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MEAN SUMMABILITY FOR ULTRASPHERICAL POLYNOMIALS[1]

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1. Introduction.

In this note we will be concerned with the mean Cesàro summability of ultraspherical polynomials. In the present section however we shall, for the sake of notational simplicity, deal only with Legendre polynomials. Let $L^p(-1,1)$ denote the space of those functions $f(x)$ defined and measurable on $(-1,1)$ for which

$$\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p \, dx\right)^{1/p}$$

is finite. If $f(x) \in L^p(-1,1)$ for some $p \geqq 1$ and if we set

$$f^\wedge(n) = \int_{-1}^{1} f(x) P_n(x) \, dx$$

then the formal Legendre series for $f(x)$ is

$$f(x) \sim \sum_0^\infty (n+\tfrac{1}{2}) f^\wedge(n) P_n(x) .$$

The partial sum of index N of this series is

$$S_N f\cdot(x) = \sum_0^N (n+\tfrac{1}{2}) f^\wedge(n) P_n(x) .$$

Pollard has proved that if $\frac{4}{3} < p < 4$ then there exists a constant $A(p)$ depending only upon p such that

$$\|S_N f\|_p \leqq A(p) \|f\|_p, \qquad N = 0,1,\ldots . \tag{1}$$

An immediate corollary of this inequality is that if $\frac{4}{3} < p < 4$

$$\lim_{N\to\infty} \|f - S_N f\|_p = 0 . \tag{2}$$

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Newman and Rudin have shown that (1) and (2) are false if $p \leqq \frac{4}{3}$ or $p \geqq 4$. Let us define

$$S_N^\alpha f\cdot(x) = (A_N^\alpha)^{-1} \sum_0^N (n+\tfrac{1}{2}) f^\wedge(n) A_{N-n}^\alpha P_n(x)$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)\dots(\alpha+n)}{n!}.$$

We will show here that if $0 \leqq \alpha \leqq \frac{1}{2}$ and if

$$\frac{4}{3+2\alpha} < p < \frac{4}{1-2\alpha} \tag{3}$$

then there exists a constant $A(p,\alpha)$ depending only upon p and α such that

$$\|S_N^\alpha f\|_p \leqq A(\alpha,p)\|f\|_p, \qquad N = 0,1,\dots . \tag{4}$$

This implies that under the same restrictions on α and p

$$\lim_{N\to\infty}\|f - S_N^\alpha f\|_p = 0 . \tag{5}$$

If $p \leqq 4/(3+2\alpha)$ or $p \geqq 4/(1-2\alpha)$ then (4) is false. The special case $\alpha = 0$ is Pollard's theorem. If $\alpha > \frac{1}{2}$ then (4) holds for all p, $1 \leqq p \leqq \infty$.

2. End point estimates.

For $\nu \geqq 0$ we set

$$2^n(\nu+\tfrac{1}{2})_n W_\nu(n,x) = (-1)^n (1-x^2)^{\frac{1}{2}-\nu}\left(\frac{d}{dx}\right)^n (1-x^2)^{n+\nu-\frac{1}{2}},$$

$$d\Omega_\nu(x) = (1-x^2)^{\nu-\frac{1}{2}}\,dx,$$

$$\omega_\nu(n) = \frac{\Gamma(\nu)(n+\nu)\Gamma(n+2\nu)}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})\Gamma(2\nu)n!}.$$

