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ELLIPTIC DIFFERENTIAL PROBLEMS WITH HIGH ORDER BOUNDARY CONDITIONS

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1, Introduction.

Let A, A_i, and B be linear homogeneous differential operators with constant coefficients of orders $2m$, m_i, and s, $s \leqq 2m$. Let the system $(A, \{A_i\})$ be elliptic in the half space $R_+{}^n$. (The definitions are given in § 2.) The usual L^p estimates for the solution of the boundary value problem

$$(1.1) \qquad Au = Bv \textit{ in } R_+{}^n \quad \textit{and} \quad A_i u = 0 \textit{ on the boundary } R^{n-1}$$

are of the form

$$(1.2) \qquad |u|_{2m-s+k,\, L^p} \leqq c|v|_{k,\, L^p}, \qquad 1 < p < \infty ,$$

for $k > m_i + s - 2m$. However, there are natural problems (see § 4) which call for such estimates for smaller k, in which case the estimates do not hold generally, but for certain functions v they may come about because of cancellations. It is our purpose to give a description of this class of functions v.

For example: If v vanishes near R^{n-1}, there is always a solution of (1.1) which satisfies (1.2) for any $k \geqq 0$.

The orders m_i of the boundary operators are completely immaterial.

Our results partly overlap recent unpublished work of S. Agmon. Agmon treats the special case where $\{A_i\}$ is a normal system of order $< 2m$. However, he does without our restriction that the operators be homogeneous with constant coefficients. We do not know if this is possible in the general case.

In § 2 we list the notations and some known properties of the Poisson kernels of Agmon, Douglis, and Nirenberg and of the singular integrals of Calderón and Zygmund.

In § 3 we give the main theorem. Its chief novelty is in the case mentioned above where the orders of some of the boundary operators are relatively high, but it holds in general. Moreover, although the inequali-

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ties are stated for the L^p norms, the proofs (as usual) involve explicit formulas in terms of singular integrals, and remain valid for various other norms such as the Hölder norms.

In § 4 we give an illustrative application to the spaces $L_m{}^p(\Omega)$.

We are indebted to L. Hörmander. Initially we considered only the particular boundary value problems arising in this application. The proper generality of the results came out in a discussion with him.

2. Preliminaries.

We use the following notation: R^n is the real n-dimensional space with points $x=(x_1,\ldots,x_n)$; C^n is the complex n-dimensional space with points $\xi=(\xi_1,\ldots,\xi_n)$; R^{n-1} is the hyperplane $x_n=0$ with points $x'=(x_1,\ldots,x_{n-1})$; $R_+{}^n$ is the open half space $x_n>0$; $\alpha=(\alpha_1,\ldots,\alpha_r)$ is a finite sequence of integers between 1 and n, and $|\alpha|=r$; $\xi^\alpha=\xi_{\alpha_1}\ldots\xi_{\alpha_r}$; $D_j=\partial/\partial x_j$; $D_\alpha=D_{\alpha_1}\ldots D_{\alpha_r}$; if a is a polynomial, then $A=a(D)$ is the differential operator obtained by replacing ξ^α by D_α.

For sufficiently regular functions u on an open set $\Omega\subset R^n$ we put

$$|u|_{k,\,L^p(\Omega)} = \left\{\sum_{|\alpha|=k}\int_\Omega |D_\alpha u|^p\,dx\right\}^{1/p},$$

$$\|u\|_{k,\,L^p(\Omega)} = \left\{\sum_{|\alpha|\leqq k}\int_\Omega |D_\alpha u|^p\,dx\right\}^{1/p}.$$

When p and Ω are fixed we may write simply $|u|_k$ and $\|u\|_k$. We will always suppose that $1<p<\infty$.

