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ON L^p ESTIMATES AND REGULARITY II

MARTIN SCHECHTER

1. Introduction.

In Part I of this work (cf. [15]) we employed abstract interpolation theorems to obtain estimates for boundary problems. We considered the spaces $H^{s,p}$, $1<p<\infty$. For a domain G in Euclidean n-space and s a non-negative integer, $H^{s,p}(G)$ is the set of functions which are in $L^p(G)$ together with all derivatives up to order s. For positive real s, $H^{s,p}(G)$ is defined by complex interpolation between consecutive integers (cf. Part I). For $s<0$ we define $H^{s,p}(G)$ as the dual of $H^{-s,p'}(G)$, $p'=p/(p-1)$.

The estimate obtained in Part I implied that for each real s

$$(1.1) \qquad \|u\|_{s,p} \leqq \text{const.}\,(\|Au\|_{s-m,p}+\|u\|_{s-m,p})$$

holds for all functions u satisfying given homogeneous boundary conditions, where A is an elliptic operator of order m and $\|\cdot\|_{s,p}$ is the norm in $H^{s,p}(G)$ (for precise hypotheses cf. [15]). In this part we extend the inequality to include boundary terms. If $B_1,\ldots,B_r$ are the given differential boundary operators, we show that for each s

$$(1.2) \qquad \|u\|_{s,p} \leqq \text{const.}\,(\|Au\|_{s-m,p}+\sum \langle B_j u\rangle_{s-m_j-1/p,\,p}+\|u\|_{s-m,p})$$

holds for all functions u, where m_j is the order of B_j. The expressions $\langle\cdot\rangle_{t,p}$ are appropriate boundary norms. For $t \geqq 1-1/p$ they are defined by

$$(1.3) \qquad \langle\varphi\rangle_{t,p} = \text{g.l.b.}\,\|u\|_{t+1/p,\,p}\,,$$

where the g.l.b. is taken over all functions u which equal φ on the boundary. (The reason for the particular choice of the index will be clear presently. For precise hypotheses used in proving (1.2) and the complete definition of the norm $\langle\cdot\rangle_{t,p}$ see Section 2.)

For s an integer $\geqq m$, inequality (1.2) was proved by Agmon–Douglis–Nirenberg [3], Browder [5], and Slobodeckii [17]. For other integers it was proved by Lions–Magenes [10] and Schechter [16].

In addition to the norms (1.3) one can define $H^{s,p}$ norms on the bound-

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ary. If ∂G denotes the boundary of G, the space $H^{s,p}(\partial G)$ can be defined by mapping ∂G locally into the unit ball S^{n-1} in $n-1$ space and considering the pre-images of elements in $H^{s,p}(S^{n-1})$ (for details of the construction cf. Section 2). The norm in $H^{s,p}(\partial G)$ is denoted by $\|\cdot\|_{s,p}^{\partial G}$.

We prove the following relationship between the norms $\langle\ \rangle_{s,p}$ and $\|\cdot\|_{s,p}^{\partial G}$, s an arbitrary real number.

(a) For $1 < p \leqq 2$, $\|\varphi\|_{s,p}^{\partial G} \leqq \text{const.}\langle\varphi\rangle_{s,p}$

(b) For $2 \leqq p < \infty$, $\langle\varphi\rangle_{s,p} \leqq \text{const.}\|\varphi\|_{s,p}^{\partial G}$.

Combining (b) with (1.2) we obtain for $2 \leqq p < \infty$

$$\|u\|_{s,p} \leqq \text{const.}(\|Au\|_{s-m,p} + \sum \|B_j u\|_{s-m_j-1/p,p}^{\partial G} + \|u\|_{s-m,p})$$

holding for all functions u. Some local variations are also given.