The $W_\nu(n,x)$ are the ultraspherical polynomials of index ν normalized by the condition $W_\nu(n,1) = 1$. We have

$$\int_{-1}^{1} W_\nu(n,x) W_\nu(m,x)\,d\Omega_\nu(x) = \delta_{n,m}/\omega_\nu(n)$$

where $\delta_{n,m}$ is 1 if $n = m$ and is 0 otherwise. Let us denote by L_ν^p the space of those functions $f(x)$ measurable on $(-1,1)$ for which

$$\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p\,d\Omega_\nu(x)\right)^{1/p}$$

is finite. Given $f(x) \in L_\nu^p$ we set

$$f\hat{\ }(n) = \int_{-1}^{1} f(x) W_\nu(n,x)\, d\Omega_\nu(x) .$$

The (formal) expansion of $f(x)$ in terms of the $W_\nu(n,x)$ is then

$$f(x) \sim \sum_{k=0}^{\infty} \omega_\nu(k) f\hat{\ }(k) W_\nu(k,x) .$$

Let us define

$$S_N{}^\alpha f\cdot(x) = (A_N{}^\alpha)^{-1} \sum_{k=0}^{N} A_{N-k}^\alpha \omega_\nu(k) f\hat{\ }(k) W_\nu(k,x) .$$

Our main objective in the present section is to prove the following result. This result is due to Kogbetliantz but the following proof is so simple and natural that we include it for completeness.

THEOREM 2a. *If $\alpha > \nu$, then there exists a constant $A(\alpha,\nu)$ depending only on α and ν such that for every $f \in L_\nu^1$*

$$\|S_N{}^\alpha f\|_1 \leqq A(\alpha,\nu) \|f\|_1 .$$

Let us set

$$C_\nu(x,y,z) = 2^{1-2\nu}\Gamma(\nu)^{-2}(1-x^2-y^2-z^2+2xyz)^{\nu-1}[(1-x^2)(1-y^2)(1-z^2)]^{\frac{1}{2}-\nu}$$

if $1-x^2-y^2-z^2+2xyz > 0$, and let $C_\nu(x,y,z)$ be zero otherwise. The following formula is due to Gegenbauer, see [1, vol. 1, p. 177]. If $-1<y<1$, $-1<z<1$, then

$$\int_{-1}^{1} W_\nu(n,x) C_\nu(x,y,z)\, d\Omega_\nu(x) = W_\nu(n,y) W_\nu(n,z) .$$

From this formula it is easily verified that if $f,g \in L_\nu^1$ and if we define

$$f*g\cdot(x) = \int_{-1}^{1}\int_{-1}^{1} f(y) g(z) C_\nu(x,y,z)\, d\Omega_\nu(y)\, d\Omega_\nu(z)$$

then

$$(f*g)\hat{\ }(n) = f\hat{\ }(n) g\hat{\ }(n) .$$

We also find that

$$\|f*g\|_1 \leqq \|f\|_1 \|g\|_1 .$$

For a more complete discussion of these formulas see [2]. It now follows that if

$$\varphi_N{}^\alpha(x) = (A_N{}^\alpha)^{-1} \sum_{k=0}^{N} A_{N-k}^\alpha \omega_\nu(k) W_\nu(k,x)$$

then

$$S_N^\alpha f\cdot(x) = f * \varphi_N^\alpha\cdot(x)$$

and thus that

$$\|S_N^\alpha f\cdot(x)\|_1 \leqq \|\varphi_N^\alpha(x)\|_1\cdot\|f\|_1\,.$$

Now it is proved in § 9.41 of [10] that if $\alpha > \nu$ then

$$(1) \qquad \|\varphi_N^\alpha(x)\|_1 \leqq A(\alpha,\nu), \qquad N = 0,1,\ldots,$$

for some finite constant $A(\alpha,\nu)$ independent of N. This proves Theorem 2a.

We also record for future use the results of Pollard, see [5], [6], and [7]:

