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THE AXIOM OF COMPREHENSION IN INFINITE VALUED LOGIC

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Introduction.

In this paper, we improve a result of Skolem [10] which states that the set Σ_1 of sentences

$$\forall x_1 \ldots x_n \, \exists y \, \forall t \, \big(t \in y \leftrightarrow U(t, x_1, \ldots, x_n)\big) ,$$

where U is an open (quantifier-free) formula, is consistent in the infinite valued logic of Łukasiewicz, hereafter denoted by the symbol Ł. We shall show that two other sets of sentences Σ_2 and Σ_3 are also consistent. Σ_2 is the set of sentences

$$\exists y \, \forall t \, \big(t \in y \leftrightarrow U(t)\big) ,$$

where U contains no free variable other than t but may contain arbitrary quantifiers, and Σ_3 is a set of sentences

$$\forall x_1 \ldots x_n \, \exists y \, \forall t \, \big(t \in y \leftrightarrow U(t, x_1, \ldots, x_n)\big) ,$$

where U may contain bound variables of a specified sort (described later). Thus we have shown that the axiom of comprehension without parameters is consistent in Ł (Theorem 2.1) and that the set of sentences Σ_1 can be considerably augmented and still remain consistent (Theorem 2.2). Whether or not the full axiom of comprehension is consistent remains an open problem. We emphasize that the axiom of comprehension is clearly inconsistent in every finite valued logic, including of course the two valued logic. Hence the question of its consistency is only open in the logic Ł.

Actually, we shall prove the consistency of Σ_2 and Σ_3 under two further assumptions about them. One is that we shall assume Ł has an identity symbol $\equiv$, and the other is that we shall assume the variable y

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[1] The author had several exchanges of correspondence with Professor Skolem about the results in this paper shortly before the untimely death of the latter. This paper is dedicated to the memory of Thoralf Skolem. This paper was written while the author held a N.S.F. Senior Postdoctoral Fellowship at The Institute for Advanced Study.

may also occur free in the formulas U. These two additional assumptions, if anything, make the proofs more difficult. As we shall see in § 3, the assumption that Ł has identity will be useful in further work along these lines. As for the fact that y may occur free in U, it follows simply as a consequence of our method of proof, and it serves to illustrate the expected difference between the two valued logic and Ł. In any case, the consistency of Σ_2 and Σ_3 under these assumptions certainly implies that Σ_2 and Σ_3 are consistent without these assumptions.

Our method of proof depends on three things: A generalization of Skolem's method of [10]; the Compactness Theorem (Theorem A of § 2) for the logic Ł [3]; and the Brouwer Fixed Point Theorem for the space $[0, 1]^m$. It is a fortunate coincidence that all three devices are available to the author.

We have put all the preliminary definitions and results in §1. The proofs of the main results are in § 2. Section 3 ends the paper with a brief discussion of generalizations and problems.

1. Preliminaries.

We recall some simple facts about the logic Ł (see, for instance, [6], [8], [2], [3]). The symbols of Ł are the following:

two binary predicate constant symbols $\equiv$ and $\in$;
an infinite number of individual variables, $v_0, v_1, \ldots$;
propositional connectives $\neg$ (unary) and $\rightarrow$ (binary);
quantifier symbol $\exists$;
improper symbols (,).

Formulas of Ł are constructed in the usual way. The sequences of symbols

$$x \equiv y, \qquad x \in y,$$

where x, y are individual variables, are the atomic formulas of Ł. If U, V are formulas, then $\neg U$ and $U \rightarrow V$ are formulas. If U is a formula and x is an individual variable, then $(\exists x)U$ is a formula. We can define, as usual, the notion of an occurrence of a variable x in a formula U, and the notions of free and bound variables of a formula U. We write $U(x_1, \ldots, x_n)$ to indicate that the free variables of U are among the variables $x_1, \ldots, x_n$. An open formula is a formula with no bound variables. A formula with no free variables is called a sentence. So far there is no difference between the formulas and sentences of Ł and the formulas and sentences of the two valued first-order predicate logic with identity. It is only in the interpretation of $\in$, $\equiv$, $\rightarrow$, and $\neg$ that the difference shows up.

Throughout this paper we let X denote the closed real unit interval $[0,1]$. We endow X with the natural topology on $[0,1]$; thus X becomes a compact Hausdorff topological space. Let $\neg$ and $\rightarrow$ be the following functions on X and X^2: for x,y in X,

$$\neg\,(x) = 1-x\,,$$
$$\rightarrow(x,y) = \min(1,1-x+y)\,.$$

The function $\exists$ is defined on the set of all non-empty subsets of X as follows: for $\emptyset \neq Y \subseteq X$,

$$\exists(Y) = \sup Y\,.$$

It shall be clear from context whether the symbols $\neg$, $\rightarrow$, $\exists$ are to stand for connectives and quantifier or for functions defined above. Notice that the functions $\neg$ and $\rightarrow$ are continuous on X and X^2, and the function $\exists$ is continuous with respect to a reasonable topology on the set of all non-empty subsets of X. Each function $\neg$, $\rightarrow$, and $\exists$ has range equal to X.

A model $M=\langle S,e\rangle$ for Ł is a pair $\langle S,e\rangle$ where S is a non-empty set and e is a mapping of $S\times S$ into X. The function e may be regarded as a subset of $(S\times S)\times X$. A partial model $M=\langle S,e\rangle$ is a pair $\langle S,e\rangle$ where S is a non-empty set and e is a partial mapping of $S\times S$ into X not necessarily defined for all pairs $\langle a,b\rangle$, a, b in S. A (partial) model $M_1=\langle S_1,e_1\rangle$ is a (partial) submodel of a (partial) model $M_2=\langle S_2,e_2\rangle$, in symbols $M_1\subseteq M_2$, if $S_1\subseteq S_2$ and $e_1=e_2\cap((S_1\times S_1)\times X)$. The relation $\subseteq$ among (partial) models is clearly reflexive and transitive.

Given a sequence of models $M_0\subseteq M_1\subseteq\ldots\subseteq M_k\subseteq\ldots$, the union of the sequence $M=\bigcup_k M_k$ is defined as follows:

$$S = \bigcup_k S_k, \quad e = \bigcup_k e_k, \quad \text{and} \quad M = \langle S,e\rangle\,.$$

We see that for a,b in S_k, $e(a,b)=e_k(a,b)$, and that $M_k\subseteq M$ for each k.

For each formula $U(x_1,\ldots,x_n)$, model $M=\langle S,e\rangle$, and interpretation of $x_1,\ldots,x_n$ as elements $a_1,\ldots,a_n$ of S, we define a real number

$$U(a_1,\ldots,a_n)[M]$$

in X by induction on the formulas as follows: (When the model M is understood, we sometimes drop the symbols $[M]$ and write $U(a_1,\ldots,a_n)$ for $U(a_1,\ldots,a_n)[M]$.)