We also prove a related regularity theorem similar to a result of Peetre [12]. Let $W^{s,p}(\partial G)$ denote the closure of $C^\infty(\partial G)$ with respect to the norm $\langle\cdot\rangle_{s,p}$ and let $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote the scalar products on G and ∂G, respectively. Then if f and g are distributions and

$$|(f, Au) + \langle g, B_1 u\rangle| \leqq c_0 \|u\|_{m-s,p'}$$

for all u satisfying the remaining boundary conditions (that is, $B_j u = 0$ on ∂G for $j \neq 1$), then $f \in H^{s,p}(G)$, $g \in W^{s-m+m_1+1-1/p,p}(\partial G)$ and

$$\|f\|_{s,p} + \langle g\rangle_{s-m+m_1+1-1/p,p} \leqq \text{const.}(c_0 + \|f\|_{s-l,p}) .$$

From this we obtain the estimate

$$\sum_{j=0}^{m-1} \langle \partial^j u/\partial n^j\rangle_{s-j-1/p,p} \leqq \text{const.}(\|Au\|_{s-m,p} + \sum \langle B_j u\rangle_{s-m_j-1/p,p} + \|u\|_{s-m,p})$$

holding for all u where $\partial^j u/\partial n^j$ denotes the normal derivative of order j.

We also note the effect of inequality (1.2) on our coerciveness results of [16]. Since we had previously been able to prove (1.2) only for s an integer, our estimates in [16] involving boundary terms were stated only for such s. Now that (1.2) has been proved in general, we may remove this restriction on the results of [16] (cf. the end of Section 2).

In proving the inclusions (a) and (b) above, we obtained certain results for the whole space E^n. We have recently learned that some of them overlap with work of Calderón and Stein (cf. the discussion at the end af Section 4).

2. Estimates for boundary problems.

We assume that the reader is familiar with the notation of [15], to which we refer as Part I. Let G be a bounded domain in Euclidean

n-space E^n with boundary ∂G of class C^∞, and let p be a fixed real number greater than one. For $\varphi \in C^\infty(\partial G)$ and s real and $\geqq 1-1/p$ we define

$$(2.1) \qquad \langle\varphi\rangle_{s,p} = \text{g.l.b.}\, \|u\|_{s+1/p,p} ,$$

where the g.l.b. is taken over all $u \in C^\infty(\bar{G})$ which equal φ on ∂G. For s real and $\leqq -1/p$ we set

$$(2.2) \qquad \langle\varphi\rangle_{s,p} = \text{l.u.b.}\, \frac{\langle\varphi,\psi\rangle}{\langle\psi\rangle_{-s,p'}},$$

where

$$\langle\varphi,\psi\rangle = \int\limits_{\partial G} \varphi\bar{\psi}\, d\sigma ,$$

$p'=p/(p-1)$, and the l.u.b. is taken over all $\psi \in C^\infty(\partial G)$. Note that we have not yet defined $\langle\varphi\rangle_{s,p}$ for s in the open interval $I \equiv (-1/p, 1-1/p)$.

LEMMA 2.1. *For s real and not in I, $\langle\varphi\rangle_{s,p}=0$ implies $\varphi\equiv 0$ on ∂G.*

The simple proof of Lemma 2.1 will be given at the end of Section 5. From it we see that $\langle\cdot\rangle_{s,p}$ is a norm for $s \notin I$. We denote the completion of $C^\infty(\partial G)$ with respect to the norm by $W^{s,p}(\partial G)$. (These spaces are not identical to the "trace spaces" of Lions–Magenes [10, III]. However when $s+1/p$ is an integer, they are equivalent (cf. Section 3).) For $s \in I$ we set

$$(2.3) \qquad W^{s,p}(\partial G) = [W^{-1/p,p}(\partial G), W^{1-1/p,p}(\partial G); \delta(s+1/p)]$$

(cf. Section 2 of Part I). For such s we let $\langle\cdot\rangle_{s,p}$ denote the norm in $W^{s,p}(\partial G)$.

Let A be a properly elliptic operator of order $m=2r$ in $\bar{G}$ and $\{B_j\}_{j=1}^r$ a set of boundary operators which covers A (cf. Section 5 of Part I). We assume that the coefficients of A and the B_j are in $C^\infty(\bar{G})$. Moreover, the orders m_j of the B_j are to be distinct and less than m, and ∂G is to be nowhere characteristic to any of the B_j.