THEOREM 2b. *If*

$$\frac{2\nu+1}{\nu+1} < p < \frac{2\nu+1}{\nu},$$

then there is a constant $A(\nu,p)$ depending only upon ν and p such that

$$\|S_N^0 f\cdot(x)\|_p \leqq A(\nu,p)\|f\|_p, \qquad N = 0,1,\ldots.$$

3. An interpolation theorem.

Let $(\mathscr{S},\mathscr{F},\mu)$ be a measure space; that is $\mathscr{S}$ is a set of points, $\mathscr{F}$ is a σ-field of subsets of $\mathscr{S}$, and μ a non-negative countably additive set function defined on the elements of $\mathscr{F}$. We denote by $L^p(\mathscr{S})$ the space of all complex $\mathscr{F}$-measurable functions for which

$$\|f\|_p = \left(\int_{\mathscr{S}} |f(x)|^p\, d\mu(x)\right)^{1/p}$$

is finite. Let $T_0, T_1, T_2, \ldots$ be a sequence of linear transformations of $\bigcup_{1\leq p\leqq\infty} L^p(\mathscr{S})$ into $\bigcap_{1\leq p\leqq\infty} L^p(\mathscr{S})$. We define a new sequence of linear transformations by setting for any complex $\alpha \neq -1, -2, -3, \ldots$,

$$S_N^\alpha = (A_N^\alpha)^{-1}\sum_{k=0}^{N} A_{N-k}^\alpha T_k, \qquad N = 0,1,\ldots.$$

We further define $\|S_N^\alpha\|_p$ to be the norm of S_N^α as a linear transformation of $L^p(\mathscr{S})$ into itself. We begin our study of $\|S_N^\alpha\|_p$ by noting the following simple result.

THEOREM 3a. *Let* $1 \leqq p \leqq \infty$, $\beta > \alpha > -1$. *Then if*

$$\|S_N^\alpha\|_p \leqq M, \qquad N = 0,1,\ldots,$$

it follows that

$$\|S_N^\beta\|_p \leqq M, \qquad N = 0,1,\ldots.$$

This is a consequence of the relations

$$A_N^\beta S_N^\beta = \sum_{k=0}^{N} A_k^\alpha S_k^\alpha A_{N-k}^{\beta-\alpha-1},$$

$$\sum_{k=0}^{N} A_k^\alpha A_{N-k}^{\beta-\alpha-1} = A_N^\beta,$$

and the fact that the various "A's" are all non-negative.

THEOREM 3b. *Let* $1 \leqq p,\ q \leqq \infty,\ -1 < \alpha, \beta < \infty$. *Suppose that*

$$\|S_N{}^\alpha\|_p \leqq M_1, \qquad N = 0, 1, \ldots,$$

and

$$\|S_N{}^\beta\|_q \leqq M_2, \qquad N = 0, 1, \ldots.$$

Then if for some θ, $0 < \theta < 1$, *we have*

$$\frac{1}{r} = (1-\theta)\frac{1}{p} + \theta\frac{1}{q},$$

$$\gamma > (1-\theta)\alpha + \theta\beta,$$

it will follow that

$$\|S_N{}^\gamma\|_r \leqq A M_1{}^{1-\theta} M_2{}^\theta$$

where A *depends only upon* α, β, p, q, θ, *and* γ.

We begin our proof of Theorem 3b by demonstrating the following lemma:

LEMMA 3c. *If for* $1 \leqq p \leqq \infty$ *and* $\delta > -1$ *we have*

$$\|S_N{}^\delta\|_p \leqq M, \qquad N = 0, 1, 2, \ldots,$$

then for any $\varepsilon > 0$

$$\|S_N{}^{\delta+\varepsilon+iy}\|_p \leqq e^{c(\varepsilon)y^2} M, \qquad N = 0, 1, \ldots, \quad -\infty < y < \infty.$$

We have

$$S_N^{\delta+\varepsilon+iy} = (A_N^{\delta+\varepsilon+iy})^{-1} \sum_{k=0}^{N} A_k^\delta S_k^\delta A_{N-k}^{\varepsilon-1+iy}$$

from which it follows that

$$\|S_N^{\delta+\varepsilon+iy}\|_p \leqq M |A_N^{\delta+\varepsilon+iy}|^{-1} \sum_{k=0}^{N} A_k^\delta |A_{N-k}^{\varepsilon-1+iy}|. \tag{1}$$

