MATH. SCAND. 13 (1963), 31–43

TRANSFORMS FOR OPERATORS AND SYMPLECTIC AUTOMORPHISMS OVER A LOCALLY COMPACT ABELIAN GROUP[1]

I. E. SEGAL

Introduction.

For the real symplectic group $\sum$, which may be defined as the group of all linear transformations on the direct sum of a real linear space $\mathfrak{M}$ with its dual $\mathfrak{M}^*$ which leave invariant the skew form

$$B(x \oplus f, x' \oplus f') = f(x') - f'(x), \qquad x, x' \in \mathfrak{M};\ f, f' \in \mathfrak{M}^*,$$

a representation as a group of *-automorphisms of a certain abstract operator algebra (C^*-algebra) $\mathfrak{A}$ was defined in [7], in connection with quantum field dynamics. When the space $\mathfrak{M}$ is finite-dimensional, the algebra $\mathfrak{A}$ is isomorphic in a natural way with that of all bounded linear operators on the Hilbert space $L_2(\mathfrak{M})$ consisting of all square-integrable functions on $\mathfrak{M}$. Since every *-automorphism of this irreducible algebra is induced by transformation by a unitary operator, a projective unitary (unitary ray) representation U of $\sum$ on $L_2(\mathfrak{M})$ results. The algebra $\mathfrak{A}$ relates to the system $(\mathfrak{M} \oplus \mathfrak{M}^*, B)$, consisting of a linear space with a distinguished non-degenerate skew-symmetric bilinear form in a manner formally analogous to that in which the Clifford algebra relates to the system consisting of a linear space together with a distinguished non-degenerate symmetric form, and the procedure for setting up the representation U is quite analogous in a formal way to that of Brauer and Weyl [1] in setting up in global form the spin representation for the orthogonal group.

It was shown in fact by David Shale [8] that the analogy with the spin representation was remarkably close. Notably, he showed that the projective representation in question was not equivalent to a full unitary representation, but could be obtained from such a representation of a double covering of the symplectic group. Again, the present representation splits, like the spin representation, into two irreducible parts

Received February 2, 1963.

[1] Research supported in part by the U. S. National Science Foundation.

which are exchanged by a type of outer transformation. Novel theoretical difficulties arise in the treatment of the present representation, due to its infinite dimensionality, and to the unboundedness of the operators corresponding to the vectors in the linear space $\mathfrak{M}\oplus\mathfrak{M}^*$. The former feature of the representation means that the Hilbert space topology must be made essential use of; the Weyl algebra $\mathfrak{A}$ in the symplectic case is not simply the linear envelopping algebra, as in the Clifford algebra case, but the maximal algebra of continuous linear transformations with the same invariant subspaces (or the weak closure). The latter feature necessitates the use in place of the Heisenberg commutation relations, which provide the direct analog to those defining the Clifford numbers, of those relations involving finite (rather than infinitesimal) elements which were first formulated by Weyl [10] and used incisively in work in the present direction by von Neumann [4]. These analytical complications and some obvious algebraic differences do not disturb the apparently fundamental analogy, which is moreover from a theoretical physical point of view very logical, being a particular mathematical manifestation of the widely accepted if somewhat metaphysical principle that any fundamental formal development applicable to systems obeying one type of quantum statistics has an analog for systems obeying the other type.

From an elementary quantum-mechanical point of view, the present representation operators are those generated by hamiltonians which are quadratic in the canonical variables. The commutators of any two such hamiltonians is again such, and an infinitesimal unitary representation of the (symplectic) group defined by the commutation relations of the finite-dimensional set of such hamiltonians is thereby obtained. This somewhat loosely formulated representation is explicitly described by van Hove [2] as a formal subrepresentation of the infinitesimal representation derived from the unitary global representation involved in his approach to the correspondence between classical and quantum-mechanical hamiltonians. (The restrictions in this work in the global treatment to functions of the canonical variables growing not too rapidly at infinity, so that polynomials are excluded, are clarified by Shale's result, which implies that the infinitesimal subrepresentation can not generate a subrepresentation of a full unitary representation such as that obtained by van Hove.)

