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# ON ALMOST PERIODIC COMPACTIFICATIONS

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In their paper [2], E. M. Alfsen and P. Holm produced the almost periodic compactification of a topological group using topological and group properties only, with no reference to extraneous structures such as spaces of continuous functions. Here, using their general ideas, we consider corresponding problems for semigroups. The main results are in Theorem 4; as corollaries we find the almost periodic compactifications of a semigroup and of a group. The last section shows that the weakly almost periodic compactification cannot be obtained by our method.

Theorem 4 appears as the last of a series of solutions to universal mapping problems (Bourbaki [3, § 3, No. 1]). The first of these is essentially the question of the Stone–Cech compactification, and we include it for completeness. It is produced in the same kind of way as below in [1].

## 1. Preliminaries.

On several occasions we shall use the same method of construction and proof, and we therefore summarize it here. Our notation for uniform structures will follow that of Bourbaki [4].

PROPOSITION 1. (i) *Let $E$ be a set, and let $\mathscr{U}_0$ be a filterbase of subsets of $E \times E$ which contain the diagonal. Define a class $\mathscr{U}$ by the condition*

> *$U \in \mathscr{U}$ if and only if there is a sequence $(V_n)$ of symmetric sets in $\mathscr{U}_0$ so that $V_n \circ V_n \subset V_{n-1} \subset U$ for all $n$.*

*Then $\mathscr{U}$ is the finest uniformity on $E$ whose filter base of vicinities is coarser than $\mathscr{U}_0$.*

(ii) *If $F$ is another set, $\mathscr{V}$ is a uniformity on $F$, and $f$ is a mapping of $E$ into $F$ with the property that if $V \in \mathscr{V}$, $(f \times f)^{-1}(V)$ contains a set of $\mathscr{U}_0$, then $f$ is uniformly continuous from $(E,\mathscr{U})$ to $(F,\mathscr{V})$.*

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PROOF. The proof of (i) is clear except perhaps that if $U_1$ and $U_2$ are in $\mathscr{U}$, so is $U_1 \cap U_2$. This follows from the fact that for any two sets $V$ and $W$ in $\mathscr{U}_0$, $V \cap W$ is in $\mathscr{U}_0$ ,and

$$(V \cap W) \circ (V \cap W) \subset (V \circ V) \cap (W \circ W) .$$

(ii) follows easily, for $(f \times f)^{-1}(\mathscr{V})$ is a uniformity on $E$, and the hypothesis ensures that its filter base of vicinities is coarser than $\mathscr{U}_0$, whence $(f \times f)^{-1}(\mathscr{V}) \subset \mathscr{U}$.

No confusion will arise if we use the

DEFINITION. $\mathscr{U}$ will be called the finest uniformity coarser than $\mathscr{U}_0$.

## 2. Totally bounded uniformities.

Given a family $\mathscr{F}$ of sets, we shall write $V_{\mathscr{F}} = \bigcup_{A \in \mathscr{F}} A \times A$.

LEMMA 1. *Let $(F,\mathscr{V})$ be a totally bounded (i.e. precompact) Hausdorff uniform space. Then for each $V \in \mathscr{V}$ there is a finite covering $\mathscr{F}$ of $F$ by open sets such that $V_{\mathscr{F}} \subset V$.*

PROOF. Let $(\bar{F},\bar{\mathscr{V}})$ be the (compact) completion of $F$, and take $\bar{V} \in \bar{\mathscr{V}}$ such that

$$\bar{V} \cap (F \times F) \subset V .$$

The uniformity $\bar{\mathscr{V}}$ consists of the neighbourhoods of the diagonal in $\bar{F} \times \bar{F}$, and so we can find a finite open covering $\bar{\mathscr{F}}$ of $\bar{F}$ such that $V_{\bar{\mathscr{F}}} \subset \bar{V}$. Then the family $\mathscr{F} = \{A \cap F : A \in \bar{\mathscr{F}}\}$ is an open covering of $F$, and $V_{\mathscr{F}} = V_{\bar{\mathscr{F}}} \cap (F \times F) \subset V$.