Let A and A_i, $i=1,\ldots,m$, be linear homogeneous differential operators of orders $2m$ and m_i with constant coefficients. The system $(A,\{A_i\})$ is elliptic in the half space $R_+{}^n$ if for each fixed real $\xi'\in R^{n-1}$, $\xi'\neq 0$, the polynomial $a(\xi)=a(\xi',\xi_n)$ has m zeros $\tau_1,\ldots\tau_m$ with positive imaginary part, and m with negative imaginary part, and the polynomials $a_i(\xi',\xi_n)$ are linearly independent modulo $a^+(\xi_n)=(\xi_n-\tau_1)\ldots(\xi_n-\tau_m)$. Equivalent to this linear independence is the property that for $\xi'\neq 0$ the equations $Au=0$ in $R_+{}^n$ and $A_iu=0$ on R^{n-1} have no bounded solution $\neq 0$ of the form

$$u(x) = e^{i(x',\xi')}\,\varphi(x_n)\,.$$

The first formulation is found in [1], where it is called the complementing condition; the second is due to Hörmander, unpublished.

Agmon, Douglis, and Nirenberg have constructed Poisson kernels K_i for an arbitrary elliptic system. The explicit formulas for these kernels and some other $K_{i,j}$ connected with them can be found in [1]. Here we simply list the necessary properties, in a somewhat different notation. To shorten the statements we shall say that a function F is almost homogeneous of degree d if $D_\alpha F$ is homogeneous of degree $d-|\alpha|$ for $|\alpha|>d$, and $|D_\alpha F(x)| \leqq c|x|^{d-|\alpha|}(1+\log^+|x|)$ for $|\alpha| \leqq d$.

(a) *For $i=1,\ldots,m$ and j an even integer $\geqq 0, K_{i,j}$ is a function of class C^∞ in the closed half space $\bar{R}_+{}^n$ except at* 0.

(b) *$K_{i,j}$ is almost homogeneous of degree $m_i+j+1-n$.*

(c) *$K_{i,j}=\Delta' K_{i,j+2}$, where Δ' is the Laplacian in R^{n-1}.*

(d) *The functions $K_i=K_{i,0}$ are Poisson kernels for the system $(A,\{A_i\})$ in the sense that*

$$w(x) = \sum_i K_i * \varphi_i = \sum_i \int_{R^{n-1}} K_i(x'-y',x_n)\varphi_i(y')\,dy'$$

satisfies $Aw=0$ in $R_+{}^n$ and $A_i w=\varphi_i$ on R^{n-1}.

Also we shall use the following results from the theory of singular integrals of Calderón and Zygmund [2, 3].

(e) *If F is almost homogeneous of degree $k-n$ and $u=D_\alpha(F*v)$, $|\alpha|=k$, then*

$$|u|_{0,L^p(R^n)} \leqq c|v|_{0,L^p(R^n)} .$$

Here the $*$ denotes convolution over R^n, while in (d) it denotes convolution over R^{n-1}. The meaning will be clear from the context.

(f) *If K is homogoneous of degree $-n$ on $R_+{}^n$ and*

$$u(x) = \int_{R_+{}^n} K(x'-y',x_n+y_n)f(y)\,dy ,$$

then

$$|u|_{0,L^p(R_+{}^n)} \leqq c|f|_{0,L^p(R_+{}^n)} .$$

It is pointed out in [1] that this results immediately from the singular integral theorems if K is extended to $R_-{}^n$ so as to be odd in x_n. Actually, we use mainly a combination of (e) and (f).

(g) *If*

$$u(x) = \int_{R_+{}^n} K(x'-y',x_n+y_n)\, D_\alpha F * v(y)\,dy ,$$

then

$$|u|_{0,L^p(R_+{}^n)} \leqq c|v|_{0,L^p(R^n)} .$$

Here K is homogeneous of degree $-n$, F is almost homogeneous of degree $k-n$, and $|\alpha|=k$.

Finally we note (see, e.g. [1])

(h) *A has a fundamental solution F which is of class C^∞ except at 0 and is almost homogeneous of degree $2m-n$; that is, $A(F*v)=v$.*

REMARK. The statements above require mild regularity conditions on the functions involved. In the use that we make of these statements very strong regularity conditions prevail. The kernels $K_{i,j}$, F, and K are all of class C^∞ except at the origin. The functions φ_i, v, and f are all of class C^k with k as large as desired (usually $k=\infty$).