The most familiar and important quadratic hamiltonian is that of the harmonic oscillator, which is distinguished mathematically as well as the generator of the center of the envelopping algebra of the restriction to the maximal compact subgroup (unitary group) of the representation of the symplectic group in question. It is therefore appropriate, as well as in harmony with the existing nomenclature "spin representation", to designate this representation as the "harmonic representation", as is done in the following. Additionally, the restriction of this representation to the orthogonal group decomposes exactly into the irreducible subspaces defined by the spherical harmonics.

Quite recently, A. Weil has introduced a representation of the symplectic group relative to an arbitrary locally compact field (of characteristic $\neq 2$), which plays a central role in his explication and development of arithmetical results of C. L. Siegel. His construction of this representation is not at all parallel to that of Brauer and Weyl for the spin representation, but it is actually identical, in the case of the real field, with the harmonic representation. (The work of Shale was not known to Professor Weil at the time of his lecture on this material at Harvard University (December, 1962). A written communication from him, following his examination of the work of Shale, expresses his agreement with the foregoing statement.) In the case of a more general field, such as the p-adic field, which is essential for the arithmetic applications, a rather direct adaptation of the earlier treatment is possible for setting up in a simple way an analogous projective unitary representation associated with those locally compact abelian groups for which the squaring operation is sufficiently regular. The present article gives a variant of this approach which avoids the use of the generalized theorem of Plessner (to the effect that a quasi-invariant measure on a locally compact group is absolutely continuous with respect to Haar measure) and is based on the development of an independently interesting transform for Hilbert–Schmidt operators on the group. There is a type of invariant Fourier representation for any such operator in terms of a square-integrable function on the direct sum of the group with its dual, the correspondence between the Hilbert space of Hilbert–Schmidt operators and that of functions being unitary; in a somewhat different form and in special cases this transform plays a significant part in [10] and [4].

The starting point is the projective Weyl relation of the type given in [7] for the real case. The general theory of induced projective representations due to G. W. Mackey [3], which subsumes among other developments his earlier extension of the Stone–von Neumann uniqueness theorem for the Schrödinger operators to the case of arbitrary locally compact groups, as well as an extension of the cited generalized theorem of Plessner, is applicable to such relations, and should provide an alternative basis for the derivation of the harmonic representation. Here the representation emerges directly from general properties of the Weyl transform, including continuity features which are less accessible from the uniqueness theorem cited.

The treatment of the harmonic representation for more general groups is of interest not only in connection with number theory but also for the clarification of the logical and mathematical structure of quantum mechanics, along the lines initiated in [5]. A theory based on a cyclic group of large prime order can be expected to approximate the conventional one based on the additive group of the reals, as well as to minimize analytical difficulties. (Some degree of confirmation of this idea has been provided by joint explorations in a Senior Seminar on quantum mechanics at the Massachusetts Institute of Technology held in the Autumn term, 1961–62.) The physical role of the p-adic group is considerably more tenuous, but the harmonic representation makes possible the formulation of analogs to quantum-mechanical motions in generalized spaces which seem of mathematical interest. In any event, the canonical commutation relations intervene in two distinct ways in quantum mechanics: in connection with Bose–Einstein quantization, and as an implicit description of the structure of quantum-mechanical microspace. It is only in the former respect that the axiom that physical observables are represented by real numbers requires the use of the real field; the precise identity of quantum-mechanical microspace with classical macrospace is physically dubious and logically expendable.

Many special features of the harmonic representation, such as its reducibility properties, the essentiality of the multiplier which intervenes, etc., appear to be quite group dependent, and are not treated, reference being made to the work of Shale [8] and forthcoming work of Weil on these matters. We venture however the suggestion that the representation of the extended symplectic group (see below) may be irreducible whenever the group is transitive, as in the case of the real vector group treated in [8].

The extended Weyl relation and transform.