THEOREM 1. (i) *Let $(E,\mathscr{T})$ be a topological space. Write*

$$\mathscr{U}_0 = \{V_{\mathscr{F}} : \mathscr{F} \text{ is a finite open covering of } E\} .$$

*Then $\mathscr{U}_0$ is a filter base on $E \times E$. The finest uniformity $\mathscr{U}$ coarser than $\mathscr{U}_0$ is totally bounded, and the topology induced on $E$ by $\mathscr{U}$ is coarser than $\mathscr{T}$.*

(ii) *If $f$ is a continuous mapping of $(E,\mathscr{T})$ into any totally bounded Hausdorff uniform space $(F,\mathscr{V})$, then $f$: $(E,\mathscr{U}) \to (F,\mathscr{V})$ is uniformly continuous.*

PROOF. (i) $\mathscr{U}_0$ satisfies the conditions of Proposition 1 (i), for if

$$\{A_i : i = 1,\ldots,n\} \quad \text{and} \quad \{B_j : j = 1,\ldots,m\}$$

are finite open coverings of $E$, so is $\{A_i \cap B_j\}$ and

$$\bigcup_{i,j} (A_i \cap B_j) \times (A_i \cap B_j) \subset \bigcup_i (A_i \times A_i) \cap \bigcup_j (B_j \times B_j) .$$

Therefore $\mathscr{U}$ is defined. Since each $U$ in $\mathscr{U}$ contains some $V_{\mathscr{F}}$, $E$ can be covered by a finite number of sections of $U$, and each section of $U$ contains an open set. This completes the proof of (i).

(ii) We show that $f$ satisfies the hypothesis of Proposition 1 (ii). Take any $V$ in $\mathscr{V}$. By Lemma 1, we can choose a finite open covering $\mathscr{G}$ of $F$ with $V_{\mathscr{G}} \subset V$; then

$$(f\times f)^{-1}(V) \supset (f\times f)^{-1}(V_{\mathscr{G}}) = (f\times f)^{-1}\Big[\bigcup_{A\in\mathscr{G}} (A\times A)\Big]$$
$$= \bigcup_{A\in\mathscr{G}} [f^{-1}(A)\times f^{-1}(A)] = V_{f^{-1}(\mathscr{G})},$$

and $f^{-1}(\mathscr{G})$ forms a finite open covering of $E$ because $f$ is continuous.

REMARKS (i). We may state the result in the form of a universal mapping theorem as follows: there is a uniformity $\mathscr{U}$ on $E$ with the properties that, the identity map

$$i: \quad (E,\mathscr{T}) \to (E,\mathscr{U})$$

is continuous and that, if

$$f: \quad (E,\mathscr{T}) \to (F,\mathscr{V})$$

is continuous there is a unique uniformly continuous map

$$h: \quad (E,\mathscr{U}) \to (F,\mathscr{V})$$

such that $f = h\circ i$. It is then clear that the completion of $(E,\mathscr{U})$ is just the Stone–Cech compactification of $(E,\mathscr{T})$.

(ii) In order that $\mathscr{U}_0 = \mathscr{U}$, it is necessary and sufficient that for each pair $A$ and $B$ of closed subsets of $(E,\mathscr{T})$ there are two disjoint open sets $U$ and $V$ such that $A\subset U$ and $B\subset V$ (Bourbaki [5, § 4, example 17]). This condition is just normality without the Hausdorff axiom. I am grateful to the referee for pointing out that the topology $\mathscr{T}$ is completely regular (again except for the Hausdorff axiom) if and only if the topology induced on $E$ by $\mathscr{U}$ is just $\mathscr{T}$ (because $\mathscr{U}$ defines the finest such topology coarser than $\mathscr{T}$).

## 3. Jointly uniformly continuous semigroups.

We shall demand that our semigroups contain an identity. This does not effect the general validity of our results, since an identity may always be algebraically adjoined, and we can then ensure that it is topologically irrelevant by making it an isolated point.