The statements also require integrability conditions at ∞. If our proofs were carried through in a completely straightforward way, these conditions would not always be met, even when the data have compact support. The precautions which have to be taken on this point are evident in the proofs.

3. The main theorem.

Let A, A_i, and B be linear homogeneous differential operators with constant coefficients of orders $2m$, m_i, and s, $s \leqq 2m$. Let the system $(A,\{A_i\})$ be elliptic in the half space $R_+{}^n$. The result stated partially in the introduction is as follows. (We treat only $k=0$. Larger k's offer no extra difficulty.)

THEOREM 1. *For each function $v \in C_0{}^\infty(R_+{}^n)$ there is a function $u \in C^\infty(\bar{R}_+{}^n)$ satisfying*

$$Au = Bv \text{ in } R_+{}^n \quad \text{and} \quad A_i u = 0 \text{ on } R^{n-1} \quad \text{and}$$
$$|u|_{2m-s} \leqq c\,|v|_0 \,.$$

The constant c depends only on certain ellipticity constants of the system $(A,\{A_i\})$. The space $C_0{}^\infty(R_+{}^n)$ is the space of functions which are of class C^∞ and which have compact support in $R_+{}^n$.

This theorem follows immediately from a more complete one which is easier to prove. Let b_i and r_i be the quotient and remainder when $a_i b$ as a polynomial in ξ_n is divided by a. Thus

$$a_i b = b_i a + r_i \quad \text{and the degree of } r_i \text{ in } \xi_n \text{ is } < 2m\,. \tag{3.1}$$

Since the coefficient of $\xi_n{}^{2m}$ in a is constant and $\neq 0$, b_i and r_i are uniquely determined polynomials in ξ. Note that if $m_i < 2m - s$, then $b_i = 0$

THEOREM 2. *For each function $v \in C_0{}^\infty(R^n)$ there is a function $u \in C^\infty(\bar{R}_+{}^n)$ satisfying*

$$Au = Bv \text{ in } R_+{}^n \quad \text{and} \quad A_i u = B_i v \text{ on } R^{n-1} \quad \text{and} \tag{3.2}$$

$$|u|_{2m-s} \leqq c\,|v|_0 \,. \tag{3.3}$$

The constant c depends only on certain ellipticity constants of the system $(A,\{A_i\})$. Concerning $|v|_0$, see Remark 2 below.

COROLLARY. *The functions v for which Theorem* 1 *holds are those which satisfy the boundary conditions $B_i v = 0$ on R^{n-1}.*

This is the description mentioned in the introduction. A completely precise description is given after Theorem 2′ below.

PROOF OF THEOREM 2. Formally the solution u is given in terms of the fundamental solution F and the Poisson Kernels K_i by

$$u = \sum_{i=0}^{m} u_i ,$$

where

$$u_0 = F*Bv \quad \text{and} \quad u_i = K_i*(B_i v - A_i u_0) = -K_i*(R_i F*v) \text{ if } i>0 .$$

In order to ensure the convergence of the convolution with K_i (which is over R^{n-1}) we suppose at the start that v has the special form $v = Ag$ with $g \in C_0^\infty(R^n)$. It follows from the results quoted in § 2 that u is of class C^∞, that u satisfies (3.2), and that u_0 satisfies (3.3).

In showing that each u_i, $i>0$, satisfies (3.3) we consider two cases. First, $m_i + s - 2m \geqq 0$. Writing D^k for a generic derivative of order k, we have

$$D^{2m-s}u_i(x) = -\int_{R^{n-1}} D^{2m-s}K_i(x'-y', x_n)\, R_i F*v(y', 0)\, dy' .$$

As in [1] we rewrite this as an integral over $R_+{}^n$ by differentiating and integrating

$$D^{2m-s}K_i(x'-y', x_n+y_n)\, R_i F*v(y', y_n)$$

with respect to y_n. We get

$$D^{2m-s}u_i(x) = \int_{R_+{}^n} D_n D^{2m-s}K_i(x'-y', x_n+y_n)\, R_i F*v(y)\, dy + \\ + \int_{R_+{}^n} D^{2m-s}K_i(x'-y', x_n+y_n)\, D_n R_i F*v(y)\, dy .$$