Let G be a given locally compact abelian group, written additively, let A be the direct sum of G with its topological dual G^*, and for any element z of A, say $z=(a,a^*)$, set $W(z)$ for the unitary operator

$$W(z):\ \ f(x) \to a^*(2x+a)f(x+a), \qquad f \in L_2(G)\,,$$

on the Hilbert space of square-integrable functions over G (relative to Haar measure). It is straightforward to verify the generalized Weyl relation

$$W(z)W(z') = \chi(z,z')W(z+z')$$

for arbitrary z and z' in A, where

$$\chi(z,z') = a^*(b)\big(b^*(a)\big)^{-1}, \quad \text{for} \quad z=(a,a^*),\ z'=(b,b^*) .$$

From the continuity of $a^*(2x+a)$ as a function of a^* and a jointly, and the continuity of the map $a \to f(x+a)$ from G into $L_2(G)$, it follows that $W(z)$ is a continuous function of z in the strong operator topology.

In the following, the element of Haar measure on a locally compact abelian group H with generic element x will be denoted as dx, and where the dual group is also involved, its Haar measure will be assumed to have been normalized so that the Fourier transform, $f(x) \to \int y^*(x)f(x)\,dx$, is unitary from $L_2(H)$ to $L_2(H^*)$, in accordance with the Plancherel–Weil theorem [9]. For any operator R on a Hilbert space $\mathfrak{H}$, $\|R\|_\infty$ will denote the usual bound, $\sup_{\|x\|<1}\|Rx\|$, while $\|R\|_2$ denotes the square root of $\Sigma_{a,b}|(Re_a,e_b)|^2$, where $\{e_a\}$ is an arbitrary maximal orthonormal set in $\mathfrak{H}$ (of the choice of which $\|R\|_2$ is independent); R is called "Hilbert–Schmidt" if $\|R\|_2$ is finite, and the set of all such operators forms a Hilbert space $\mathfrak{K}(\mathfrak{H})$ relative to the (invariant) inner product, $(R,S)= \Sigma_{a,b}(Re_a,Se_a)$. The product of two Hilbert–Schmidt operators is said to be of absolutely convergent trace, and to have trace (denoted "tr") (R,S^*).

THEOREM 1. *If the map $x \to 2x$ transforms Haar measure into its multiple by a constant c^2, then for any function f in $L_1(A)\cap A_2(A)$, the operator $\int W(z)f(z)\,dz$ is Hilbert–Schmidt, and the map*

$$f \to c \int W(z)f(z)\,dz$$

extends uniquely to an isometric transformation (called henceforth the Weyl Transform) *from $L_2(A)$ into the Hilbert space of all Hilbert–Schmidt operators on $L_2(G)$. If the doubling map is an automorphism, the Weyl transform is unitary (onto), and any operator T of absolutely convergent trace on $L_2(G)$ is the Weyl transform of the function $f(z)$ given by the equation*

$$f(z) = c\,\mathrm{tr}\big(W(-z)T\big) .$$

To clarify the role of the absolute continuity condition, it may be noted that if the map $x \to 2x$ is singular, e.g. if every element is of order 2, the operator $\int W(z)f(z)\,dz$ will in general not be Hilbert–Schmidt. (If G is the additive group of an infinite Boolean ring in the discrete topology, W_f is equivalent via the Fourier transform to a multiplication operator in $L_2(G^*)$, which is Hilbert–Schmidt only if $f=0$.) On the other hand, it is sufficient, — but not necessary, as the case of the circle group shows, — that $x \to 2x$ be an automorphism of G.