Definition. (i) If U is the atomic formula $x_i\equiv x_j$, $1\leqq i$, $j\leqq n$, define

$$U(a_1,\ldots,a_n) = 1 \quad \text{if} \quad a_i = a_j\,,$$
$$U(a_1,\ldots,a_n) = 0 \quad \text{if} \quad a_i \neq a_j\,.$$

(ii) If U is the atomic formula $x_i \in x_j$, $1 \leqq i, j \leqq n$, define

$$U(a_1,\ldots,a_n) = e(a_i,a_j) .$$

(iii) If U is the formula $\neg V$, define

$$U(a_1,\ldots,a_n) = \neg \big(V(a_1,\ldots,a_n)\big) .$$

(iv) If U is the formula $V \to W$, define

$$U(a_1,\ldots,a_n) = \to \big(V(a_1,\ldots,a_n),\ W(a_1,\ldots,a_n)\big) .$$

(v) If U is the formula $(\exists x_{n+1})V(x_1,\ldots,x_{n+1})$, define

$$U(a_1,\ldots,a_n) = \exists\big(\{V(a_1,\ldots,a_n,b) : b \text{ in } S\}\big) .$$

In this way $U(a_1,\ldots,a_n)[M]$ is a uniquely defined real number in X. In case U is a sentence, $U(a_1,\ldots,a_n)[M]$ is independent of the sequence $a_1,\ldots,a_n$, and we shall denote it simply by $U[M]$. In case U is an open formula and $M_1 \subseteqq M_2$,

$$U(a_1,\ldots,a_n)[M_1] = U(a_1,\ldots,a_n)[M_2]$$

for any interpretation of $x_1,\ldots,x_n$ as elements $a_1,\ldots,a_n$ of S_1.

From the truth functions $\neg$ and $\to$, we can define by compositions the following truth functions: for x,y in X,

$$\min(x,y) = \neg [\to (x, \neg [\to (x,y)])] ,$$
$$\max(x,y) = \neg \min(\neg x, \neg y) .$$

We let $\wedge$ and $\vee$ denote the propositional connectives corresponding to the functions min and max, and we let $U \leftrightarrow V$ denote the formula $(U \to V) \wedge (V \to U)$. Thus, for formulas U and V, model M, and interpretation of $x_1,\ldots,x_n$ as $a_1,\ldots,a_n$ of S, we have

$$(U \wedge V)(a_1,\ldots,a_n) = \min\big(U(a_1,\ldots,a_n),\ V(a_1,\ldots,a_n)\big) ,$$
$$(U \vee V)(a_1,\ldots,a_n) = \max\big(U(a_1,\ldots,a_n),\ V(a_1,\ldots,a_n)\big) ,$$
$$(U \leftrightarrow V)(a_1,\ldots,a_n) = 1 - |U(a_1,\ldots,a_n) - V(a_1,\ldots,a_n)| .$$

Consequently,

$$(U \wedge V)(a_1,\ldots,a_n) = 1 \text{ if and only if } U(a_1,\ldots,a_n) = 1 \text{ and } V(a_1,\ldots,a_n) = 1 ,$$
$$(U \vee V)(a_1,\ldots,a_n) = 1 \text{ if and only if } U(a_1,\ldots,a_n) = 1 \text{ or } V(a_1,\ldots,a_n) = 1 ,$$
$$(U \leftrightarrow V)(a_1,\ldots,a_n) = 1 \text{ if and only if } U(a_1,\ldots,a_n) = V(a_1,\ldots,a_n) .$$

We define $(\forall x)U$ as the formula $\neg (\exists x) \neg U$, and we see easily that for a formula $U(x_1,\ldots,x_{n+1})$, we have

$$(\forall x_{n+1})U(a_1,\ldots,a_n) = \inf\{U(a_1,\ldots,a_{n+1}): a_{n+1} \text{ in } S\}\,.$$

Thus,

$$(\exists x_{n+1})U(a_1,\ldots,a_n) = 1 \text{ if and only if for every } k \geqq 1\,,$$
$$U(a_1,\ldots,a_{n+1}) \geqq (1-1/k) \text{ for some } a_{n+1} \text{ in } S\,,$$

and

$$(\forall x_{n+1})U(a_1,\ldots,a_n) = 1 \text{ if and only if for all } a_{n+1} \text{ in } S\,,$$
$$U(a_1,\ldots,a_{n+1}) = 1\,.$$

We say that a sentence U holds on a model M if $U[M]=1$. A model M satisfies a set Σ of sentences if each sentence U of Σ holds on M. If M satisfies Σ, we say that Σ is satisfied by M. A set Σ of sentences of Ł is said to be consistent if Σ is satisfied by some model for Ł. The precise meaning of the result of Skolem in [10] cited in the introduction is now clear: *In Ł (without identity) the set Σ_1 is consistent.*

We need to know the following form of the Compactness Theorem for the logic Ł.

THEOREM A. *For a set Σ of sentences to be consistent it is necessary and sufficient that every finite subset of Σ be consistent.*

This theorem is not directly connected with the subject at hand and so we shall not give a proof of it here. Theorem A can be found in the abstracts [3] and a more general version of it can be found in [5]. (A theorem embodying both Theorems *A* and *B* (below) will be proved in a forthcoming monograph by Chang and Keisler.) Theorem A is needed in an essential way for the proof of Theorem 2.1. The proof of our Theorem 2.2 can be simplified somewhat by the use of a stronger version of Theorem A, namely:

THEOREM B. *For a set Σ of sentences to be consistent it is necessary and sufficient that for every finite subset Σ' of Σ and every $k \geqq 1$, there exists a model M such that*

$$U[M] \geqq 1-1/k \quad \textit{for each} \quad U \textit{ in } \Sigma'\,.$$

Our proof of Theorem 2.2 will require neither Theorems A nor B. It is not known if a proof of Theorem 2.1 can also be given directly and not via Theorem A. We might also mention that the following is also true:

THEOREM C. *If a set Σ of sentences is consistent then it is satisfied by a countable (finite or infinite) model.*

Thus, in applying Theorem A to Theorem 2.1, we can always be sure of finding a denumerable model for Σ_2.

We continue the preliminaries by introducing some auxiliary but necessary notions.

A formula $U(x_1,\ldots,x_n)$ is said to be valid if for all models M and all interpretations of $x_1,\ldots,x_n$ as elements $a_1,\ldots,a_n$ of S, $U(a_1,\ldots,a_n)[M] = 1$. We use the symbols $\models U$ to denote that U is valid. This notion is, of course, the semantical counterpart of the syntactical notion of provability (see, however, [1], [2], [4], [8], [9] and § 3 for the difficulties encountered in syntax). It turns out that most of the familiar rules and axioms of two valued first-order predicate logic remain true. In particular the following semantical results concerning the symbol $\models$ can be easily proved.

Rule of substitution for bound variables. Let U be a formula and let U' be obtained from U by replacing a bound variable of U by a variable not occurring in U. Then $\models U \leftrightarrow U'$.

Rule of substitution for formulas. Suppose that $\models U \leftrightarrow U'$. Let V and V' be such that V and V' are alike except that some occurrences of U in V are replaced by occurrences of U' in V'. Then $\models V \leftrightarrow V'$.

Prenex normal form. Let U be a formula. There exists a formula $V = QW$ where W is an open formula and Q is a (possibly empty) string of quantifiers ($\exists$ or $\forall$) such that $\models U \leftrightarrow V$.

Substitutivity of the identity. Let U and V be such that U and V are alike except that U contains free ocurrences of the variable x wherever V contains free occurrences of the variable y. Then $\models x \equiv y \rightarrow (U \leftrightarrow V)$. (This last rule is a consequence of Definition (i).)