Let V be the set of those $u \in C^\infty(\bar{G})$ which satisfy

$$(2.4) \qquad B_j u = 0 \quad \text{on} \quad \partial G, \qquad 1 \leqq j \leqq r ,$$

and V' the set of $v \in C^\infty(\bar{G})$ such that $(Au,v)=(u,A'v)$ for all $u \in V$ (recall that A' is the formal adjoint of A). As in Part I we make use of the following norms for s real and non-negative

$$(2.5) \qquad |w|'_{-s,p} = \underset{v\in V'}{\text{l.u.b.}}\, \frac{|(w,v)|}{\|v\|_{s,p'}}$$

$$|w|'_{s,p} = \|w\|_{s,p}$$

The closure of V' with respect to the norm $|\cdot|'_{s,p}$ is denoted by $V'^{s,p}(G)$.

THEOREM 2.1. *For each real s*

$$(2.6)\qquad \|u\|_{s,p} \leqq \text{const.}\left(|Au|'_{s-m,p}+\sum_{j=1}^{r}\langle B_ju\rangle_{s-m_j-1/p,p}+\|u\|_{s-m,p}\right)$$

holds for all $u \in C^\infty(\bar{G})$.

THEOREM 2.2. *Let f be a distribution in* $H^{s-l,p}(G)$ $(l>0)$, *g a distribution on* ∂G, *and assume that*

$$(2.7)\qquad |(f,Au)+\langle g,B_1u\rangle| \leqq c_0\|u\|_{m-s,p'}$$

for all $u \in C^\infty(\bar{G})$ *satisfying*

$$(2.8)\qquad B_ju = 0 \quad in \quad \partial G, \qquad 2 \leqq j \leqq r\,.$$

Then $f \in V'^{s,p}(G)$, $g \in W^{s-m+m_1+1-1/p,p}(\partial G)$, *and*

$$(2.9)\qquad |f|'_{s,p}+\langle g\rangle_{s-m+m_1+1-1/p,p} \leqq \text{const.}(c_0+\|f\|_{s-l,p})$$

where the constant does not depend on f, g, or c_0.

THEOREM 2.3. *For each real s the inequality*

(2.10)

$$\sum_{j=0}^{m-1}\langle \partial^j u/\partial n^j\rangle_{s-j-1/p,p} \leqq \text{const.}\left(\|Au\|_{s-m,p}+\sum_{j=1}^{r}\langle B_ju\rangle_{s-m_j-1/p,p}+\|u\|_{s-m,p}\right)$$

holds for all $u \in C^\infty(\bar{G})$, *where* $\partial^j u/\partial n^j$ *denotes the normal derivative of order j.*

We now consider the spaces $H^{s,p}(\partial G)$ (cf. [10, III, p. 66]). Since ∂G is of class C^∞, $\bar{G}$ can be covered by an interior subdomain G_0 and a finite set of open "boundary patches" $\{N_\nu\}$ with the following property:

(*) For each ν there is a C^∞ homeomorphim T_ν which maps $\bar{N}_\nu$ onto the ball $|x| \leqq 1$ and $N_\nu \cap \partial G$ into the hysperplane $x_n = 0$.

The sets $N_\nu \cap \partial G$ form a $n-1$ dimensional open covering of ∂G. Let $\sum \zeta_\nu{}^2 = 1$ be a partition of unity subordinate to this covering. We may assume that each $\zeta_\nu \in C_0{}^\infty(N_\nu)$. If S^k denotes the unit ball $|x|<1$ in E^k, then $T_\nu(N_\nu) = S^n$ and $T_\nu(N_\nu \cap \partial G) = S^{n-1}$ by (*). For $g \in C^\infty(\partial G)$ we set