For the proof, consider first the case of a single function $f(z)$ of the form

$$f(z) = g(a)h(a^*), \qquad z = (a,a^*) .$$

W_f then carries a general element $F(x)$ of $L_2(G)$ into

$$\iint a^*(a)\, a^*(2x)\, g(a)\, h(a^*) F(x+a)\, da\, da^* .$$

On replacing a by $a-x$ and integrating first over a^* this becomes

$$\int \hat{h}(a+x)\, g(a-x) F(a)\, da ,$$

where $\hat{h}$ is the Fourier transform of h. Thus W_f may be described as the integral operator $F(x) \to \int K(x,y)F(y)\,dy$ with kernel

$$K(x,y) = \hat{h}(y+x)\, g(y-x) .$$

Such an operator is Hilbert–Schmidt provided $\iint |K(x,y)|^2\, dx dy < \infty$, and here this integral is

$$\iint |\hat{h}(y+x) g(y-x)|^2\, dx\, dy .$$

Replacing $y+x$ by y, and integrating first with respect to x gives

$$\int |\hat{h}(y)|^2 \left[\int |g(y-2x)|^2\, dx\right] dy .$$

Replacing $2x$ by x and using the relation $d(2x) = c^2 dx$, the last integral is readily evaluated as $c^{-2}\|g\|^2\|\hat{h}\|^2$ (where $\|\cdot\|$ indicates the L_2-norm), which by the Plancherel–Weil theorem [9] is the same as $c^{-2}\|f\|^2$. Thus W_f is indeed a Hilbert–Schmidt operator.

In view of the linearity of W_f as a function of f, it follows that W_f is Hilbert–Schmidt whenever f is a finite linear combination of product functions of the type just treated. To show unitarity of the map $f \to cW_f$ from the domain of all such finite linear combinations to the Hilbert–Schmidt operators $\mathfrak{K}$ on $L_2(G)$, it suffices by linearity to treat the case of two product functions, — i.e. to show that if

$$f_i(a,a^*) = g_i(a)\, h_i(a^*), \qquad i = 1,2 ,$$

with the g_i and h_i continuous and vanishing outside compact sets, then

$$(f_1, f_2) = c^2\, \mathrm{tr}(W^*_{f_2} W_{f_1}) .$$

To this end, recall that the product of two Hilbert–Schmidt operators which are integral operators with kernels $K(x,y)$ and $L(x,y)$ is of trace $\iint K(x,y)L(y,x)\,dx\,dy$. This gives

$$\operatorname{tr}(W^*_{f_2}W_{f_1}) = \iint \hat{h}_1(y+x)\,\bar{\hat{h}}_2(y+x)\,g_1(y-x)\,\bar{g}_2(y-x)\,dx\,dy\,.$$

Integrating first over x and using the Plancherel–Weil theorem as above gives

$$(h_1,h_2)(g_1,g_2)\,c^{-2} = c^{-2}(f_1,f_2)\,.$$

Thus the map $f \to cW_f$ from the dense domain of all finite linear combinations of product functions of the type indicated, in $L_2(A)$ to $\mathfrak{K}$, preserves inner products, and so extends uniquely to an isometric transformation W of $L_2(A)$ into a closed linear manifold in $\mathfrak{K}$. To show that this manifold is all of $\mathfrak{K}$, when doubling is an automorphism, it suffices to show the range of W includes a spanning subset in $\mathfrak{K}$. Now these operators with kernels $K(x,y)$ of the form $K(x,y)=p(x)q(y)$, with p and q ranging over spanning subsets of $L_2(G)$ are well-known to form such a spanning set. The image of this spanning set under any unitary transformation on $\mathfrak{K}$ will likewise be spanning. Consider now the unitary transformation U on $\mathfrak{K}$:

$$K(x,y) \to c^{-1}K(x+y,y-x)\,.$$

This transforms the kernel

$$p(y+x)\,q(y-x)$$

into the kernel

$$c^{-1}p(2y)\,q(2x)\,.$$

From the assumption that $d(2x)=c^2dx$ it follows that if $\{p(x)\}$ is a spanning set in $L_2(G)$, then so also is $\{p(2x)\}$. Since those p's of the form $p=\hat{h}$ for some continuous h on G^* vanishing outside a compact set are dense in $L_2(G)$, by virtue of the density in $L_2(G^*)$ of those h's, U^{-1} carries a spanning set into a set of kernels for the W_f with f a product function on A of the indicated type, showing that such W_f span $\mathfrak{K}$.