LEMMA 2. *Let $(F, \mathscr{V})$ be a semigroup in which multiplication is jointly uniformly continuous, that is, $(x,y) \to xy$ is uniformly continuous. Then for each $V \in \mathscr{V}$ we can find $V_0 \in \mathscr{V}$ with the property that*

$$\bigcup_{a,b \in F} \{(axb, ayb) : (x,y) \in V_0\} \subset V .$$

PROOF. Since $(x,y,z) \to xyz$ is uniformly continuous,

$$V_0 = \{(x,y) : (axb, ayb) \in V \text{ for all } a, b \in F\}$$

is a vicinity for $\mathscr{V}$.

In the same way as Theorem 1, our next result is a universal mapping theorem.

THEOREM 2. (i) *Let $E$ be a semigroup, and $\mathscr{U}$ a uniformity on $E$. For each $U \in \mathscr{U}$, write*

$$U' = \bigcup_{a,b \in E} \{(axb, ayb) : (x,y) \in U\} ,$$

*and let $\mathscr{U}_0'$ be the set of all such $U'$. Then $\mathscr{U}_0'$ is a filter base. The finest uniformity $\mathscr{U}'$ coarser than $\mathscr{U}_0'$ is also coarser than $\mathscr{U}$, and multiplication is jointly uniformly continuous in $\mathscr{U}'$.*

(ii) *Any uniformly continuous homomorphism $f$ of $(E, \mathscr{U})$ into any jointly uniformly continuous semigroup $(F, \mathscr{V})$ is also uniformly continuous with respect to $\mathscr{U}'$.*

PROOF. (i) $\mathscr{U}_0'$ is a filter base because if $U_1$ and $U_2$ are in $\mathscr{U}$, $U_1 \cap U_2$ is in $\mathscr{U}$, and $(U_1 \cap U_2)' \subset U_1' \cap U_2'$. So $\mathscr{U}'$ is defined, it is coarser than $\mathscr{U}$ because $E$ contains an identity, and multiplication is jointly uniformly continuous in $\mathscr{U}'$ because if $(x,y)$ and $(z,t)$ are in $U'$ (where $U'$ has the form given in the statement of the theorem), then also $(xz, yz)$ and $(yz, yt)$ are in $U'$, whence $(xz, yt) \in U' \circ U'$.

(ii) Given $V$ in $\mathscr{V}$, choose $V_0$ according to Lemma 2, and then we can choose $U$ in $\mathscr{U}$ so that $(f \times f)(U) \subset V_0$ since $f$ is $\mathscr{U}$-uniformly continuous. Then since $f$ is a homomorphism, $(f \times f)(U') \subset V$, so that $f$ satisfies the condition of Proposition 1 (ii), and is therefore $\mathscr{U}'$-uniformly continuous.

## 4. Separately uniformly continuous semigroups.

Multiplication in a semigroup is said to be separately (uniformly) continuous if for each $z$ both the map $x \to xz$ and the map $x \to zx$ are (uniformly) continuous. The results of this section parallel those of the last.

Lemma 3. *Let $(F,\mathscr{V})$ be a separately uniformly continuous semigroup. Then for each $V \in \mathscr{V}$ we can find a mapping $(a,b) \to V(a,b)$ of $F \times F \to \mathscr{V}$ with the property that*

$$\bigcup_{a,b\in F} \{(axb,ayb):\ (x,y) \in V(a,b)\} \subset V .$$

Proof. Since $x \to axb$ is uniformly continuous for each pair $(a,b)$,

$$V(a,b) = \{(x,y):\ (axb,ayb) \in V\}$$

is a vicinity for $\mathscr{V}$. The mapping $(a,b) \to V(a,b)$ satisfies our conditions.