Now we note that if $\xi > -1$,

$$|A_k{}^{\xi+iy}/A_k{}^\xi| = \left| \frac{(\xi+iy+1)(\xi+iy+2)\ldots(\xi+iy+n)}{(\xi+1)(\xi+2)\ldots(\xi+n)} \right|.$$

We thus see that

(2) $$|A_k^{\xi+iy}/A_k^{\xi}| \geqq 1\,,$$

and that

$$|A_k^{\xi+iy}/A_k^{\xi}| \leqq \left[1+\left(\frac{y}{\xi+1}\right)^2\right]^{\frac{1}{2}}\cdots\left[1+\left(\frac{y}{\xi+n}\right)^2\right]^{\frac{1}{2}}.$$

Since

$$\left[1+\left(\frac{y}{\xi+k}\right)^2\right]^{\frac{1}{2}} \leqq \exp\left(\frac{y^2}{2(\xi+k)^2}\right)$$

it follows that if

$$c = \tfrac{1}{2}\sum_{k=1}^{\infty}(\xi+k)^{-2}$$

then

(2′) $$|A_k^{\xi+iy}/A_k^{\xi}| \leqq e^{cy^2}\,.$$

Using (2) and (2′) in (1) we obtain our desired result.

We now return to the proof of Theorem 3b. Choose $\varepsilon > 0$ so that

$$\gamma = (1-\theta)(\alpha+\varepsilon) + \theta(\beta+\varepsilon)\,.$$

Fix N. By Lemma 3c

$$\|S_N^{\alpha+\varepsilon+iy}\|_p \leqq e^{c(\varepsilon)y^2}M_1\,,$$

$$\|S_N^{\beta+\varepsilon+iy}\|_q \leqq e^{c(\varepsilon)y^2}M_2\,.$$

By Stein's interpolation theorem, see [12, Chapter XII], we have

$$\|S_N^{\gamma}\|_r \leqq AM_1^{1-\theta}M_2^{\theta}$$

as desired.

Theorem 3b is essentially due to Stein and Weiss [8].

4. The main theorem.

On combining the theorems of sections 2 and 3 we see that we have proved a result of the desired type.

THEOREM 4a. *If* $0 \leqq \nu < \infty$, $0 \leqq \alpha \leqq \nu$, *and if*

(1) $$\frac{2\nu+1}{\nu+1+\alpha} < p < \frac{2\nu+1}{\nu-\alpha}\,,$$

then there exists a constant A depending only upon ν, α, *and* p *such that*

(2) $$\|S_N\cdot f\|_p \leqq A\|f\|_p, \qquad N = 0,1,2,\ldots\,.$$

The result quoted in the introduction is the special case $\nu = \frac{1}{2}$.

In the remainder of this section we will prove, using a variation of the method of Newman and Rudin [4], that if

$$p \leqq (2\nu+1)/(\nu+1+\alpha) \quad \text{or} \quad p \geqq (2\nu+1)/(\nu-\alpha)$$

then (2) is false.

In what follows we shall use A for any positive constant independent of N. The constant A may have different values in different relations or even within one relation. We require the following preliminary result.

LEMMA 4b. *If*

$$1.\ \varphi(x) = \sum_{k=0}^{N} \varphi^{\wedge}(k)\, \omega_\nu(k) W_\nu(k,x)\ ,$$

$$2.\ 1 \leqq p < \infty,$$

then

$$\|\varphi\|_\infty \leqq A(N+1)^{(2\nu+1)/p} \|\varphi\|_p\ .$$

We have trivially that

$$\|\varphi\|_\infty \leqq \|\varphi\|_\infty\ . \tag{3}$$

Since $|W_\nu(k,x)| \leqq 1$ we see that since

$$\varphi^{\wedge}(k) = \int_{-1}^{1} \varphi(x) W_\nu(x,k)\, d\Omega_\nu(x)$$

it follows that

$$|\varphi^{\wedge}(k)| \leqq \|\varphi\|_1\ .$$

Consequently

$$\|\varphi\|_\infty \leqq \|\varphi\|_1 \sum_{k=1}^{N} \omega_\nu(k)\ ,$$

$$\|\varphi\|_\infty \leqq A(N+1)^{2\nu+1} \|\varphi\|_1 \tag{4}$$

where A depends only on ν. Applying a polynomial interpolation theorem of Stein [9] to (3) and (4) we obtain our desired result.