For the converse, define the skew convolution $f *_\chi g$ of two functions f and g on A relative to χ as the function h given by the equation

$$h(z) = \int \chi(z,z')\,f(z-z')\,g(z')\,dz'\,.$$

The circumstance that $\chi(z,z')$ is of absolute value one permits the derivation from the Fubini theorem of analogues to the familiar results for the case when χ is replaced by unity; in particular, if f and g are integrable, then their convolution exists almost everywhere and

$$\|f *_\chi g\|_1 \leqq \|f\|_1 \|g\|_1 ,$$

where the subscript "1" indicates the usual L_1 norm, while if f and g are in L_2, then the convolution exists everywhere and

$$|(f *_\chi g)(z)| \leqq \|f\|_2 \|g\|_2 .$$

The special properties of χ are however involved in the

LEMMA. *If f and g are in $L_2(A)$, then $f *_\chi g$ is in $L_2(A)$ and*

$$W_{f *_\chi g} = W_f W_g .$$

If f and g are integrable as well as square-integrable, this follows from the Fubini theorem in the indicated fashion. But additionally for such f and g,

$$\|f *_\chi g\| = \|W_f W_g\|_2 ,$$

which is dominated by $\|W_f\|_\infty \|W_g\|_2$, which in turn is dominated by $\|W_f\|_2 \|W_g\|_2$, since the operator bound is dominated by the Hilbert–Schmidt norm. If now f and g are arbitrary in L_2, sequences $f_n \to f$ and $g_n \to g$ in L_2 with the f_n and g_n in $L_1 \cap L_2$ may be chosen; then $f_n *_\chi g_n \to f * g$ by the preceding inequality, while

$$(f_n *_\chi g_n)(z) \to (f *_\chi g)(z)$$

by the bound given above for the latter expression.

To conclude the proof, let T be an arbitrary operator on $L_2(G)$ of absolutely convergent trace, and hence also in the Hilbert–Schmidt class, and let it be written in the form $T = RS^*$, with R and S likewise Hilbert–Schmidt (as is clearly possible from the polar decomposition of T). Then $R = W_f$, $S = W_g$, and $T = W_h$, for f, g, and h in $L_2(A)$, and

$$h = f *_\chi g^*, \qquad g^*(z) = \bar{g}(-z) .$$

Now $\mathrm{tr}\big(W(-z)T\big) = \mathrm{tr}\big(\big(W(-z)R\big)S^*\big) = c^2 \int f_z(z')\bar{g}(z')dz'$, with $W(-z)R = W_{f_z}$. From the latter equation it follows that

$$f_z(z') = f(z+z')\chi(-z,z') ,$$

i.e.

$$\mathrm{tr}\big(W(-z)T\big) = c^{-2} \int \chi(-z,z') f(z+z') \bar{g}(z') \, dz' ,$$

which on replacing z' by $-z'$ is seen to be identical with $(f *_\chi g^*)(z) c^{-2}$.

The harmonic representation of the symplectic group.

A (bicontinuous) automorphism of A which leaves invariant the bicharacter χ may be called a *symplectic* transformation, and the set of all such transformations the symplectic group Σ for G; when χ is carried into its inverse, the automorphism may be called *anti-symplectic*, these together with the symplectic transformations forming the extended symplectic group Σ^+ containing Σ as a subgroup of index 2. In the compact-open topology, in which a transformation T on A is close to the identity if it carries a preassigned compact set into a preassigned neighborhood of the identity, these are topological groups. A projective representation of a group on a Hilbert space is a homomorphism of the group into the group of all invertible bounded linear transformations on the Hilbert space, modulo the subgroup of multiplications by non-zero scalars. The latter group is topologized by regarding an element S as close to the unit in case its action $X \to SXS^{-1}$ on the space of Hilbert–Schmidt operators is close to the identity in the strong operator topology for this space of operators. A continuous projective representation of a topological group on a Hilbert space is then one which is continuous in this topology; for full unitary representations, this agrees with the usual notion of continuity (i.e. continuity in the strong operator topology).