Each term in $r_i(\xi)$ has degree at least $m_i+s-2m+1$ in ξ'. Therefore, we can integrate by parts m_i+s-2m times in the first integral and $m_i+s-2m+1$ times in the second to obtain terms of the form

$$\int_{R_+{}^n} D^{m_i+1}K_i(x'-y', x_n+y_n)\, D^{2m}F*v(y)\, dy .$$

The required inequality then follows from (b), (d), (g), and (h) of § 2.

Now suppose that $m_i+s-2m<0$, and choose an integer j so that

$2j \geqq 2m-m_i-s$. Using the fact that $K_i=\Delta'^j K_{i,2j}$ and using the same device as before to produce an integral over $R_+{}^n$ we have

$$D^{2m-s}u_i(x) = \int_{R_+{}^n} D_n D^{2m-s}\Delta'^j K_{i,2j}(x'-y', x_n+y_n)\, R_iF*v(y)\, dy + \\ + \int_{R_+{}^n} D^{2m-s}\Delta'^j K_{i,2j}(x'-y', x_n+y_n)\, D_nR_iF*v(y)\, dy\ .$$

Now we can use the derivatives in Δ'^j to integrate by parts $2m-m_i-s$ times in the first integral and $2m-m_i-s-1$ times in the second to obtain terms of the form

$$\int_{R_+{}^n} D^{2j+m_i+1}K_{i,2j}(x'-y', x_n+y_n)\, D^{2m}F*v(y)\, dy\ .$$

As before the required inequality follows from § 2.

Remark 1. This part of the proof shows that there is no additional difficulty in proving

(3.4) $$|u|_{2m-s+k} \leqq c|v|_k \quad \textit{for} \quad k \geqq 0\ .$$

All that is needed is a sufficiently high power of Δ'.

Now let v be an arbitrary function in $C_0{}^\infty(R^n)$, not necessarily of the special form Ag, and let k be a large integer. As is well known, there is a sequence of functions v_n of the special form such that

$$\|v-v_n\|_{k, L^p(R^n)} \to 0\ .$$

Let u_n be the corresponding solution given by what has been proved. The inequality (3.4) shows that for every α with $2m-s \leqq |\alpha| \leqq 2m-s+k$, $D_\alpha u_n$ converges in $L^p(R_+{}^n)$ to some function $u_\alpha \in L^p(R_+{}^n)$. The u_α are uniquely determined by v, are bounded and of class C^∞ on $\bar{R}_+{}^n$, and satisfy the relations necessary for the existence of a function u of class C^∞ on $\bar{R}_+{}^n$ with $D_\alpha u=u_\alpha$. Thus

(3.5) $$\|D_\alpha u - D_\alpha u_n\|_{2m-s+k-|\alpha|} \to 0 \quad \textit{for} \quad 2m-s \leqq |\alpha| \leqq 2m-s+k\ .$$

When k is sufficiently large this implies that

(3.6) $Au=Bv$ *in* $R_+{}^n$, *and* $A_iu=B_iv$ *on* R^{n-1} *if* $m_i \geqq 2m-s$,

(3.7) $A_iu=q_i$ *on* R^{n-1} *if* $m_i<2m-s$,

where q_i is a polynomial of degree $<2m-s-m_i$. Indeed, consider for example (3.7). If D_α is any derivative of order $2m-s-m_i$ which depends only on x', then $D_\alpha A_iu_n=0$ on R^{n-1}. Therefore by (3.5), $D_\alpha A_iu=0$ on

R^{n-1}. Since this is true for every such derivative D_α, it follows that on R^{n-1}, $A_i u$ is a polynomial of degree $< 2m-s-m_i$. Since (3.3) and (3.6) are unaffected if u is changed by a polynomial of degree $< 2m-s$ and since $B_i = 0$ when $m_i < 2m-s$, the proof will be finished by the following lemma. (The homogeneity gives the right degrees.)