Theorem 3. (i) *Let $E$ be a semigroup, and $\mathscr{U}$ a uniformity on $E$. For each mapping $(a,b) \to U(a,b)$ of $E \times E$ into $\mathscr{U}$, write*

$$U' = \bigcup_{a,b\in E} \{(axb,ayb):\ (x,y) \in U(a,b)\} ,$$

*and let $\mathscr{U}_0'$ be the set of all such $U'$. Then $\mathscr{U}_0'$ is a filter base. The finest uniformity $\mathscr{U}'$ coarser than $\mathscr{U}_0'$ is also coarser than $\mathscr{U}$, and multiplication is separately uniformly continuous in $\mathscr{U}'$.*

(ii) *Any uniformly continuous homomorphism $f$ of $(E,\mathscr{U})$ into any separately uniformly continuous semigroup $(F,\mathscr{V})$ is also uniformly continuous with respect to $\mathscr{U}'$.*

Proof. (i) To show that whenever $U_1'$ and $U_2'$ are in $\mathscr{U}_0'$, $U_1' \cap U_2'$ contains a set of $\mathscr{U}_0'$, we use the mapping

$$(a,b) \to U_1(a,b) \cap U_2(a,b) .$$

Then, as in Theorem 2, $\mathscr{U}'$ is a uniformity coarser than $\mathscr{U}$. We indicate how to prove that multiplication is separately uniformly continuous by showing that, for a given $z$ in $E$, the mapping $x \to zx$ is uniformly continuous.

In fact, we can apply Proposition 1 (ii); it is enough to show that the inverse image of a given $U$ in $\mathscr{U}'$ under the mapping $x \to zx$ contains a set of $\mathscr{U}_0'$. Choose some $U'$ in $\mathscr{U}_0'$ which is contained in $U$; write $W(a,b) = U(za,zb)$, and use the mapping $(a,b) \to W(a,b)$ to form a set $W'$ in $\mathscr{U}_0'$. Then if $(x,y) \in W'$, we can find $(a,b) \in E \times E$ and $(u,v) \in W(a,b)$ so that $(x,y) = (aub,avb)$; but then

$$(zx,zy) = (zaub,zavb) \quad \text{and} \quad (u,v) \in W(a,b) = U(za,zb)$$

so that $(zx,zy) \in U' \subset U$, and we have finished.

(ii) We may use the proof of Theorem 2 (ii) after replacing "Lemma 2" by "Lemma 3".

REMARKS. (i) If $(E,\mathscr{T})$ is a topological semigroup with separately continuous multiplication, and $\mathscr{U}$ is the uniformity produced on $E$ by Theorem 1, then $\mathscr{U}'=\mathscr{U}$ in Theorem 3. We have only to show that multiplication is separately uniformly continuous in $\mathscr{U}$, and this follows immediately from Theorem 1 (ii) on taking $(F,\mathscr{V})$ to be $(E,\mathscr{U})$ and considering the mappings $x\to zx$ and $x\to xz$.

(ii) The constructions in Theorems 2 and 3 are particular cases of a general process. Let $\mathscr{A}$ be any collection of mappings of $E\times E$ into $\mathscr{U}$, and write

$$\mathscr{U}_0' = \bigcup_{\alpha\in\mathscr{A}} \{ \bigcup_{a,b\in E} (axb,ayb) : (x,y)\in\alpha(a,b)\} .$$

Then we are in the situation of Theorem 3 (resp. Theorem 4) if we take $\mathscr{A}$ to be the set of all constant mappings (resp. all mappings).

## 5. Compactifications and precompactifications.

THEOREM 4. *Let $E$ be a semigroup and $\mathscr{T}$ a topology on $E$. Then there exist a totally bounded Hausdorff semigroup $(E^*,\mathscr{U}^*)$ in which multiplication is jointly uniformly continuous (resp. separately uniformly continuous) and a continuous mapping*

$$i:\quad (E,\mathscr{T})\to(E^*,\mathscr{U}^*)$$

*with the property that if $f$ is any continuous homomorphism of $(E,\mathscr{T})$ into any totally bounded Hausdorff semigroup $(F,\mathscr{V})$ with jointly uniformly continuous (resp. separately uniformly continuous) multiplication, there is a unique uniformly continuous homomorphism*