THEOREM 2. *If the map $x \to 2x$ on the locally compact abelian group G is an automorphism, then for any symplectic (resp. anti-symplectic) transformation T there exists a unitary (resp. anti-unitary) transformation $\Gamma(T)$ on $L_2(G)$, unique within multiplication by a scalar factor, such that*

$$W(Tz) = \Gamma(T)\, W(z)\, \Gamma(T)^{-1}, \qquad z \in A\ .$$

The map $T \to \Gamma(T)$ gives a continuous projective representation of the symplectic group.

Observe first that a symplectic or anti-symplectic transformation is Haar measure preserving; since the square of an anti-symplectic transformation is symplectic, it suffices to consider the case of a symplectic transformation T. Then for any integrable function f on A and for some constant k,

$$(**) \qquad \int f(T^{-1}z)\, dz = k \int f(z)\, dz\ .$$

For f in $L_1 \cap L_2$, define a transform $\chi f = F$ by the equation

$$F(z) = \int \chi(z,z')\, f(z')\, dz'\ ;$$

by the Plancherel–Weil theorem, $\|F\| = \|f\|$. On the other hand,

$$F(T^{-1}z) = \int \chi(T^{-1}z,z')\, f(z')\, dz' = k \int \chi(z,z')\, f(T^{-1}z')\, dz' ,$$

by (**) and the invariance of χ under T.

Thus, $F_T = k\chi f_T$, where the subscript "T" indicates the function obtained by replacing the variable z by $T^{-1}z$, showing that $\|F_T\| = k\|f_T\|$. On the other hand, from (**) it results that $\|f_T\|^2 = k\|f\|^2$ for any f (including F). It follows that $k = 1$.

Now let T be symplectic. The map $f \to f_T$ carries $L_2(A)$ onto itself, and it results from the invariance of χ and of Haar measure under T that this mapping is an automorphism of $L_2(A)$ as an algebra relative to skew convolution as multiplication. Since $(f^*)_T = (f_T)^*$, where $f^*(z) = \bar{f}(-z)$, it is a *-automorphism. It follows, noting that $(W_f)^* = W_{f^*}$, that the map $W_f \to W_{f_T}$ is a *-automorphism $\theta(T)$ of $\mathfrak{R}$. Appealing to the result that any such automorphism is transformation by a fixed unitary operator, unique within multiplication by a scalar factor (see Appendix following Corollary 2.2), the existence of the $\Gamma(T)$ described in Theorem 2 follows.

Continuity of Γ means that $\theta(T)$ depends continuously on T, in the strong operator topology, which is equivalent to the continuity of the map $T \to f_T$ from Σ into $L_2(A)$ for any fixed f in $L_2(A)$. If f is continuous and vanishes outside a compact set, this follows directly from the definition of the topology on Σ. For arbitrary f in $L_2(A)$, it results from the fact that if $\{f_n\}$ is a sequence of continuous functions vanishing outside compact sets convergent in $L_2(A)$ to f, then $\{(f_n)_T\}$ is uniformly convergent on Σ to f_T.

Now consider the case when T is anti-symplectic. The map $f \to \bar{f}_T$, which is anti-linear, is then an anti-linear automorphism of $L_2(A)$ as an algebra over the reals relative to skew convolution as multiplication, which leaves invariant the set of all self-adjoint elements, and the existence of the required anti-unitary $\Gamma(T)$ follows.

COROLLARY 2.1. *If G and G' are locally compact abelian groups, and if T and T' are symplectic transformations on $G \oplus G^*$ and $G \oplus G'^*$ respectively, then*

$$\Gamma(T \oplus T') \cong \Gamma(T) \otimes \Gamma(T') ,$$

where Γ is the representation described in Theorem 2, while $T \oplus T'$ is the automorphism of $(G \oplus G') \oplus (G^* \oplus G'^*)$ extending both T and T', and the unitary equivalence between $L_2(G \oplus G')$ and $L_2(G) \otimes L_2(G')$ is that extending the correspondence $f(a)g(a') \leftrightarrow f \otimes g$.