LEMMA 1. *If $q_i(x')$ are any polynomials, there is a polynomial $p(x)$ satisfying $A_i p = q_i$ on R^{n-1}.*

It is convenient to prove the lemma with weaker hypotheses on the A_i than the ones resulting from ellipticity.

LEMMA 2. *Let $a_1(\xi), \ldots, a_k(\xi)$ be polynomials which are linearly independent over $C(\xi')$, the field of rational functions of ξ'. If $q_i(x')$ are any polynomials, there is a polynomial $p(x)$ satisfying $A_i p = q_i$ on R^{n-1}.*

In proving Lemma 2 we will use a third lemma.

LEMMA 3. *If $a'(\xi')$ and $q(x')$ are polynomials, there is a polynomial $r(x')$ satisfying $A'r = q$.*

The proof of Lemma 3 is a simple induction on the dimension.

PROOF OF LEMMA 2. Let $d-1$ be the highest degree in ξ_n of any of the polynomials a_i, and choose additional polynomials $a_{k+1}, \ldots, a_d$ so that $a_1, \ldots, a_d$ is a basis over $C(\xi')$ for the polynomials of degree $< d$ in ξ_n. Then

$$a_i = \sum_{j=1}^{d} a_{ij} \xi_n^{\,j-1} \qquad \textit{and} \qquad a' \xi_n^{\,j-1} = \sum_{k=1}^{d} a'_{jk} a_k \, ,$$

where a_{ij}, a'_{jk}, and $a' = \det\{a_{ij}\}$ are all polynomials in ξ'. By Lemma 3 there are polynomials $r_i(x')$ such that $A' r_i = q_i$ on R^{n-1}. If we define $p(x)$ so that

$$D_n^{\,j-1} p = \sum_{k=1}^{d} A'_{jk} r_k \quad \textit{on} \quad R^{n-1} \, ,$$

then

$$A_i p = \sum_{j,k} A_{ij} A'_{jk} r_k = A' r_i = q_i \quad \textit{on} \quad R^{n-1} \, .$$

REMARK 2. It would appear at first that the norm of v in (3.3) should be taken over the whole space R^n. Actually, the norm over $R_+{}^n$ suffices. The proof is as follows. Since the derivatives of u of order $2m-s$ are bounded,

$$u(x) = O(|x|^{2m-s}) \qquad \text{as} \qquad |x| \to \infty \, .$$

Let k be a large integer, and let v_k be of class $C_0{}^k(R^n)$, $v_k = v$ on $R_+{}^n$, and

$$\|v_k\|_{j, L^p(R^n)} \leqq c \|v\|_{j, L^p(R_+{}^n)} \quad \textit{for} \quad 0 \leqq j \leqq k \, .$$

The procedure above leads to a corresponding solution u_k which is sufficiently regular and satisfies $u_k(x)=O(|x|^{2m-s})$. A special case of the uniqueness theorem in [1, p. 662] states that any solution to $Au=0$ in $R_+{}^n$ and $A_iu=0$ on R^{n-1} which is sufficiently regular and has polynomial growth must be a polynomial. Therefore, $u-u_k$ is a polynomial, which must have degree $<2m-s$ since $|u-u_k|_{2m-s}<\infty$. Hence

$$|u|_{2m-s} = |u_k|_{2m-s} \leqq c\,|v_k|_{0,L^p(R^n)} \leqq c\,|v|_{0,L^p(R_+{}^n)} .$$

REMARK 3. The solution we have found may not be unique. The function u in (3.5) is only determined up to a polynomial of degree $<2m-s$, and the polynomial p in Lemma 1 may not be unique. In both cases a finite number of additional relations can be used to fix the determination. Thus we have a slightly more precise version of Theorem 2.

THEOREM 2′. *There is a linear transformation T from $C_0{}^\infty(R^n)$ into $C^\infty(\bar{R}_+{}^n)$ such that $u=Tv$ satisfies* (3.2) *and* (3.3) (with the norm over $R_+{}^n$) *and has polynomial growth at* ∞.