$$h:\quad (E^*,\mathscr{U}^*)\to(F,\mathscr{V})$$

*such that $f=h\circ i$.*

PROOF. Starting with $(E,\mathscr{T})$, we construct on $E$ first the uniformity $\mathscr{U}$ provided by Theorem 1, and then the uniformity $\mathscr{U}'$ provided by Theorem 2 (resp. Theorem 3); we then take $(E^*,\mathscr{U}^*)$ to be the Hausdorff space associated with $(E,\mathscr{U}')$, and $i$ to be given by the natural map

$$(E,\mathscr{U}')\to(E^*,\mathscr{U}^*) .$$

It will follow that $(E^*,\mathscr{U}^*)$ is a semigroup with the desired continuity properties if we can prove that the equivalence relation $i(x)=i(y)$, that is,

$$(x,y)\in\bigcap_{U'\in\mathscr{U}'} U' ,$$

is compatible with multiplication in $E$. Take any couples

$$(x,y) \text{ and } (z,t) \quad\text{in}\quad \bigcap_{U'\in\mathscr{U}'} U';$$

we shall show that, given any $U' \in \mathscr{U}'$, $(xz,yt) \in U'$, which is enough. Choose $W \in \mathscr{U}'$ with $W \circ W \subset U'$. If multiplication is separately uniformly continuous (and this case obviously includes the jointly continuous one), we can find $W_1 \in \mathscr{U}'$ such that $(u,v) \in W_1$ implies $(uz,vz) \in W$, and $W_2 \in \mathscr{U}'$ such that $(u,v) \in W_2$ implies $(yu,yv) \in W$. Now $(x,y) \in W_1$ and $(z,t) \in W_2$; therefore both $(xz,yz)$ and $(yz,yt)$ lie in $W$; but then

$$(xz,yt) \in W \circ W \subset U' .$$

It is now immediate from Theorems 1 and 2 (resp. 3) that $(E^*,\mathscr{U}^*)$ has the required universal factorization property.

COROLLARY 1. *Let $(E,\mathscr{T})$ be a semigroup with a topology. Then there is a jointly (uniformly) continuous compact semigroup $(\hat{E},\hat{\mathscr{U}})$ and a continuous map*

$$i: \quad (E,\mathscr{T}) \to (\hat{E},\hat{\mathscr{U}})$$

*such that if $f$ is any continuous homomorphism of $(E,\mathscr{T})$ into any jointly (uniformly) continuous compact semigroup $(F,\mathscr{V})$, there is a (uniformly) continuous homomorphism*

$$h: \quad (\hat{E},\hat{\mathscr{U}}) \to (F,\mathscr{V})$$

*such that $f = h \circ i$.*

PROOF. We first point out that the completion $(\hat{S},\hat{\mathscr{W}})$ of a jointly uniformly continuous semigroup $(S,\mathscr{W})$ is again such a semigroup; for the uniformly continuous mapping $(x,y) \to xy$ of

$$(S \times S, \mathscr{W} \times \mathscr{W}) \quad \text{into} \quad (S,\mathscr{W})$$

has a unique uniformly continuous extension to

$$(\hat{S} \times \hat{S}, \hat{\mathscr{W}} \times \hat{\mathscr{W}}) \quad \text{into} \quad (\hat{S},\hat{\mathscr{W}}) .$$

In particular, in the jointly continuous case of Theorem 4, we can replace the precompact semigroup $(F,\mathscr{V})$ by its compact completion. Then the completion $(\hat{E},\hat{\mathscr{U}})$ of the semigroup $(E^*,\mathscr{U}^*)$ provided by this theorem is also a compact semigroup, and any uniformly continuous homomorphism of $(E^*,\mathscr{U}^*)$ into $(F,\mathscr{V})$ has a unique extension to $(\hat{E},\hat{\mathscr{U}})$ which is again a homomorphism.