For each formula U in prenex normal form and variables $z_1,\ldots,z_k$ not occurring in U, we define the k-transform of U, written $T_k(U)$, by induction on the number of quantifiers occurring in U. If U is an open formula, $T_k(U) = U$. If $U = (\exists x)V$, then

$$T_k(U) = T_k{}^1(V) \vee \ldots \vee T_k{}^k(V),$$

where $T_k{}^j(V)$ for $1 \leqq j \leqq k$ is obtained from $T_k(V)$ by replacing the variable x by the variable z_j. If $U = (\forall x)V$, then

$$T_k(U) = T_k{}^1(V) \wedge \ldots \wedge T_k{}^k(V),$$

where the $T_k{}^j(V)$, $1 \leqq j \leqq k$, has the same meaning as before. We see by an easy induction that if U is a formula in prenex normal form, then $T_k(U)$ is an open formula. Furthermore, if $x_1,\ldots,x_n$ are the only free variables of U, then $x_1,\ldots,x_n$, $z_1,\ldots,z_k$ are the only free variables of $T_k(U)$. The following lemma can be established by a simple induction on the number of quantifiers in U.

LEMMA 1.1. *Let* $U(x_1,\ldots,x_n)$ *be a formula in prenex normal form, let* $M=\langle S,e\rangle$ *be a model with exactly* k *elements,* $S=\{a_1,\ldots,a_k\}$, *and let* $z_1,\ldots,z_k$ *be interpreted as the elements* $a_1,\ldots,a_k$ *of* S. *Then, under any interpretation of* $x_1,\ldots,x_n$ *as* $b_1,\ldots,b_n$ *of* S,

$$U(b_1,\ldots,b_n) = T_k(U)(b_1,\ldots,b_n,a_1,\ldots,a_k) .$$

For each open formula $U(z_1,\ldots,z_k)$, we define the function $P(U)$ of (at most) $2\cdot k^2$ real variables d_{ij} and e_{ij}, $1\leqq i,\ j\leqq k$, by induction as follows:

$$P(z_i \equiv z_j) = d_{ij}, \quad 1 \leqq i,j \leqq k ,$$
$$P(z_i \in z_j) = e_{ij}, \quad 1 \leqq i,j \leqq k ,$$
$$P(\neg\, U) = \neg\, P(U) ,$$
$$P(U \to V) = \to \big(P(U), P(V)\big) .$$

The function $P(U)$ is a continuous mapping of $X^{2\cdot k^2}$ into X. Since X is a compact Hausdorff space, the function $P(U)$ is uniformly continuous. That is to say, for every positive integer m, there exists a positive integer r such that if

$$|d_{ij}-d'_{ij}| \leqq 1/r, \quad |e_{ij}-e'_{ij}| \leqq 1/r \quad \text{for } 1 \leqq i,j \leqq k ,$$

then

$$|P(U)(d_{ij},e_{ij};\, 1 \leqq i,j \leqq k)-P(U)(d'_{ij},e'_{ij};\, 1 \leqq i,j \leqq k)| \leqq 1/m .$$

Given two sequences $\sigma=\langle a_1,\ldots,a_k\rangle$ and $\sigma'=\langle a'_1,\ldots,a'_k\rangle$ of elements of a model $M=\langle S,e\rangle$, and an open formula $U(z_1,\ldots,z_k)$. We say that the sequences σ and σ' are within $1/r$ modulo the formula U, written symbolically

$$|\sigma-\sigma'| \leqq 1/r \quad (\text{mod}\, U) ,$$

if for every pair $\langle i,j\rangle$, $1\leqq i,\ j\leqq k$, if the atomic formula $z_i\equiv z_j$ occurs in the formula U then

$$a_i = a_j \quad \text{if and only if} \quad a_i' = a_j' ,$$

and if the atomic formula $z_i \in z_j$ occurs in the formula U then

$$|e(a_i,a_j)-e(a_i',a_j')| \leqq 1/r .$$

It should be quite clear that if

$$|\sigma-\sigma'| \leqq 1/r \quad (\text{mod}\, U)$$

then

$$|U(a_1,\ldots,a_k)[M]-U(a_1',\ldots,a_k')[M]| \leqq 1/m .$$

From the function $P(U)$, we define another function $R(U)$ of (at most) k^2 real variables e_{ij}, $1 \leqq i, j \leqq k$, by setting each $d_{ii}=1$ and $d_{ij}=0$ if $i \neq j$. The following lemma is clear.

LEMMA 1.2. *Let $U(z_1,\ldots,z_k)$ be an open formula, let $M=\langle S,e\rangle$ be a model, and let $a_1,\ldots,a_k$ be distinct elements of S. Then*

$$U(a_1,\ldots,a_k) = R(U)\big(e(a_i,a_j);\ 1 \leqq i,j \leqq k\big)\,.$$

Let $U_{ij}(z_1,\ldots,z_k)$, $1 \leqq i, j \leqq k$, be an array of open formulas, let $M=\langle S,e\rangle$ be a partial model, and let $a_1,\ldots,a_k$ be distinct elements of S. A sequence of real numbers t_{ij}, $1 \leqq i, j \leqq k$, is said to be a fixed point of the array U_{ij}, $1 \leqq i, j \leqq k$, with respect to the partial model M and the elements $a_1,\ldots,a_k$ of S, if for $1 \leqq i, j \leqq k$,

$$t_{ij} = e(a_i,a_j) \text{ if } e(a_i,a_j) \text{ is defined}\,,$$
$$t_{ij} = R(U_{ij})(t_{ij};\ 1 \leqq i,j \leqq k) \text{ if } e(a_i,a_j) \text{ is not defined}\,.$$

LEMMA 1.3. *For each array $U_{ij}(z_1,\ldots,z_k)$, $1 \leqq i, j \leqq k$, of open formulas, each partial model $M=\langle S,e\rangle$, and every sequence of distinct elements $a_1,\ldots,a_k$ of S, there exists a fixed point.*

PROOF. If e is defined for all pairs $\langle a_i,a_j\rangle$, $1 \leqq i, j \leqq k$, then the sequence $e(a_i,a_j)$, $1 \leqq i, j \leqq k$, is a fixed point. Suppose, therefore, that e is not defined for some pairs $\langle a_i,a_j\rangle$. Let

$$I = \{\langle i,j\rangle \colon e(a_i,a_j) \text{ is not defined and } 1 \leqq i,j \leqq k\}\,.$$

Let m be the number of elements of I. For $\langle i_0,j_0\rangle$ in I, we denote by $R'(U_{i_0j_0})$ that function of (at most) m variables obtained from the function $R(U_{i_0j_0})$ by replacing the variable e_{ij} by the real number $e(a_i,a_j)$ whenever $\langle i,j\rangle$ is not in I. Consider now the m equations

$$R'(U_{ij})(e_{ij};\, i,j \text{ in } I) = e_{ij}, \quad \langle i,j\rangle \text{ in } I\,,$$

in the m unknowns $e_{ij}, \langle i,j\rangle$ in I. Let F be the continuous mapping of X^m into X^m defined by the m equations

$$F(e_{ij};\, \langle i,j\rangle \text{ in } I)(\langle i,j\rangle) = R'(U_{ij})(e_{ij};\, \langle i,j\rangle \text{ in } I), \quad \langle i,j\rangle \text{ in } I\,.$$

From the Brouwer Fixed Point Theorem, F has a fixed point r_{ij}, $\langle i,j\rangle$ in I. Clearly the sequence

$$t_{ij} = e(a_i,a_j) \quad \text{if} \quad \langle i,j\rangle \text{ is not in } I\,,$$
$$t_{ij} = r_{ij} \qquad \text{if} \quad \langle i,j\rangle \text{ in } I\,,$$

is the required fixed point for the array U_{ij}.

