\}$ such that $x_i \in E_{n_i} \setminus E_{n_{i-1}}$. Let $\{U_n\}$ be a decreasing fundamental system for the neighbourhoods of 0 in F. Then we can find for each x_i an $\alpha_i > 0$ such that $\alpha_i x_i \in U_i$. Hence $\alpha_i x_i \rightarrow 0$ in the metric topology, and therefore in the topology of E. This implies that $\{\alpha_i x_i\}$ is a bounded subset of E, and consequently [2, Chap. III, p. 8] $\{\alpha_i x_i\} \subset E_n$ for some n. This contradiction gives us the proof.

From now on every E_n is supposed to be metric. Let E'' (E''_n) be the bidual of E (E_n). We equip both E'' and E''_n with their "natural" topologies and hence we may and shall consider E''_n as a topological subspace of E'' [4, p. 84]. Furthermore, since E''_n is a Fréchet-space [4, p. 62], E''_n is a closed subspace of E''_{n+1} and E'' is the union of E''_n. We also notice that since E and E''_n are topological subspaces of E'', they both induce the same topology on $E \cap E''_n$.

PROPOSITION 3. $\lim_{\rightarrow} E_n = \lim_{\rightarrow} E''_n \cap E$.

PROOF. It is immediate that the topology of $\lim_{\rightarrow} E''_n \cap E$ is coarser than that of $\lim_{\rightarrow} E_n$. On the other hand, since $E''_n \cap E$ is a metric subspace of E, it follows from Lemma 1 that $E''_n \cap E \subset E_k$ for some k, and hence the topology of $\lim_{\rightarrow} E_n$ is coarser than the topology of $\lim_{\rightarrow} E''_n \cap E$.

COROLLARY 1. $\lim_{\rightarrow} E_n$ *is a dense topological subspace of the* (LF)*-space* $\lim_{\rightarrow} \overline{E''_n \cap E}^n$. (Here $^{-n}$ denotes closure in E''_n.)

PROOF. This follows directly from the propositions 1 and 3.

COROLLARY 2. *Every E_n is complete if and only if for each n there exists a k such that*

$$\overline{E_n'' \cap E^n} \subset E_k \,. \tag{1}$$

PROOF. If (1) is true, then $E = \lim_{\rightarrow} \overline{E_n'' \cap E^n}$. Hence E is complete [5, p. 257], and therefore the closed subspace E_n is complete. Conversely, if every E_k is complete, then $E = \lim_{\rightarrow} E_n$ is complete. Hence

$$\overline{E_n'' \cap E^n} \subset \bigcup_{k=1}^{\infty} \overline{E_k'' \cap E^k} = \bigcup_{k=1}^{\infty} E_k \,,$$

and the conclusion follows from Lemma 1.

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