The boundary operators B_i are the only ones for which such a theorem is true. In fact, suppose it were true for some others B_i' and a linear transformation T'. When $v\in C_0{}^\infty(R_+{}^n)$, $B_iv=B_i'v=0$. Hence, by the uniqueness theorem used in Remark 2, $Tv-T'v$ is a polynomial, which must have degree $<2m-s$. Given $v\in C_0{}^\infty(R^n)$, let $v_n\in C_0{}^\infty(R_+{}^n)$ and

$$|v-v_n|_{0,L^p(R_+{}^n)} \to 0 .$$

For $|\alpha|=2m-s$

$$D_\alpha Tv - D_\alpha T'v = \lim (D_\alpha Tv_n - D_\alpha T'v_n) = 0$$

so that $Tv-T'v$ is a polynomial. Hence, on R^{n-1}

$$B_iv - B_i'v = A_iTv - A_iT'v$$

is a polynomial, and this is not possible for an arbitrary $v\in C_0{}^\infty(R^n)$ unless $B_i=B_i'$.

An argument very much like this one leads to the following precise version of the corollary to Theorem 2.

COROLLARY. *Let C be a linear class of functions, $C_0{}^\infty(R_+{}^n)\subset C\subset C_0{}^\infty(R^n)$. Suppose there is a linear transformation $S\colon C\to C^\infty(\bar{R}_+{}^n)$ such that for $v\in C$, the function $u=Sv$ satisfies*

(a) *$Au=Bv$ in $R_+{}^n$ and $A_iu=0$ on R^{n-1},*

(b) $|u|_{2m-s,R_+{}^n}\leqq c\,|v|_{0,R_+{}^n}$,

(c) *u has at most polynomial growth at ∞.*

Then every function $v\in C$ satisfies the boundary conditions $B_iv=0$ on R^{n-1}.

4. An application to the spaces $L_m{}^p(\Omega)$.

Let Ω be a bounded open set in R^n with boundary of class C^m. The class $L_m{}^p(\Omega)$ of functions whose derivatives of orders $\leqq m$ belong to L^p on Ω is a Banach space under the norm $\|\cdot\|_{m, L^p(\Omega)}$. Some of its properties are given in [1, 4, 5]. We shall give a general representation theorem about the linear forms on this Banach space.

Let $\{P_j\}$ be a finite set of linear differential operators of orders $\leqq m$ with coefficients sufficiently regular in $\bar{\Omega}$. Let p_j be the part of the characteristic polynomial of P_j of order m. (If P_j has order $< m$, then $p_j = 0$.) We assume:

(a) If $x \in \Omega$, the $p_j(x,\xi)$ have no common real zero $\xi \neq 0$; and

(b) If $x \in \partial\Omega$, the $p_j(x,\xi)$ have no common complex zero $\xi \neq 0$ with $\operatorname{Im}\xi$ orthogonal to $\partial\Omega$ at x.

THEOREM 3. *For every linear form φ on $L_m{}^p(\Omega)$ which vanishes on the common null space of the P_j there is a function $v \in L_m{}^{p'}(\Omega)$ such that*

$$\varphi(u) = \sum_j \int_\Omega P_j u \overline{P_j v}\, dx \qquad \textit{for all} \qquad u \in L_m{}^p(\Omega)\,.$$

Lions and Magenes [5] have obtained this result by other methods, at least when the set $\{P_j\}$ is the set $\{D_\alpha\}$, $|\alpha| \leqq m$.

We shall not give the proof in detail, but we shall show its connection with Theorems 1 and 2. In addition to these theorems the main fact needed is the inequality

$$\sum_j \int_\Omega |P_j u|^p\, dx + \int_\Omega |u|^p\, dx \geqq c\|u\|^p_{m, L^p(\Omega)}\,. \tag{4.1}$$

It has been shown by Agmon (unpublished) and by Smith (unpublished) that conditions (a) and (b) on the P_j are necessary and sufficient for such an inequality. In [6] there is a proof of the sufficiency when (b) is replaced by the slightly stronger condition: (b′) If $x \in \partial\Omega$, the $p_j(x,\xi)$ have no common complex zero $\xi \neq 0$. If the coefficients of the p_j are constant, (b) and (b′) are equivalent.