THEOREM 4c. *If* $0 \leqq \nu < \infty$, $0 \leqq \alpha < \nu$, *and if* $1 \leqq p \leqq (2\nu+1)/(\nu+1+\alpha)$ *or* $(2\nu+1)/(\nu-\alpha) \leqq p \leqq \infty$, *then* (2) *does not hold.*

It is sufficient to prove that (2) is false for the single value

$$p_1 = (2\nu+1)/(\nu-\alpha)\ .$$

For it then follows using the Riesz–Thorin convexity theorem that it is false for $p_1 \leqq p \leqq \infty$. Let q_1 be the index conjugate to p_1

$$q_1 = (2\nu+1)/(\nu+1+\alpha)\ .$$

That (2) is false for $1 \leqq p \leqq q_1$ then follows by simple duality considerations from the fact it is false for $p_1 \leqq p \leqq \infty$.

Let us consider the functional

$$T_N^\alpha \cdot f = S_N^\alpha f \cdot (1)$$

from L_ν^p to the real numbers. Since

$$T_N^\alpha f = \int_{-1}^{1} f(x)\varphi_N^\alpha(x)\, d\Omega_\nu(x) ,$$

a standard result on functionals on L_ν^p gives

$$\|T_N^\alpha\|_p = \|\varphi_N^\alpha(x)\|_q , \qquad \frac{1}{p}+\frac{1}{q} = 1 . \tag{5}$$

If (2) holds for p then we have

$$\|S_N^\alpha f\|_p \leqq A\|f\|_p ,$$

and therefore using Lemma 4b

$$\|T_N^\alpha f\|_p \leqq A(N+1)^{(2\nu+1)/p}\|f\|_p ,$$

which implies that

$$\|T_N^\alpha\|_p \leqq A(N+1)^{(2\nu+1)/p} . \tag{6}$$

We will prove that (6) and (5) with $p=p_1$ are contradictory. To obtain a minorant for $\|\varphi_N^\alpha\|_{q_1}$ we use an expression for φ_N^α derived by Szegö [10; 9.41.13]:

$$\varphi_N^\alpha(x) = \frac{\Gamma(N+2\nu+\alpha+1)\,\Gamma(2N+2\nu+\alpha+2)\,P_N^{(\alpha+\nu+\frac{1}{2},\,\nu-\frac{1}{2})}(x)}{2^{2\nu}\Gamma(\nu+\frac{1}{2})\,\Gamma(N+\nu+\frac{1}{2})\,\Gamma(2N+2\nu+2\alpha+2)\,A_N^\alpha} +$$

$$+\sum_{r=1}^{\infty}(-1)^{r-1}\binom{\alpha+r}{r}\frac{\alpha(\alpha-1)\ldots(\alpha-r+1)}{(2N+2\nu+\alpha+2)\ldots(2N+2\nu+\alpha+r+1)}\,\frac{A_N^{\alpha+r}}{A_N^\alpha}\,\varphi_N^{\alpha+r}(x) .$$

Set

$$R_N^\alpha = \sum_{r=1}^{\infty}\binom{\alpha+r}{r}\frac{|\alpha(\alpha-1)\ldots(\alpha-r+1)|}{(2N+2\nu+\alpha+2)\ldots(2N+2\nu+\alpha+r+1)}\,\frac{A_N^{\alpha+r}}{A_N^\alpha}\,\|\varphi_N^{\alpha+r}\|_{q_1} .$$