$$\tau_\nu g(x) = g(T_\nu{}^{-1}x), \qquad x \in S^{n-1}$$

and

$$(2.11)\qquad \|g\|^{\partial G}_{s,p} = \left[\sum_\nu (\|\tau_\nu\zeta_\nu g\|^{S^{n-1}}_{s,p})^p\right]^{1/p}$$

where we consider S^{n-1} as a domain in E^{n-1}, and the norm on the right hand side is that of $H^{s,p}(S^{n-1})$. Clearly, $\|\cdot\|^{\partial G}_{s,p}$ is a norm; we denote the completion of $C^\infty(\partial G)$ with respect to it by $H^{s,p}(\partial G)$. It is not difficult to show that all possible choices of the N_ν and ζ_ν give equivalent spaces (cf. [10, III]).

An important relationship between the spaces $H^{s,p}(\partial G)$ and $W^{s,p}(\partial G)$ is

THEOREM 2.4. *If $p \geqq 2$, then for each real s*

$$H^{s,p}(\partial G) \subseteq W^{s,p}(\partial G) . \tag{2.12}$$

If $p \leqq 2$, then

$$H^{s,p}(\partial G) \supseteq W^{s,p}(\partial G) \tag{2.13}$$

Both inclusions are continuous.

Combining Theorems 2.1 and 2.4 we have

THEOREM 2.5. *If $p \geqq 2$, then for each real s*

$$\|u\|_{s,p} \leqq \text{const.}\left(\|Au\|_{s-m,p} + \sum_{j=1}^{r} \|B_j u\|^{\partial G}_{s-m_j-1/p,p} + \|u\|_{s-m,p}\right) \tag{2.14}$$

for all $u \in C^\infty(\bar{G})$.

We give now some local versions of Theorem 2.2. Let Γ be a subset of ∂G which is open in the topology of ∂G and such that the boundary of Γ consists of a finite number of C^∞ manifolds of dimension $n-2$. The set $C_0^\infty(G \cup \Gamma)$ will consist of those functions in $C^\infty(\bar{G})$ which vanish near $\partial G - \Gamma$. We say that $u \in H^{s,p}_{\text{loc}}(G \cup \Gamma)$ if $\zeta u \in H^{s,p}(G)$ for every $\zeta \in C_0^\infty(G \cup \Gamma)$. Similarly ,we say that $g \in W^{s,p}_{\text{loc}}(\Gamma)$ if $\varphi g \in W^{s,p}(\partial G)$ for every $\varphi \in C_0^\infty(\Gamma)$, the set of those $\varphi \in C^\infty(\partial G)$ which vanish on and near $\partial G - \Gamma$.

THEOREM 2.6. *If f is a distribution on G and*

$$|(f, Au)| \leqq \text{const.}\, \|u\|_{m-s,p'}$$

for all $u \in C^\infty(G \cup \Gamma)$ satisfying

$$B_j u = 0 \quad \text{on} \quad \Gamma, \qquad 1 \leqq j \leqq r , \tag{2.15}$$

then $f \in H^{s,p}_{\text{loc}}(G \cup \Gamma)$.

THEOREM 2.7. *If f is a distribution on G, g a distribution on ∂G and*

$$|(f, Au) + \langle g, B_1 u\rangle| \leqq \text{const.}\, \|u\|_{m-s,p'}$$

for all $u \in C_0^\infty(G \cup \Gamma)$ satisfying

$$B_j u = 0 \quad \text{on} \quad \Gamma, \qquad 2 \leqq j \leqq r , \tag{2.16}$$

then

$$f \in H^{s,p}_{\text{loc}}(G \cup \Gamma) \qquad \text{and} \qquad g \in W^{s-m+m_1+1-1/p,\,p}_{\text{loc}}(\Gamma) .$$

For s an integer $\geqq m$, (2.6) was proved by Agmon–Douglis–Nirenberg [3], Browder [5] and Slobodeckii [17]. For s an arbitrary integer, it was proved in [16]. Related results employing different norms were established by Lions–Magenes [10]. A slightly weaker form of Theorem 2.2 was first proved by Peetre [12] for the case $p=2$ and s greater than the maximum order of the B_j.