LEMMA 1.4. *Let t_{ij}, $1 \leqq i, j \leqq k$, be a fixed point as in Lemma* 1.3. *Then the function e can be extended to a function e′ defined on all pairs $\langle a_i, a_j \rangle$, $1 \leqq i, j \leqq k$, in such a way that for each $\langle i,j \rangle$ in I,*

$$U_{ij}(a_1,\ldots,a_k)[M'] = e'(a_i,a_j) ,$$

where M' is the partial model $\langle S,e' \rangle$.

PROOF. By Lemmas 1.2 and 1.3.

2. Proofs of the main theorems.

Let Σ_2 be the set of sentences

$$\exists y \, \forall t \, (t \in y \leftrightarrow U(t,y)) ,$$

where U is an arbitrary formula of Ł with at most the variables t and y free.

THEOREM 2.1. *Σ_2 is consistent.*

PROOF. By Theorem A, it is sufficient to show that every finite subset of Σ_2 is consistent. Let, therefore, $U_1(t,y),\ldots,U_k(t,y)$ be formulas of Ł so that the sentences V_j,

$$V_j = \exists y \, \forall t \, (t \in y \leftrightarrow U_j(t,y)) ,$$

form a finite subset of Σ_2. We may assume that each U_j, $1 \leqq j \leqq k$ is already in prenex normal form. We shall prove that there exists a model $M = \langle S,e \rangle$ with exactly k elements, $S = \{1,\ldots,k\}$, such that

$$(1) \qquad V_j[M] = 1 \quad \text{for} \quad 1 \leqq j \leqq k .$$

To prove (1), we first show that in the model M to be constructed

$$(2) \qquad U_j(i,j) = e(i,j)$$

for every interpretation of t,y as elements i,j of S. Let $z_1,\ldots,z_k$ be variables not occurring in any of the formulas U_j, $1 \leqq j \leqq k$. Consider the k-transforms $T_k(U_j)$ for $1 \leqq j \leqq k$. Each transform $T_k(U_j)$ will contain at most the free variables $t,y,z_1,\ldots,z_k$. For each pair $\langle i,j \rangle$, $1 \leqq i, j \leqq k$, define the formula $W_{ij}(z_1,\ldots,z_k)$ by replacing t by z_i and y by z_j in the formula $T_k(U_j)$. Thus, schematically,

$$(3) \qquad W_{ij}(z_1,\ldots,z_k) = T_k(U_j)(z_i,z_j,z_1,\ldots,z_k), \quad 1 \leqq i,j \leqq k .$$

Let $S = \{1,\ldots,k\}$ and let $M = \langle S,e \rangle$ be the partial model where e is not defined at all. Consider the array of open formulas W_{ij}, $1 \leqq i, j \leqq k$. By Lemma 1.3, let the sequence t_{ij}, $1 \leqq i, j \leqq k$, be a fixed point for the array with respect to the partial model M and the distinct elements $1,\dots,k$ of S. Define

$$e(i,j) = t_{ij}, \quad 1 \leqq i,j \leqq k\,.$$

$M = \langle S, e\rangle$ now becomes a model. Under the interpretation of $z_1,\dots,z_k$ as the distinct elements $1,\dots,k$ of S, we have, by Lemma 1.4,

$$W_{ij}(1,\dots,k) = e(i,j), \quad 1 \leqq i,j \leqq k; \tag{4}$$

hence, by (3),

$$T_k(U_j)(i,j,1,\dots,k) = e(i,j), \quad 1 \leqq i,j \leqq k\,. \tag{5}$$

By Lemma 1.1 and (5), we have

$$U_j(i,j) = e(i,j), \quad 1 \leqq i,j \leqq k\,,$$

and (2) is proved. From (2) it follows that

$$\big(i \in j \leftrightarrow U_j(i,j)\big) = 1, \quad 1 \leqq i,j \leqq k\,.$$

Hence

$$\forall t\,\big(t \in j \leftrightarrow U_j(t,j)\big) = 1, \quad 1 \leqq j \leqq k\,,$$

and, finally,

$$\exists y\,\forall t\,\big(t \in y \leftrightarrow U_j(t,y)\big) = 1, \quad 1 \leqq j \leqq k\,.$$

Thus (1) is fulfilled and the theorem is proved.

Let Σ_3 be the set of sentences

$$\forall x_1 \dots x_n\, \exists y\, \forall t\, \big(t \in y \leftrightarrow U(t,y,x_1,\dots,x_n)\big)\,,$$

where U is a formula of Ł with at most the variables $t,y,x_1,\dots,x_n$ free and such that in every atomic formula $u \in v$ of U, if u is a bound variable of U then $u = v$. Putting it in another way, this means no bound variable u of U can occur in the first place of an atomic formula of the form $u \in v$ unless u is already the variable v. Notice that no restriction is placed on the atomic formulas of the form $u \equiv v$. It is clear that Σ_3 contains Σ_1 as a subset.

THEOREM 2.2. *Σ_3 is consistent.*

PROOF. Let S be the set of positive integers arranged in the natural order,

$$S = \{1,\dots,k,\dots\}\,.$$

We shall define three increasing sequences of finite subsets of S,

$$\begin{aligned} S_0 &\subseteqq \dots \subseteqq S_k \subseteqq \dots\,, \\ A_0 &\subseteqq \dots \subseteqq A_k \subseteqq \dots\,, \\ B_0 &\subseteqq \dots \subseteqq B_k \subseteqq \dots\,, \end{aligned}$$

and one increasing sequence of functions,

$$e_0 \subseteqq \cdots \subseteqq e_k \subseteqq \cdots ,$$

satisfying the following conditions:

$$A_0 = \emptyset; \quad B_0 = S_0 = \{1\};$$
$$A_k \cap B_k = \emptyset; \quad A_k \cup B_k = S_k;$$
$$S = \bigcup_k S_k;$$
e_k is defined on $(S_k \times S_k) \cup (S \times B_k)$ taking values in X.

The model we construct is $M = \langle S, e \rangle$ where $e = \bigcup_k e_k$. Let

$C = \{\langle Y, U \rangle : Y$ is a finite (possibly empty) sequence, $Y = \langle b_1, \ldots, b_n \rangle$, of distinct elements of S, and $U = U(t, y, x_1, \ldots, x_n)$ is a formula in prenex normal form of the sort described and in which the variables $x_1, \ldots, x_n$ appear free$\}$.

Notice that the variables t and y need not appear in U. Clearly C is a denumerably infinite set. Let

$$C_1 = \langle Y_1, U_1 \rangle, \; C_2 = \langle Y_2, U_2 \rangle, \ldots, C_k = \langle Y_k, U_k \rangle, \ldots$$

be an enumeration of the elements of C in such a way that each element $\langle Y, U \rangle$ of C occurs an infinite number of times. This is always possible.