REMARK. $(\hat{E},\hat{\mathscr{U}})$ is known as the almost periodic compactification of $(E,\mathscr{T})$.

COROLLARY 2. *If $E$ is a group, $(\hat{E},\hat{\mathscr{U}})$ is a topological group.*

PROOF. The homomorphic image $i(E)$ of $E$ is dense in $\hat{E}$ and a group. Multiplication is jointly continuous in $\hat{E}$ by construction; we show that each $x \in \hat{E}$ has an inverse and that inversion is continuous by adapting slightly a proof in [7] (Lemma 2 on page 813).

Suppose $x \in \hat{E}$, and let $\mathscr{F}$ be any filter base of subsets of $i(E)$ which converges to $x$ in $\hat{E}$. Then the filter base $\mathscr{F}^{-1} = \{A^{-1} : A \in \mathscr{F}\}$ has cluster points in the compact set $\hat{E}$. Let $y$ be any one of these, and let $\mathscr{G}$ be a refinement of $\mathscr{F}^{-1}$ consisting of subsets of $i(E)$ which converges to $y$. Then $\mathscr{G}^{-1}$ is a refinement of $\mathscr{F}$, and therefore converges to $x$. Since multiplication is jointly continuous, $\mathscr{G}^{-1}\mathscr{G}$ converges to $xy$; but every set in $\mathscr{G}^{-1}\mathscr{G}$ contains the identity $e$ of $E$, so $xy = e$. Similarly, $yx = e$. Therefore $y$ must be the unique inverse of $x$, and since $y$ was any cluster point of $\mathscr{F}^{-1}$, we deduce that $\mathscr{F}^{-1}$ converges to $y$.

Now that we know that $\hat{E}$ is a group, we may apply the same argument to deduce that if a filter $\mathscr{F}$ of subsets of $\hat{E}$ converges to $x$, $\mathscr{F}^{-1}$ converges to $x^{-1}$, i.e. inversion is continuous.

## 6. Remarks on the weakly almost periodic compactification.

We might hope that by carrying out procedures modelled on Corollary 1 for the separately continuous case, we would obtain the weakly almost periodic compactification (i.e. a compact semigroup $(\hat{E}, \hat{\mathscr{U}})$ for which the statement of Corollary 1 holds with "jointly continuous" replaced everywhere by "separately continuous") [6]. However, this is not so, mainly because the completion of a separately uniformly continuous semigroup need not be a semigroup of the same form.

A simple example of this is provided by the real line $R$ with the uniformity induced by the usual two-point compactification (the extended real line $\bar{R}$). Addition is separately uniformly continuous in $R$, but it is easy to see that no way of defining addition for the two infinities makes the operation separately continuous in $\bar{R}$.

We shall use this example again (after Proposition 2 below) to show that the completion of the separately uniformly continuous semigroup produced by Theorem 4 need not be such a semigroup.

LEMMA 4. *Let $S$ be a compact semigroup with separately continuous multiplication, let $T$ be a semigroup with a compact topology, and let $h$ be a continuous homomorphism of $S$ onto $T$. Then multiplication is separately continuous in $T$.*

PROOF. For each $x$ in $S$, write $h(x) = x'$; since $h$ is onto $T$, every element of $T$ is of this form. We shall show that for a given $a$ in $S$, the mapping $\alpha' : x' \to a'z'$ is continuous. Let $\alpha : x \to ax$ for some $a$ with $h(a) = a'$. Then

$$h(\alpha(x)) = h(ax) = h(a)h(x) = a'h(x) = \alpha'(h(x)) .$$

So, for any subset $F$ of $T$,

$$\alpha'^{-1}(F) = h(h^{-1}(\alpha'^{-1}(F))) = h(\alpha^{-1}(h^{-1}(F))) .$$

If $F$ is closed, $\alpha^{-1}(h^{-1}(F))$ is a closed subset of $S$ (because the mapping $x \to h(\alpha(x))$ is continuous) and is therefore compact; since $h$ is continuous, it follows that $\alpha'^{-1}(F)$ is compact, and hence closed. So $\alpha'$ is continuous.