The relation (2.13) was essentially proved by Calderón [6] for $s \geqq 1-1/p$ (cf. Lemma 4.8).

The proof of Theorem 2.4 is given in Section 5 with the remaining proofs given in Section 6. In Section 3 we prove some abstract interpolation results from which we obtain

$$W^{s_1+\theta(s_2-s_1),\,p}(\partial G) = [W^{s_1,\,p}(\partial G), W^{s_2,\,p}(\partial G); \delta(\theta)]$$

for any real numbers s_1, s_2. The relationship between the spaces $H^{s,p}(E^k)$ and $W^{s,p}(E^k)$ is discussed in Section 4. In Section 5 we consider the spaces $H^{s,p}(\partial G)$. In the original draft of the paper we included some estimates of the Schauder type. However, we have decided to postpone their publication until our results are more complete.

We now consider some remarks concerning our study of coerciveness in L^p. We refer the reader to [16]. In that paper we proved that for s an integer

$$\|u\|_{s,p} \leqq \text{const.}\,(\textstyle\sum \|A_k u\|_{s-h_k,p} + \sum \langle B_j u\rangle_{s-m_j-1/p,\,p} + \|u\|_{s-m,\,p}) \tag{2.17}$$

holds for all u under certain assumptions on the operators A_k (hypotheses (a)–(c) of Section 3 in [16]. In (2.17) h_k is the order of A_k. The notation is slightly different from that of [16].) The precise statement is given in Theorem 3.1 of [16]. The proof of (2.17) relied on (2.6) which had then been proved only for s an integer. As soon as (2.6) is known to hold for all real values of s, it follows that the same is true for (2.7). We state this as

THEOREM 2.8. *Under hypotheses* (a)–(e) *Section* 3 *of* [16], *inequality* (2.17) *holds for each real* s. *In particular if* $p \geqq 2$ *we have*

$$\|u\|_{s,p} \leqq \text{const.}\,(\textstyle\sum \|A_k u\|_{s-h_k,\,p} + \sum \|B_j u\|^{\partial G}_{s-m_j-1/p,\,p} + \|u\|_{s-m,\,p})\,. \tag{2.18}$$

The same remarks apply to our theorems for bilinear forms in Section 8 of [16]. We refer to that section for definitions.

THEOREM 2.9. *Under the hypotheses of Theorem* 8.1 *of* [16], *the inequality*

$$\|u\|_{s,p} \leqq \text{const.}\,([u]_{s-m,\,p} + \textstyle\sum \langle B_j u\rangle_{s-m_j-1/p,\,p} + \|u\|_{s-m,\,p})$$

holds for each real s. An inequality corresponding to (2.18) holds when $p \geqq 2$.

3. Some Theorems of Interpolation Spaces.

In this section we shall discuss the spaces $W^{s,p}(\partial G)$. We shall show that they have very desirable properties when complex interpolation methods are employed. Specifically, we shall prove

THEOREM 3.1. *For any real numbers* s_1, s_2

$$W^{s_3,p}(\partial G) = [W^{s_1,p}(\partial G), W^{s_2,p}(\partial G); \delta(\theta)] , \tag{3.1}$$

where $s_3 = s_1 + \theta(s_2 - s_1)$.

In proving Theorem 3.1 we shall make use of abstract results in complex interpolation theory. In addition to the lemmas of Section 2 of Part I, we shall employ

THEOREM 3.2. *Let* X_0 *and* X_1 *be Banach spaces and set*

$$X_\theta = [X_0, X_1; \delta(\theta)], \qquad 0 < \theta < 1 .$$

Then for any θ_0 such that $0 < \theta_0 < 1$

$$X_{\theta\theta_0} \subseteqq [X_0, X_{\theta_0}; \delta(\theta)] .$$

with continuous injection.

