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ON A PROBLEM OF ALFSEN AND FENSTAD

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In [2] the authors closed with the following question: Does every p-equivalence class of uniform structures have a finest member? The purpose of the present note is to give a negative answer to this question. Thus completion of proximity spaces is not equivalent to completion of uniform spaces.

Let (X,p) be a general proximity space [3]. Let $\mathscr{U}$ be the class of all pseudometrics ϱ on $X \times X$ which satisfy

$$(1) \qquad \varrho(A,B) = 0 \quad \text{for all subsets } A,B \text{ of } X \text{ with } A\, p\, B\,.$$

($\mathscr{U}$ is the "gauge system" of [5].) Let $\mathscr{T}$ consist of all totally bounded pseudometrics in $\mathscr{U}$. We shall consider uniform structures to be classes of pseudometrics with the appropriate properties (see [4, Chapter 15]). From this point of view $\mathscr{T}$ is a uniform structure [1]. We shall prove (Theorem 2) that $\mathscr{U}$ need not be a uniform structure.

From [1] it follows that a uniform structure $\mathscr{S}$ belongs to the equivalence class determined by p if, and only if,

$$(2) \qquad \mathscr{T} \subseteqq \mathscr{S} \subseteqq \mathscr{U}\,.$$

LEMMA I. *Let $\mathscr{R}$ be any non-empty subclass of $\mathscr{U}$ such that*

$$(3) \qquad \varrho_1 \textit{ and } \varrho_2 \textit{ in } \mathscr{R} \textit{ imply } \varrho_1 \vee \varrho_2 \textit{ is in } \mathscr{R}\,.$$

Then the uniform structure $\mathscr{S}$ generated by $\mathscr{R}$ is a subclass of $\mathscr{U}$.

PROOF. In view of (3), $\mathscr{S}$ consists of all pseudometrics which are uniformly continuous with respect to $\mathscr{R}$. Since $\mathscr{U}$ contains every pseudometric uniformly continuous with respect to $\mathscr{U}$ and since $\mathscr{R}$ is contained in $\mathscr{U}$, $\mathscr{U}$ contains every pseudometric uniformly continuous with respect to $\mathscr{R}$.

LEMMA II. *Given any pseudometric ϱ in $\mathscr{U}$ there exists a uniform structure $\mathscr{S}$ containing ϱ such that* (2) *holds.*

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PROOF. Let $\mathscr{R}$ consist of all $\varrho \vee \beta$ with β in $\mathscr{T}$. By Lemma 1 of [5], $\mathscr{R}$ is contained in $\mathscr{U}$. Thus Lemma II follows from Lemma I.

THEOREM 1. *For a general proximity space the following conditions are equivalent:*

(i) *The equivalence class of uniform structures determined by p has a finest (i.e. largest) member.*

(ii) *$\mathscr{U}$ is a uniform structure.*

(iii) *ϱ_1 and ϱ_2 in $\mathscr{U}$ imply $\varrho_1 \vee \varrho_2$ is in $\mathscr{U}$.*

PROOF. The equivalence of (i) and (ii) follows from (2) and Lemma II. That (iii) implies (ii) follows from Lemma I. The converse is a consequence of the definition [4] of uniform structure.

THEOREM 2. *There exist proximity spaces for which the conditions* (i), (ii), (iii) *fail to hold.*

PROOF. Let X be the cartesian product $X_1 \times X_2$ where $X_1 = X_2$ is any infinite set. Let P_t be the canonical projection of X onto X_t:

$$P_t x = x_t \quad \text{for} \quad x = (x_1, x_2) \,. \tag{4}$$

For A, B subsets of X define $A \; p \; B$ to mean:

(5) Given any finite coverings $A_1, \ldots, A_m$ of A and $B_1, \ldots, B_n$ of B there exist A_i and B_j such that $P_t A_i$ meets $P_t B_j$ for $t = 1, 2$.

One can verify directly that p is a proximity relation. (Indeed p is the product proximity relation over the product of two discrete proximity spaces [6], [1].) Now p is not the discrete proximity relation. In particular, for D the diagonal in X we contend

$$D \; p \; X - D \,. \tag{6}$$

To prove (6) consider (5) with $A = D$ and $B = X - D$. Since D is infinite, some A_i from the given covering of D must contain at least two distinct points (x_1, x_1) and (x_2, x_2) of D. Thus (x_1, x_2) is in $X - D$, hence in some B_j from the given covering of $X - D$. Thus for $t = 1, 2$ we have x_t in both $P_t A_i$ and $P_t B_j$. So (5) holds, giving (6).

Now we contend that (iii) of Theorem 1 fails to hold for the class $\mathscr{U}$ of pseudometrics defined by (1). To show this define for $x = (x_1, x_2)$ and $y = (y_1, y_2)$

$$\varrho_t(x, y) = \begin{cases} 0 & \text{if } x_t = y_t \\ 1 & \text{if } x_t \neq y_t \,. \end{cases} \tag{7}$$

Clearly each ϱ_t is in $\mathscr{U}$ since by (5) $A\ p\ B$ implies P_tA meets P_tB, which by (7) implies $\varrho_t(A,B)=0$. Now for $\varrho=\varrho_1 \vee \varrho_2$ we have

$$\varrho(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y\,. \end{cases} \tag{8}$$

Thus,

$$\varrho(D,X-D) = 1\,. \tag{9}$$

Comparison of (9) with (6) shows that ϱ is not in $\mathscr{U}$ since (1) fails to hold.

NOTE ADDED IN PROOF: Theorem 2 has been proved by Alfsen and Njåstad in *Proximity and generalized uniformity*, Fund. Math. 52 (1963), 235–252.

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