PROPOSITION 2. *Let $S$ and $T$ be two semigroups with totally bounded Hausdorff uniformities and separately uniformly continuous multiplications, and let $h$ be a uniformly continuous homomorphism of $S$ onto $T$. If in the completion $\hat{S}$ of $S$ we can define a separately continuous multiplication which extends the multiplication in $S$, we can also define a separately continuous multiplication in $\hat{T}$ which extends the multiplication in $T$.*

PROOF. The mapping $h$ has a unique continuous extension $\hat{h}$ mapping $\hat{S}$ into $\hat{T}$; but $\hat{h}(\hat{S})$ is compact and contains $T$, so $\hat{h}(\hat{S}) = \hat{T}$.

We see from Lemma 4 that any multiplication in $\hat{T}$ which makes $\hat{h}$ a homomorphism is separately continuous. We have therefore finished if we can define $x'y' = (xy)'$, where $x' = h(x)$, and this is permissible if the equivalence relation $\hat{h}(x) = \hat{h}(y)$ is compatible with the multiplication in $\hat{S}$. So we must show that if both $\hat{h}(x) = \hat{h}(y)$ and $\hat{h}(z) = \hat{h}(t)$, then $\hat{h}(zx) = \hat{h}(ty)$.

In fact we shall only show that $\hat{h}(zx) = \hat{h}(zy)$; a repetition of the argument then gives $\hat{h}(zy) = \hat{h}(ty)$ to complete the proof. First, take $z$ in $S$. The mapping $x' \to z'x'$ is uniformly continuous on $T$, and so has a unique extension $x' \to z'(x')$ from $\hat{T}$ into $\hat{T}$. The mapping

$$x \to \hat{h}(x) \to z'(\hat{h}(x))$$

of $S$ into $\hat{T}$ is uniformly continuous, and so has a unique extension to $\hat{S}$; but it may also be considered as the composition of the mappings

$$x \to \hat{h}(z)\hat{h}(x) = h(z)h(x) = h(zx)$$

of $S$ into $T$, and the injection of $T$ into $\hat{T}$. Since multiplication is separately continuous in $\hat{S}$, this mapping has the continuous extension $x \to \hat{h}(zx)$ to $\hat{S}$. It follows from the uniqueness of the extensions that $\hat{h}(zx) = z'(\hat{h}(x))$. So for $z$ in $S$, we have that

$$\hat{h}(x) = \hat{h}(y) \quad \text{implies} \quad \hat{h}(zx) = \hat{h}(zy) .$$

Since multiplication is separately uniformly continuous in $\hat{S}$, the mapping $z \to \hat{h}(zx)$ is continuous on $\hat{S}$ for each $x$ in $\hat{S}$. So if $\hat{h}(zx) = \hat{h}(zy)$ for all $z$ in the dense subspace $S$ of $\hat{S}$, the equality also holds for all $z$ in $\hat{S}$, and the proof is finished.

Now consider $R$ with any topology $\mathscr{T}$ finer than the usual one, and form the separately continuous $(R^*, \mathscr{U}^*) = (R, \mathscr{U}^*)$ as in Theorem 4. The identity map of $R$ into $\bar{R}$ (see above) must be uniformly continuous in $\mathscr{U}^*$, and therefore by Lemma 4, if the completion of $(R^*, \mathscr{U}^*)$ is a separately uniformly continuous semigroup, so must $\bar{R}$ be; but we have seen that this is not so.

Finally, we remark that if we ask only for a precompact separately uniformly continuous semigroup $(E^*, \mathscr{U}^*)$ such that every continuous homomorphism of $(E, \mathscr{T})$ into any *compact* separately continuous semigroup can be factored through $(E^*, \mathscr{U}^*)$, we may not obtain a unique result; for Theorem 4 provides one solution, and the weakly almost periodic compactification provides another; however, we can assert that every solution lies (in an obvious sense) between these two.

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