Since

$$A_N^{\alpha+r}\varphi_N^{\alpha+r}(x) = \sum_{k=0}^{N} A_k^{\alpha+1}\varphi_k^{\alpha+1}(x)A_{N-k}^{r-2}$$

and

$$\sum_{k=0}^{N} A_k^{\alpha+1}A_{N-k}^{r-2} = A_N^{\alpha+r}$$

and the "A's" are positive, we see that

$$\|\varphi_N^{\alpha+r}\|_{q_1} \leqq \|\varphi_N^{\alpha+1}\|_{q_1}, \qquad r = 1,2,\ldots .$$

By (5), $\|\varphi_N^{\alpha+1}\|_{q_1} = \|T_N^{\alpha+1}\|_{q_1}$. Since (2) holds for $p=p_1$ if α is replaced by $(\alpha+1)$ the argument used to establish (6) proves that $\|T_N^{\alpha+1}\|_{q_1} \leqq A(N+1)^{(2\nu+1)/p_1}$. Consequently

$$\|\varphi_N^{\alpha+1}\|_{q_1} \leqq A(N+1)^{(2\nu+1)/p_1} .$$

Thus

$$R_N^{\alpha} \leqq A(N+1)^{(2\nu+1)/p_1} \sum_{r=1}^{\infty} \binom{\alpha+r}{r} \frac{|\alpha(\alpha-1)\dots(\alpha-r+1)|}{(2N+2\nu+\alpha+2)\dots(2N+2\nu+\alpha+r+1)} \frac{A_N^{\alpha+r}}{A_N^{\alpha}} .$$

It is easily seen that

$$\sum_{r=1}^{\infty} \binom{\alpha+r}{r} \frac{|\alpha(\alpha-1)\dots(\alpha-r+1)|}{(2N+2\nu+\alpha+2)\dots(2N+2\nu+\alpha+r+1)} \frac{A_N^{\alpha+r}}{A_N^{\alpha}} \leqq A ,$$

where A is independent of N. Thus

$$R_N^{\alpha} \leqq A(N+1)^{(2\nu+1)/p_1} .$$

On the other hand by [10; 8.21.17] we see that

$$\|P_N^{(\alpha+\nu+\frac{1}{2},\, \nu-\frac{1}{2})}\|_{q_1} \geqq A(N+1)^{-\frac{1}{2}}(\log N)^{1/q_1} .$$

By Minkowski's inequality

$$\|\varphi_N^{\alpha}\|_{q_1} \geqq A(N+1)^{\nu-\alpha}(\log N)^{1/q_1} - A(N+1)^{(2\nu+1)/p_1} .$$

Since $\nu-\alpha = (2\nu+1)/p_1$, we see that for N large

$$\|\varphi_N^{\alpha}\|_{q_1} \geqq A(N+1)^{(2\nu+1)/p_1}(\log N)^{1/q_1} . \tag{7}$$

The inequalities (5), (6), and (7) are contradictory which proves that (2) must be false if $p=(2\nu+1)/(\nu-\alpha)$.

5. Other polynomials.

The question posed in the introduction can be asked for other orthogonal sets of polynomials. Pollard has shown that the question of mean convergence can be answered completely for Jacobi, Hermite, and Laguerre polynomials. Mean summability for Jacobi series is still open because the analogue of Theorem 2a is unproven at present. For Hermite and Laguerre polynomials Pollard has shown that mean convergence holds only if $p=2$. We will show that the same is true for mean summability.

Let $L_\mu^p(-\infty,\infty)$ be the space of functions defined and measurable on $(-\infty,\infty)$ with

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p e^{-x^2}\, dx\right)^{1/p}$$

finite. The Hermite polynomials are

$$H_n(x) = e^{x^2}(-1)^n \left(\frac{d}{dx}\right)^n e^{-x^2} .$$

If for $f \in L_\mu{}^p$ we set

$$f\hat{\ }(n) = \pi^{-\frac{1}{2}} 2^{-n}(n!)^{-1} \int_{-\infty}^{\infty} f(x)H_n(x)e^{-x^2}\, dx$$

then

$$f(x) \sim \sum_{n=0}^{\infty} f\hat{\ }(n)H_n(x) .$$

Let us define

$$S_N^\alpha f\cdot(x) = (A_N^\alpha)^{-1} \sum_{n=0}^{N} A_{N-n}^\alpha f\hat{\ }(n)H_n(x) .$$