We define e_0 on $(S_0 \times S_0) \cup (S \times S_0)$ as follows,

$$e_0(k, 1) = 1/k, \quad 1 \leqq k .$$

Assume that the sets

$$S_0, \ldots, S_m ,$$
$$A_0, \ldots, A_m ,$$
$$B_0, \ldots, B_m ,$$

and the functions

$$e_0, \ldots, e_m ,$$

have already been defined with the help of the sequence $C_1, \ldots, C_m$. Suppose that S_m has $(k-1)$ elements, so that $S_m = \{1, \ldots, k-1\}$. We proceed to the definition of S_{m+1}, A_{m+1}, B_{m+1}, and e_{m+1} with the help of $C_{m+1} = \langle Y_{m+1}, U_{m+1} \rangle$. There are two cases. In case the sequence $Y_{m+1} = \langle b_1, \ldots, b_n \rangle$ contains elements not belonging to S_m, we let

$$A_{m+1} = A_m, \quad B_{m+1} = B_m, \quad S_{m+1} = S_m, \quad e_{m+1} = e_m .$$

The case where every element of the sequence Y_{m+1} is an element of S_m will occupy our attention for some time. So let us assume that $b_i \leqq (k-1)$ for $1 \leqq i \leqq n$.

First we let $A_{m+1}=A_m\cup\{k\}$. Thus $A_{m+1}-A_m$ is the singleton $\{k\}$.

Before defining B_{m+1} and e_{m+1}, we would like to give the reader some idea of what we intend to do. It should be clear by now that the element k just introduced in A_{m+1} is intended to be that element in S so that

$$\forall t\,\big(t\in k\leftrightarrow U_{m+1}(t,k,b_1,\ldots,b_n)\big)[M] = 1$$

for the given sequence $\langle b_1,\ldots,b_n\rangle$. If we can do this for all sequences $\langle b_1,\ldots,b_n\rangle$ and all formulas $U(t,y,x_1,\ldots,x_n)$, then M shall be the desired model. Since Brouwer's Fixed Point Theorem applies only to finite powers of X, we can only define the model M by finite approximations. Now the possible quantifiers in U cause a great deal of trouble. This is because the value of $U_{m+1}(t,k,b_1,\ldots,b_n)$ would change depending on how much of M we have already defined. It turns out that due to the special restriction on the bound variables of U we can add enough extra elements, the elements of $B_{m+1}-B_m$, in such a way that the value of $U_{m+1}(t,k,b_1,\ldots,b_n)$ will not change appreciably as we extend the definition of e. This is why the definitions of B_{m+1} and e_{m+1} seem very complicated.

Next we shall define B_{m+1}. Let

$$U_{m+1} = Q_q u_q\ldots Q_1 u_1 V(t,y,x_1,\ldots,x_n,u_1,\ldots,u_q)$$

where $x_1,\ldots,x_n$ appear free in U_{m+1} and V is an open formula, $Q_1,\ldots,Q_q$ are the quantifiers ($\exists$ or $\forall$) of U_{m+1}, and $u_1,\ldots,u_q$ are the bound variables of U_{m+1}. Consider the function $P(V)$ of (at most) $2\cdot(2+n+q)^2$ real variables. By the uniform continuity of $P(V)$, there exists a positive integer r such that if

$$|d_{ij}-d'_{ij}| \leqq 1/r,\quad |e_{ij}-e'_{ij}| \leqq 1/r,\quad \text{for } 1\leqq i,j\leqq 2+n+q\,,$$

then

$$|P(V)(d_{ij},e_{ij};\,1\leqq i,j\leqq 2+n+q)-P(V)(d'_{ij},e'_{ij};\,1\leqq i,j\leqq 2+n+q)| \leqq 1/(m+1)\,.$$

Let $p=r^{(n+3)}$, let $l=k+p\cdot(q+1)$, let

$$B_{m+1} = B_m\cup\{k+1,\ldots,l\}\,,$$

and let $S_{m+1}=A_{m+1}\cup B_{m+1}$. We have already fulfilled the conditions that

$$A_m\subsetneqq A_{m+1},\quad B_m\subsetneqq B_{m+1},\quad S_m\subsetneqq S_{m+1}\,,$$
$$A_{m+1}\cap B_{m+1} = \emptyset,\quad A_{m+1}\cup B_{m+1} = S_{m+1}\,.$$

To define e_{m+1} we extend e_m in two stages. First we shall define e_{m+1} on the set $S\times B_{m+1}$. To do this we only need to assign values to all pairs of $S\times(B_{m+1}-B_m)$. We divide the $(l-k)=p\cdot(q+1)$ elements of

$B_{m+1}-B_m$ into p blocks each having $(q+1)$ elements. For each j, $l \leqq j \leqq p$, let

$$D_j = \{k+s : s \equiv j \pmod{p} \text{ and } 1 \leqq s \leqq p\cdot(q+1)\} .$$

We see that $D_i \cap D_j = \emptyset$ if $i \neq j$, each D_j has exactly $(q+1)$ elements, and

$$B_{m+1}-B_m = D_1 \cup \ldots \cup D_p .$$

We define e_{m+1} on $S \times D_j$ for each j, $1 \leqq j \leqq p$. Let $g_1, \ldots, g_p$ be an enumeration of all functions mapping the set with $n+3$ elements $\{0, k, b_1, \ldots, b_n, -1\}$ into the set of rational numbers $\{1/r, 2/r, \ldots, 1\} \subseteqq X$. For each element a in D_j, define

$$\begin{aligned} e_{m+1}(b_i, a) &= g_j(b_i) \text{ for } 1 \leqq i \leqq n , \\ e_{m+1}(k, a) &= g_j(k) , \\ e_{m+1}(a, a) &= g_j(-1) , \\ e_{m+1}(b, a) &= g_j(0) \text{ for all } b \neq k, b_1, \ldots, b_n, a . \end{aligned}$$

In this way, we have extended e_m to be defined on all of $S \times B_{m+1}$.

Notice that given any element b of S (b need not be distinct from $k, b_1, \ldots, b_n$, and may be in $B_{m+1}-B_m$), and given any $(n+3)$-sequence of rational numbers of the form s/r, with $s \geqq 1$, there will always be at least q elements a of $B_{m+1}-B_m$ distinct from b so that the sequence

$$\langle e_{m+1}(b,a), e_{m+1}(k,a), e_{m+1}(b_1,a), \ldots, e_{m+1}(b_n,a), e_{m+1}(a,a)\rangle$$

is the given sequence.

To extend the definition of e_m to include all of $S_{m+1} \times S_{m+1}$ it is only necessary to define e_{m+1} on the set

$$S_{m+1} \times \{k\} \cup (S_{m+1}-S_m) \times A_m ,$$

since e_{m+1} is already defined on

$$(S_{m+1} \times S_{m+1}) - (S_{m+1} \times \{k\} \cup (S_{m+1}-S_m) \times A_m) .$$

Let $z_1, \ldots, z_l$ be individual variables not occurring in any of the formulas $U_1, \ldots, U_{m+1}$. Consider the l-transforms $T_l(U_1), \ldots, T_l(U_{m+1})$. We may write

$$T_l(U_{m+1}) = T_l(U_{m+1})(t, y, x_1, \ldots, x_n, z_1, \ldots, z_l) .$$

For each pair $\langle i, k\rangle$ in $S_{m+1} \times \{k\}$, we define the formula

$$W_{ik}(z_1, \ldots, z_l) = T_l(U_{m+1})(z_i, z_k, z_{b_1}, \ldots, z_{b_n}, z_1, \ldots, z_l) .$$