SKETCH OF THE PROOF. By virtue of (4.1) the common null space N of the P_j is finite dimensional, and the mapping

$$u \to (P_1 u, P_2 u, \ldots)$$

is an isomorphism of the quotient $L_m{}^p(\Omega)/N$ into a product of spaces $L^p(\Omega)$. Consequently, any linear form φ on $L_m{}^p(\Omega)$ which vanishes on N has the form

$$\varphi(u) = \sum_j \int_\Omega P_j u \bar{f}_j \, dx ,$$

where the f_j are functions in $L^{p'}(\Omega)$. Therefore we must find a function $v \in L_m^{p'}(\Omega)$ such that

$$\sum_j \int_\Omega P_j u \overline{P_j v} \, dx = \sum_j \int_\Omega P_j u \bar{f}_j \, dx \quad \textit{for all} \quad u \in L_m^{p}(\Omega) . \tag{4.2}$$

If we can show in addition that

$$\|v\|_{m, L^{p'}(\Omega)} \leqq c \sum_j |f_j|_{0, L^{p'}(\Omega)} , \tag{4.3}$$

then by continuity we will only have to consider f_j's which lie in a dense set in $L^{p'}(\Omega)$. We will take $f_j \in C_0^\infty(\Omega)$. (For the reason see Remark 4 below).

If D_ν is the normal derivative to $\partial\Omega$ and if boundary operators C_{ij} are chosen so that (P_j^* denoting the adjoint of P_j)

$$\int_\Omega P_j u \bar{w} \, dx = \int_\Omega u \overline{P_j^* w} \, dx + \sum_{i=0}^{m-1} \int_{\partial\Omega} D_\nu^{m-1-i} u \overline{C_{ij} w} \, dx' ,$$

then (4.2) becomes

$$\sum_j P_j^* P_j v = \sum_j P_j^* f_j \ \textit{in} \ \Omega, \quad \textit{and} \quad \sum_j C_{ij} P_j v = 0 \ \textit{on} \ \partial\Omega . \tag{4.4}$$

Hence, if we take

$$A = \sum_j P_j^* P_j, \qquad A_i = \sum_j C_{ij} P_j ,$$

f to be one of the f_j's, and B to be the corresponding P_j^*, then we need a solution to

$$Av = Bf \ \textit{in} \ \Omega \quad \textit{and} \quad A_i v = 0 \ \textit{on} \ \partial\Omega ,$$

$$\|v\|_{m, L^{p'}(\Omega)} \leqq c |f|_{0, L^{p'}(\Omega)} . \tag{4.5}$$

This problem is similar to the one considered in Theorem 1 (with $s = m$ and $m_i = m + i$) except in the following respects. The operators are not homogeneous with constant coefficients, Ω is not a half space, and (4.5) involves the norm $\|\cdot\|_m$ rather than the semi-norm $|\cdot|_m$. However, in the beginning the problem can be localized and transformed to a half space by the usual methods, so that what is really needed is a local version of Theorem 1 which involves the norm $\|\cdot\|_m$. Such a local version is easily established by means of the first part of the proof of Theorem 2. Consideration of the functions of the special form Ag and of the polynomials is unnecessary. The ellipticity of the system $(A, \{A_i\})$ is not difficult to verify, especially if the definition of Hörmander (see § 2) is used.

REMARK 4. If we take $f_j \in C^{\infty}(\bar{\Omega})$, the boundary conditions in (4.4) become

$$\sum_j C_{ij} P_j v = \sum_j C_{ij} f_j .$$

Theorem 2 could be applied here, but it would have to be verified that if $B = P_j{}^*$, then $B_i = C_{ij}$. This is avoided by taking $f_j \in C_0{}^{\infty}(\Omega)$ and using Theorem 1.

REMARK 5. The proof of Theorem 3 is much less elementary than the theorem itself. The right proof should give the theorem for rather general domains Ω — perhaps, for example, those with Lipschitz boundaries.

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