This follows in a straightforward way from uniqueness, since $\Gamma(T) \otimes \Gamma(T')$ is readily seen to have the defining properties of $\Gamma(T \oplus T')$.

COROLLARY 2.2. *$\Gamma \otimes_i \Gamma^* \cong \Omega$, where Ω is the unitary representation of the extended symplectic group given by the equation*

$$\Omega(T): \ f(z) \to f(T^{-1}z), \qquad f \in L_2(G \oplus G^*) ,$$

and "$\otimes_i$" indicates the internal direct product.

For any representation Γ of a group on a Hilbert space, $\Gamma \times \Gamma^*$ is isomorphic to the induced action in the Hilbert space of all Hilbert–Schmidt operators, $X \to \Gamma(a) X \Gamma(a)^*$. By Theorem 1 this action is equivalent, via the Weyl transform, to the stated action.

APPENDIX. Any *-automorphism $\theta(T)$ of $\mathfrak{R}$ is transformation by a fixed unitary operator, unique within multiplication by a scalar factor. This quite elementary result may be established in the following way. Any projection P in $\mathfrak{R}$ is carried by a given linear automorphism θ of $\mathfrak{R}$ preserving the set of self-adjoint elements into a projection Q, and θ maps the subring of operators X such that $XP = PX = X$ into that of operators Y such that $YQ = QY = Y$. These subrings are *-isomorphic to the complete rings of all linear transformations on the respective ranges of P and Q, so that by the familiar finite-dimensional result there exists a unitary transformation S_P between these ranges, unique apart from an ambiguous scalar factor, implementing the isomorphism. The ambiguity may be removed by requiring an arbitrary fixed non-zero vector in the range of P to go into a particular representative for the corresponding ray of vectors in the range of Q. Doing this once for all, and then considering only $P' \geqq P$, a net $S_{P'}$ is obtained, each $S_{P'}$ being a well-defined unitary transformation from the range of P' onto that of $\theta(P')$ and with $S_{P''}$ an extension of $S_{P'}$ when $P' \leqq P''$, since the restriction of $S_{P''}$ to the range of P' has the defining properties of $S_{P'}$. Defining S as the unique transformation on the Hilbert space extending all the $S_{P'}$, S is a unitary operator implementing the given automorphism. The structure of the general anti-linear automorphism leaving invariant the set of all self-adjoint elements follows from this by composing it with a fixed such automorphism which is transformation by an anti-unitary operator (e.g. any conjugation).

Comments on the foregoing.

1. *Strict conformity with the conventional Weyl relation.* In the interest of treating more general groups G, in the formulation of the Weyl operators $W(z)$ above, certain factors of 2 intervene above in a different fashion from the treatment in [7]. When the doubling operation on G

is an automorphism, the procedure followed is entirely equivalent to one directly parallel to that followed in [7]. Specifically, the fixed unitary transformation

$$V:\quad f(x) \to c'f(\varepsilon x), \qquad c'^2 = d(\varepsilon x)/dx\ ,$$

where $2\varepsilon x = x$ for all x, carries the operators $W(z)$ defined above into the operators $W_0(\varrho z)$, $VW(z)V^{-1} = W_0(\varrho z)$, where

$$W_0(z):\quad f(x) \to a^*(x+\varepsilon a)f(x+a), \qquad \varrho:\quad (a,a^*) \to (2a,a^*)\ .$$

The relation satisfied by the $W_0(z)$ is

$$W_0(z)W_0(z') = \chi_0(z,z')W_0(z,z')\ ,$$

where $\chi_0(z,z') = a^*(\varepsilon b)\big(b^*(\varepsilon a)\big)^{-1}$, for $z=(a,a^*)$ and $z'=(b,b^*)$.