For each pair $\langle i, j\rangle$ in $(S_{m+1}-S_m) \times A_m$, let h be the unique index such that $j \in A_h - A_{h-1}$. We may write

$$T_l(U_h) = T_l(U_h)(t, y, x_1, \ldots, x_s, z_1, \ldots, z_l) ,$$

where $C_h = \langle Y_h, U_h \rangle$ and $Y_h = \langle c_1, \ldots, c_s \rangle$ is a sequence of elements of S_{h-1}. We define the formula

$$W_{ij}(z_1, \ldots, z_l) = T_l(U_h)(z_i, z_j, z_{c_1}, \ldots, z_{c_s}, z_1, \ldots, z_l) .$$

In this way, an open formula $W_{ij}(z_1, \ldots, z_l)$ is defined for each pair $\langle i,j \rangle$ in

$$S_{m+1} \times \{k\} \cup (S_{m+1} - S_m) \times A_m .$$

For any other pair $\langle i,j \rangle$ in $S_{m+1} \times S_{m+1}$, we define W_{ij} arbitrarily. We now have an array of formulas W_{ij}, $1 \leqq i, j \leqq l$. Let $\langle S_{m+1}, e_{m+1} \rangle$ be the partial model on the set S_{m+1} with the e_{m+1} defined so far. Let t_{ij}, $1 \leqq i,j \leqq l$, be a fixed point of the array W_{ij}, $1 \leqq i,j \leqq l$, with respect to $\langle S_{m+1}, e_{m+1} \rangle$ and the distinct elements $1, \ldots, l$ of S_{m+1}. We now extend e_{m+1} to the rest of $S_{m+1} \times S_{m+1}$ by the definition $e_{m+1}(i,j) = t_{ij}$ for $\langle i,j \rangle$ in

$$S_{m+1} \times \{k\} \cup (S_{m+1} - S_m) \times A_m .$$

By mathematical induction the sequences of sets and functions S_k, A_k, B_k, and e_k are defined for each $k \geqq 0$. As we have already noted, we let $e = \bigcup_k e_k$ and $M = \langle S, e \rangle$. We also let $M_k = \langle S_k, e_k \rangle$, where only that part of e_k on $S_k \times S_k$ is used. Each M_k is a submodel of M.

To complete the proof of the theorem we require the following lemmas. Each of the lemmas is proved for an arbitrary integer $m \geqq 0$.

LEMMA 2.3. *If $\langle i,k \rangle$ is in $S_{m+1} \times (A_{m+1} - A_m)$ and $Y_{m+1} = \langle b_1, \ldots, b_n \rangle$, then*

$$U_{m+1}(i, k, b_1, \ldots, b_n)[M_{m+1}] = e(i,k) .$$

PROOF. It is quite clear that, by the definition of e_{m+1},

$$W_{ik}(1, \ldots, l)[M_{m+1}] = e_{m+1}(i,k) = e(i,k) .$$

By the definition of W_{ik},

$$W_{ik}(1, \ldots, l)[M_{m+1}] = T_l(U_{m+1})(i, k, b_1, \ldots, b_n, 1, \ldots, l)[M_{m+1}] .$$

By Lemma 1.1,

$$U_{m+1}(i, k, b_1, \ldots, b_n)[M_{m+1}] = T_l(U_{m+1})(i, k, b_1, \ldots, b_n, 1, \ldots, l)[M_{m+1}] .$$

So, putting everything together, we have

$$U_{m+1}(i, k, b_1, \ldots, b_n)[M_{m+1}] = e(i,k) .$$

LEMMA 2.4. *If $\langle i,j \rangle$ is in $(S_{m+1} - S_m) \times A_m$ and h is the unique index such that $j \in A_h - A_{h-1}$, then*

$$U_h(i,j,c_1,\ldots,c_s)[M_{m+1}] = e(i,j) ,$$

where $Y_h = \langle c_1,\ldots,c_s \rangle$.

PROOF. By the definition of e_{m+1},

$$W_{ij}(1,\ldots,l)[M_{m+1}] = e_{m+1}(i,j) = e(i,j) .$$

By the definition of W_{ij},

$$W_{ij}(1,\ldots,l)[M_{m+1}] = T_l(U_h)(i,j,c_1,\ldots,c_s, 1,\ldots,l)[M_{m+1}] .$$

By Lemma 1.1,

$$U_h(i,j,c_1,\ldots,c_s)[M_{m+1}] = T_l(U_h)(i,j,c_1,\ldots,c_s, 1,\ldots,l)[M_{m+1}] .$$

So

$$U_h(i,j,c_1,\ldots,c_s)[M_{m+1}] = e(i,j) .$$

LEMMA 2.5. *Suppose that* $h \geqq m+1$. *If* $\langle i,k \rangle$ *is in* $S_h \times (A_{m+1} - A_m)$ *and* $Y_{m+1} = \langle b_1,\ldots,b_n \rangle$, *then*

$$|U_{m+1}(i,k,b_1,\ldots,b_n)[M_h] - U_{m+1}(i,k,b_1,\ldots,b_n)[M]| \leqq 1/(m+1) .$$

PROOF. We recall that the formula U_{m+1} is written as

$$U_{m+1}(t,y,x_1,\ldots,x_n) = Q_q u_q \ldots Q_1 u_1 V(t,y,x_1,\ldots,x_n,u_1,\ldots,u_q) ,$$

where V is an open formula, $Q_1,\ldots,Q_q$ are quantifiers of U_{m+1}, and $u_1,\ldots,u_q$ are the bound variables of U_{m+1}. Recall also the function $P(V)$ and the pair of positive integers r, $m+1$ with respect to which $P(V)$ is uniformly continuous. For each $j, 0 \leqq j \leqq q$, we let

$$V_j(t,y,x_1,\ldots,x_n,u_{j+1},\ldots,u_q)$$

denote the formula $Q_j u_j \ldots Q_1 u_1 V(t,y,x_1,\ldots,x_n,u_1,\ldots,u_q)$. We now prove: For each integer $j, 0 \leqq j \leqq q$, each sequence

$$\sigma = \langle i,k,b_1,\ldots,b_n,a_{j+1},\ldots,a_q \rangle$$

of elements of S_h, and each sequence

$$\sigma' = \langle i,k,b_1,\ldots,b_n,a'_{j+1},\ldots,a'_q \rangle$$

of elements of S, if

$$|\sigma - \sigma'| \leqq 1/r \pmod V$$

then

$$|V_j(i,k,b_1,\ldots,b_n,a_{j+1},\ldots,a_q)[M_h] - V_j(i,k,b_1,\ldots,b_n,a'_{j+1},\ldots a'_q)[M]| \leqq 1/(m+1) .$$

The case when $j=0$ is trivial, because of the definitions of the function $P(V)$ and the integer r, and because V is an open formula and M_h is a submodel of M. So, let us assume that the case for j is proved. We consider the case $j+1$. Let

$$\sigma = \langle i,k,b_1,\ldots,b_n,a_{j+2},\ldots,a_q\rangle$$

be a sequence of elements of S_h and let

$$\sigma' = \langle i,k,b_1,\ldots,b_n,a'_{j+2},\ldots,a'_q\rangle$$

be a sequence of elements of S. Suppose that

$$|\sigma-\sigma'| \leqq 1/r \quad (\operatorname{mod} V) .$$

We first show that

(1) for each a in S_h, there exists a' in S such that

$$|\langle i,k,b_1,\ldots,b_n,a,a_{j+2},\ldots,a_q\rangle - \langle i,k,b_1,\ldots,b_n,a',a'_{j+2},\ldots,a'_q\rangle| \leqq 1/r \quad (\operatorname{mod} V) .$$