We will show that if $1 \leqq p < 2$, there is a function $f \in L_\mu{}^p$ for which

$$\|S_N{}^\alpha f\|_p \leqq A\|f\|_p, \qquad N = 0, 1, \ldots , \tag{1}$$

does not hold if A is to be independent of N. Let $f(x) = e^{cx^2}$ where $\frac{1}{2} < c < 1/p$. Then $f \in L_\mu{}^p$. It is easily seen that

$$f\hat{\ }(2n+1) = 0 ,$$

$$f\hat{\ }(2n) = K(c)\left(\frac{c}{1-c}\right)^n \frac{(-1)^n}{4^n n!}, \tag{2}$$

where $K(c)$ is independent of n. We have

$$A_N^0 S_N^0 f\cdot(x) = \sum_{k=0}^{N} f\hat{\ }(n)H_n(x) = \sum_{k=0}^{N} A_k^\alpha S_k^\alpha f\cdot(x)\, A_{N-k}^{-\alpha-1}$$

so that if (1) were valid it would follow that

$$\left\|\sum_{k=0}^{N} f\hat{\ }(n)H_n(x)\right\|_p \leqq A\|f\|_p \sum_{k=0}^{N} A_k^\alpha |A_{N-k}^{-\alpha-1}| \leqq A\|f\|_p (N+1)^\alpha .$$

By Minkowski's inequality we find

$$\|f\hat{\ }(N)H_N(x)\|_p \leqq A\|f\|_p(N+1)^\alpha . \tag{3}$$

Now it is easily shown, see [6], that

$$\|H_{2N}\|_p \geqq \|H_{2N}\|_1 \geqq A\frac{(2N)!}{N!} .$$

Thus

$$|f\hat{\ }(2N)|\ \|H_{2N}\|_p \geqq A\left(\frac{c}{1-c}\right)^N \frac{(2N)!}{(2^N N!)^2} .$$

This contradicts (3).

REFERENCES

1. A. Erdélyi et al., *Higher transcendental functions*, vol. 1 and 2, New York, 1953.
2. I. I. Hirschman, Jr., *Harmonic analysis and ultraspherical polynomials*, Symposium on harmonic analysis and related integral transforms, Cornell University, 1956, 24 pp.
3. E. Kogbetliantz, *Recherches sur la sommabilité des séries ultrasphériques par la méthode des moyennes arithmétiques*, J. Math. Pures Appl. (9) 3 (1924), 107–187.
4. J. Newman and W. Rudin, *Mean convergence of orthogonal series*, Proc. Amer. Math. Soc. 3 (1952), 219–222.
5. H. Pollard, *The mean convergence of orthogonal series*, Trans. Amer. Math. Soc. 62 (1947), 387–403.
6. H. Pollard, *The mean convergence of orthogonal series* II, Trans. Amer. Math. Soc. 63 (1948), 355–367.
7. H. Pollard, *The mean convergence of orthogonal series* III, Duke Math. J. 16 (1949), 189–191.
8. E. Stein and G. Weiss, *On the interpolation of analytic families of operators acting on H^p spaces*, Tôhoku Math. J. 9 (1957), 318–339.
9. E. Stein, *Interpolation in polynomial classes and Markoff's inequality*, Duke Math. J. 24 (1957), 467–476.
10. G. Szegö, *Orthogonal polynomials*, New York, 1959.
11. G. Szegö and A. Zygmund, *On certain mean values of polynomials*, J. Analyse Math. 3 (1954), 225–244.
12. A. Zygmund, *Trigonometric series*, vol. 1 and 2, Cambridge, 1959.

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