2. *The inhomogeneous symplectic group.* The harmonic representation is the restriction to the symplectic group of a projective unitary representation of the larger group obtained by extending Σ^+ by the translations on A. The inhomogeneous symplectic transformation $S:\ z \to Tz + z_0$, where $T \in \Sigma^+$, is represented by the unitary operator $\Gamma(S)$ on $L_2(G)$ characterized as that transforming W_f into W_{f_S}, where $f_S(z) = f(S^{-1}z)$. From an elementary quantum mechanical viewpoint the corresponding infinitesimal group is that of all hamiltonians which are at most quadratic in the canonical variables.

3. *The exponential of certain representations.* Corollary 2.1 is a type of exponential law for the representation Γ, which is significant for quantum theory. It is noteworthy that in the case of a real vector group, the restriction Γ' of Γ to the maximal compact subgroup (the unitary group, within conjugacy) can in fact be expressed by a series having a formal analogy with the exponential series:

$$\Gamma'(U) = I \oplus U \oplus (U\otimes U)_{sy} \oplus \ldots \oplus (U\otimes U\otimes \ldots)_{sy} \oplus \ldots\ ,$$

where for any unitary operator U, $(U\otimes U\otimes\ldots)_{sy}$ with n factors represents the n-fold symmetric tensor product. (Cf. [6], where analogous features to this and Corollaries 2.1 and 2.2 are also discussed.) The foregoing representation has however only a specious similarity to the representation of the full symplectic group

$$T \to I \oplus T \oplus (T\otimes T)_{sy} \oplus \ldots \oplus (T\otimes T\otimes \ldots)_{sy} \oplus \ldots;$$

the tensor products involved here are relative to the real rather than the complex field as in the case of Γ'. It would seem of interest to determine more generally the decomposition into irreducible constituents of the restriction of Γ to a maximal compact subgroup; in the real vector case this is multiplicity-free, and forms the basis of the so-called occupation number formalism in quantum theory, the proper functions under a maximal abelian subgroup of the maximal compact group being the familiar hermite functions which intervene in this connection.

ADDED IN PROOF. (January 16, 1964). In amplification of the indication earlier as to the relevance of the Stone–von Neumann–Mackey theorem, it may be mentioned that the use of the extended Weyl relations in place of those which are the direct concern of this theorem, makes possible a very short derivation from it of the existence, but not the continuity or other finer features, of the harmonic representation. On the other hand, Mackey has since independently made essentially the same observation, and in addition derived from his general theory certain continuity properties for the representation, as well as a natural extension of it to the case of an arbitrary locally compact abelian group, for which squaring is not necessarily an automorphism [to be submitted for publication].

REFERENCES

1. R. Brauer and H. Weyl, *Spinors in n dimensions*, Amer. J. Math. 57 (1935), 425–449.
2. L. van Hove, *Sur certaines représentations unitaires d'un groupe infini de transformations*, Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° 26, no. 6 (1951).
3. G. W. Mackey, *Unitary representations of group extensions* I, Acta Math. 99 (1958), 265–311.
4. J. von Neumann, *Die Eindeutigkeit der Schrödingerschen Operatoren*, Math. Ann. 104 (1931), 570–578.
5. I. E. Segal, *A class of operator algebras which are determined by groups*, Duke Math. J. 18 (1951) 221–265.
6. I. E. Segal, in Report of the Lille (1957) *Colloque sur les problèmes mathématiques de la théorie quantique des champs*, Paris, 1959, 57–103.
7. I. E. Segal, *Foundations of the theory of dynamical systems of infinitely many degrees of freedom* I, Mat.-fys. Medd. Danske Vid. Selsk. 31, No. 12, 1–39 (1959).
8. D. Shale, *Linear symmetries of free boson fields*, Trans. Amer. Math. Soc. 103 (1962), 149–167.
9. A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris, 1938.
10. H. Weyl, *Quantenmechanik und Gruppentheorie*, Z. Physik 46 (1927), 1–46.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE 39, MASS., U.S.A.