Suppose a is one of the elements in the sequence σ, then let a' be the corresponding element in the sequence σ'. Suppose a is different from every element in the sequence σ. Recall that u_{j+1} is a bound variable of U_{m+1} and hence, in order to satisfy (1), it is sufficient to find an element a' in S such that, first of all, a' is different from every element in the sequence σ' and, secondly, the sequences of values

(2) $$\langle e(i,a),\, e(k,a),\, e(b_1,a),\ldots,e(b_n,a),\, e(a,a)\rangle$$

and

(3) $$\langle e(i,a'),\, e(k,a'),\, e(b_1,a'),\ldots,e(b_n,a'),\, e(a',a')\rangle$$

are term by term within $1/r$ of each other. We know that there are at least q elements a' of $B_{m+1}-B_m$ distinct from i such that the sequences of (2) and (3) are within $1/r$ of each other. Since the elements $k,b_1,\ldots,b_n$ are not in $B_{m+1}-B_m$, and there are at most $(q-1)$ elements $a'_{j+2},\ldots,a_q'$, we can easily find an element a' satisfying (1). The same argument will show that (1) remains true if the roles of S_h and S are interchanged. Now, independent of whether Q_{j+1} is $\exists$ or $\forall$, we have from (1) and the inductive hypothesis

$$|Q_{j+1}\{V_j(i,k,b_1,\ldots,b_n,a,a_{j+2},\ldots a_q)[M_h]:a \text{ in } S_h\} - \\ - Q_{j+1}\{V_j(i,k,b_1,\ldots,b_n,a',a'_{j+2},\ldots,a'_q)[M]:a' \text{ in } S\}| \leqq 1/(m+1) .$$

This implies

$$|V_{j+1}(i,k,b_1,\ldots,b_n,a_{j+2},\ldots,a_q)[M_h] - \\ - V_{j+1}(i,k,b_1,\ldots,b_n,a'_{j+2},\ldots,a'_q)[M]| \leqq 1/(m+1)$$

and the induction is complete. The lemma is proved with $j=q$.

LEMMA 2.6. *If* $\langle i,k\rangle$ *is in* $S_{m+1}\times(A_{m+1}-A_m)$ *and* $Y_{m+1}=\langle b_1,\ldots,b_n\rangle$, *then*

$$|U_{m+1}(i,k,b_1,\ldots,b_n)[M]-e(i,k)| \leqq 1/(m+1) .$$

PROOF. This follows from Lemmas 2.3 and 2.5.

LEMMA 2.7. *If* $\langle i,j\rangle$ *is in* $(S_{m+1}-S_m)\times A_m$ *and* h *is the unique index such that* $j\in A_h-A_{h-1}$, *then*

$$|U_h(i,j,c_1,\ldots,c_s)[M]-e(i,j)| \leqq 1/h ,$$

where $Y_h=\langle c_1,\ldots,c_s\rangle$.

PROOF. In Lemma 2.5, read h for $(m+1)$, $(m+1)$ for h, and j for k. The result follows from this and Lemma 2.4.

We now return to the proof of the theorem. Let V be a sentence of Σ_3 of the form

$$\forall x_1\ldots x_n\,\exists y\,\forall t\,\bigl(t\in y \leftrightarrow U(t,y,x_1,\ldots,x_n)\bigr)$$

where U is a formula in prenex normal form of the sort described. We show that $V[M]=1$. Let $b_1,\ldots,b_n$ be elements in S. Without loss of generality, we assume that $b_1,\ldots,b_n$ are distinct. In order to show $V[M]=1$, it is sufficient to show

$$\exists y\,\forall t\,\bigl(t\in y \leftrightarrow U(t,y,b_1,\ldots,b_n)\bigr)[M] = 1 .$$

This means we have to prove that for every positive integer h there exists an element k in S such that for all elements i in S

$$|U(i,k,b_1,\ldots,b_n)[M]-e(i,k)| \leqq 1/h . \tag{4}$$

Let a positive integer h be given. By the way in which the enumeration $C_1,C_2,\ldots,C_k,\ldots$ was chosen, there exists an integer m so that

$$Y_{m+1} = \langle b_1,\ldots,b_n\rangle,\ U_{m+1} = U ,$$
$$C_{m+1} = \langle Y_{m+1},U_{m+1}\rangle ,$$
$$b_1,\ldots,b_n \text{ are in } S_m, \text{ and } h \leqq m+1 .$$

Let $\{k\}=A_{m+1}-A_m$. We show that (4) holds for all i in S. If i is in S_{m+1}, then by Lemma 2.6,

$$|U(i,k,b_1,\ldots,b_n)[M]-e(i,k)| \leqq 1/(m+1) \leqq 1/h .$$

If i is not in S_{m+1}, let m' be the index so that i is in $S_{m'+1} - S_{m'}$. Clearly $m' \geqq (m+1)$. So by Lemma 2.7, reading m' for m, $m+1$ for h and k for j, we have

$$|U(i,k,b_1,\ldots,b_n)[M] - e(i,k)| \leqq 1/(m+1) \leqq 1/h \,.$$

Hence (4) holds for all i in S and the theorem is proved.

3. Generalizations and problems.

In this section we shall discuss various aspects of our results, progressing from things we know to be true (or false) to things which may be true (or false).

Our proofs have already established that the formulas U occurring in sentences of Σ_2 and Σ_3 can have free occurrences of the variable y. It may turn out that some of the problems we pose later on cannot be carried out with this liberal assumption. Therefore, whenever we speak of the sets Σ_2 and Σ_3 in what follows, it is with the understanding that they may have to be curtailed by not allowing free y's to occur in the U's. Before we go on, let us denote by Σ_0 the set of all sentences of the form

$$\forall x_1 \ldots x_n \, \exists y \, \forall t \, \big(t \in y \leftrightarrow U(t,x_1,\ldots,x_n)\big)$$

with no restriction placed on the formulas U. The set Σ_0 is referred to as the full axiom schema of comprehension.

Our results can be generalized in various ways. For instance the logic Ł can be enriched by adding a countable number of new propositional functions, provided they are all continuous. The more propositional functions we have in a language, the more we add to the power of expression of the language. It is almost immediate that our results hold in any such extension of Ł. We should also mention that the corresponding generalizations of Theorems A, B and C also hold in any such extension of Ł.

The space X of truth values need not be confined to the interval $[0,1]$, see for instance [4], [5]. It is difficult to put precise conditions on those spaces Y for which our results hold. However, at the very least, Y must have the following properties:

Y is compact Hausdorff;
Y^m has the fixed point property for each positive integer m;
Y contains the proper analogs of the continuous functions min, max, inf, and sup.

We feel that at the moment we should try to extend our results to sets of sentences more general than Σ_2 and Σ_3 with the fixed space X, rather than to investigate more general spaces for which our results hold. For instance, the very first such problem might be to find out if the set $\Sigma_2 \cup \Sigma_3$ is consistent.

The next question that concerns us is whether the axiom of extensionality in its classical form,

$$\mathrm{A}_1: \qquad \forall xy\ (x \equiv y \leftrightarrow \forall t\ (t \in x \leftrightarrow t \in y))$$

can be consistently added to Σ_2 and Σ_3. This would amount to saying that a model M will have to be constructed so that Σ_2 (or Σ_3) is satisfied on M, and whenever a,b in M and $a \neq b$, then

$$\inf_{c \text{ in } M}(1-|e(c,a)-e(c,b)|) = 0 .$$

This happens only if

$$(1) \qquad \sup_{c \text{ in } M}|e(c,a)-e(c,b)| = 1 \text{ for } a \neq b .$$

Suppose that M satisfies Σ_2 (or Σ_3). Then the two sentences

$$\exists y\, \forall t\ (t \in y \leftrightarrow t \in t) ,$$
$$\exists y\, \forall t\ (t \in y \leftrightarrow (\neg\ t \in t \rightarrow t \in t))$$

must have value 1 on M. From this we can easily find two elements a,b in M so that $a \neq b$ and (1) does not hold. Hence the axiom A_1 cannot be consistently added to Σ_2 or Σ_3.

In many set theories the following holds:

$$\mathrm{A}_1': \qquad \forall xy\ (x \equiv y \leftrightarrow \forall t\ (x \in t \leftrightarrow y \in t)) .$$

We can show that A_1' holds in any model M satisfying Σ_3. For suppose a,b in M and $a \neq b$. We would like to show that

$$(2) \qquad \sup_{c \text{ in } M}|e(a,c)-e(b,c)| = 1 .$$

We have

$$(3) \qquad \exists y\, \forall t\ (t \in y \leftrightarrow t \equiv a)[M] = 1 ,$$

and

$$(4) \qquad \exists y\, \forall t\ (t \in y \leftrightarrow t \equiv b)[M] = 1 .$$

Since $a \neq b$, (3) and (4) easily imply (2). Hence the set $\Sigma_3 \cup \{\mathrm{A}_1'\}$ is consistent. We can also show, without a great deal of trouble, that the set $\Sigma_2 \cup \{\mathrm{A}_1'\}$ is also consistent. This can be done, for instance, by adding a sufficient number of extra elements in each of the finite models constructed in Theorem 2.1 with preassigned values for e (0's or 1's) in such a way that (2) holds for any two distinct elements a and b of that finite model. Then extend the function e to the rest of the model by a fixed point argument.

The question then is open as to what form of the axiom of extensionality we should accept. It may be that the condition (1) is unreasonable to impose on a and b if $a \neq b$. Certainly the fact that (1) should hold in two valued logic may simply be a consequence that only the values 0 and 1 are permitted. We can, of course, weaken our notion of identity somewhat by relaxing our fixed interpretation of the symbol $\equiv$. Let us temporarily mean by a weak model M an ordered triple $M = \langle S, e, \equiv \rangle$ where both e and $\equiv$ are interpreted as arbitrary functions of two arguments mapping $S \times S$ into X. By a weak model with identity we mean a weak model M such that whenever a,b are in M,

$$\equiv (a,b) = 1 \quad \text{if and only if} \quad a = b\,.$$

It makes sense now to ask whether any of the sets

$$\Sigma_2 \cup \{A_1\}, \quad \Sigma_2 \cup \{A_1, A_1'\}\,,$$
$$\Sigma_3 \cup \{A_1\}, \quad \Sigma_3 \cup \{A_1, A_1'\}$$

can be satisfied by a weak model or by a weak model with identity.

The only thing that we can state positively is that $\Sigma_2 \cup \{A_1\}$ can be satisfied by a weak model. The argument goes as follows. We simply repeat the procedure of Theorem 2.1 except that whenever the atomic formula $x \equiv y$ occurs in the formula U, we replace it by the formula

$$\forall t\, (t \in x \leftrightarrow t \in y)\,.$$

In this way, we get rid of all occurrences of the atomic formulas $x \equiv y$ in U. We next find the fixed points of the array W_{ij}, and finally we calculate the values of $\equiv (a,b)$ by using the equation

$$\equiv (a,b) = \forall t\, (t \in a \leftrightarrow t \in b)[M]\,.$$

In connection with this argument we should mention that Theorems A, B, and C hold for weak models but not for weak models with identity.

We should also warn the reader that weak models (with or without identity) may not satisfy the rule of substitution for identity mentioned in § 1. However, if a weak model should satisfy both A_1 and A_1' then it can be proved that it must also satisfy the rule of substitution for identity. Thus it is more important to determine whether or not the sets $\Sigma_2 \cup \{A_1, A_1'\}$ and $\Sigma_3 \cup \{A_1, A_1'\}$ can be satisfied by weak models.

To continue we note that two other classical axioms of set theory, namely the axiom of union,

A_2: $$\forall x\, \exists y\, \forall t\, \big(t \in y \leftrightarrow \exists z\, (t \in z \wedge z \in x)\big)\,,$$

and the power axiom,

$$\mathrm{A}_3: \qquad \forall x\, \exists y\, \forall t\, (t \in y \leftrightarrow \forall z\, (z \in t \rightarrow z \in x)),$$

do not have the required form to be included in either Σ_2 or Σ_3. It would be interesting if it could be proved that the set $\Sigma_3 \cup \{\mathrm{A}_1, \mathrm{A}_1', \mathrm{A}_2, \mathrm{A}_3\}$ is satisfied by a weak model.

One can even speculate that perhaps the classical axioms of regularity and infinity can also be consistently added to the set $\Sigma_3 \cup \{\mathrm{A}_1, \mathrm{A}_1', \mathrm{A}_2, \mathrm{A}_3\}$. It should be clear that axioms A_1', A_2, and A_3, and even the axiom of replacement, will be satisfied in every model of Σ_0. Thus, the main open problem is still whether or not Σ_0 is consistent.

Finally, we would like to re-examine the whole question of the definition of consistency. We have defined a set Σ of sentences of Ł to be consistent if for some model M, $U[M]=1$ for every sentence U in Σ. Since there are more than two truth values in X and since X admits a simple ordering, this notion of consistency is clearly susceptible to generalizations. Suppose that r is a rational number in X. We say that the set Σ is $[r]$-consistent ((r)-consistent) if for some model M, $U[M] \geqq r$ ($U[M] > r$) for every sentence U in Σ. Clearly (r)-consistency implies $[r]$-consistency, and if $r < s$, $[s]$-consistency implies (r)-consistency. In particular $[1]$-consistency is our old notion. In some respects the study of (r)-consistency may be more important than the study of $[1]$-consistency. This is because of the following reasons.

It is known (from [9] and unpunlished results of the author) that for any positive rational r in X the set of sentences U of Ł such that

$$U[M] \geqq r \quad \text{for every model } M$$

is not recursively enumerable. On the other hand, for any rational r in X the set of sentences U such that

$$U[M] > r \quad \text{for every model } M \tag{5}$$

is recursively enumerable ([7]) and is, in fact, axiomatizable by some simple axioms and rules of inference ([1]). Now, in case the set Σ_0 turns out to be inconsistent, it would be quite natural and significant to ask if there exists any rational $r \geqq \frac{1}{2}$ in X such that Σ_0 is (r)-consistent. It is easily seen by letting e be the constant function $\frac{1}{2}$ that Σ_0 is $[\frac{1}{2}]$-consistent; it is not even known if Σ_0 is $(\frac{1}{2})$-consistent. The hope here is of course that one might be able to develop a syntactical system of set theory using the axiomatizability of the set of sentences U in (5) and the (r)-consistency of Σ_0 for some $r \geqq \frac{1}{2}$.

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