COROLLARY 3.1. *If* X_0 *and* X_1 *are reflexive, then*

$$X_{\theta\theta_0} = [X_0, X_{\theta_0}; \delta(\theta)] .$$

We shall prove Theorem 3.1 and Corollary 3.1 at the end of this section. Employing them we have

COROLLARY 3.2. *If* X_0 *and* X_1 *are reflexive, then for* $0 \leqq \theta_1 \leqq \theta_2 \leqq 1$

$$[X_{\theta_1}, X_{\theta_2}; \delta(\theta)] = [X_0, X_1; \delta(\theta_1 + \theta(\theta_2 - \theta_1))] .$$

PROOF. By Corollary 3.1,

$$X_{\theta_1} = [X_0, X_{\theta_2}; \delta(\theta_1/\theta_2)] .$$

Hence by formula (2.1) of Part I

$$X_{\theta_1} = [X_{\theta_2}, X_0; \delta(1 - \theta_1/\theta_2)] .$$

Thus

$$\begin{aligned}
[X_{\theta_1}, X_{\theta_2}; \delta(\theta)] &= [X_{\theta_2}, X_{\theta_1}; \delta(1 - \theta)] \\
&= [X_{\theta_2}, X_0; \delta((1-\theta)(1 - \theta_1/\theta_2))] \\
&= [X_0, X_{\theta_2}; \delta(\theta + \theta_1/\theta_2 - \theta\theta_1/\theta_2)] \\
&= [X_0, X_1; \delta(\theta\theta_2 + \theta_1 - \theta\theta_1)] ,
\end{aligned}$$

where we have employed Corollary 3.1 twice more.

THEOREM 3.3. *Let $\{X_k\}$ be a sequence of reflexive Banach spaces, $k=0, \pm 1, \pm 2, \ldots$, such that*

$$X_{k_3} = [X_{k_1}, X_{k_2}; \delta(\theta)] \tag{3.2}$$

whenever k_1, k_2, and $k_3 = k_1 + \theta(k_2 - k_1)$ are integers. For any real number s let i be the integer such that $i \leqq s < i+1$ and set

$$X_s = [X_i, X_{i+1}; \delta(s-i)] .$$

Then for any real numbers s_1, s_2 and $s_3 = s_1 + \theta(s_2 - s_1)$, we have

$$X_{s_3} = [X_{s_1}, X_{s_2}; \delta(\theta)] .$$

PROOF. We consider three cases.

Case I; the numbers s_1 and s_2 are integers and s_3 is arbitrary. Let i be the integer such that $i \leqq s_3 < i+1$. Then by Corollary 3.2 and (3.2)

$$X_{s_3} \equiv [X_i, X_{i+1}; \delta(s_3 - i)] = \left[X_{s_1}, X_{s_2}; \delta\left(\frac{i-s_1}{s_2-s_1} + \frac{s_3-i}{s_2-s_1}\right)\right] = [X_{s_1}, X_{s_2}; \delta(\theta)]$$

since $\theta = (s_3 - s_1)/(s_2 - s_1)$.

Case II; the number s_1 is an integer, s_2, s_3 are arbitrary. Let i be the integer such that $i - 1 \leqq s_2 < i$. Then by Case I,

$$X_{s_2} = \left[X_{s_1}, X_i; \delta\left(\frac{s_2-s_1}{i-s_1}\right)\right] . \tag{3.3}$$

Hence by Corollary 3.1

$$[X_{s_1}, X_{s_2}; \delta(\theta)] = \left[X_{s_1}, X_i; \delta\left(\theta \frac{s_2-s_1}{i-s_1}\right)\right] = X_{s_3}$$

by another application of Case I.

Case III; all of the s_i are arbitrary. By Case II and (2.1) of Part I, the theorem holds when s_2 is an integer. Thus (3.3) holds and the theorem follows by repeating the proof of Case II. This completes the proof.

When $s_i + 1/p$ are integers, $i = 1, 2, 3$, Theorem 3.1 was proved by Lions–Magenes [10, V, Theorem 1.1]. Actually they did not employ our spaces $W^{s,p}(\partial G)$. However, their spaces (called "trace spaces") are equivalent to ours when $s + 1/p$ is an integer (cf. Theorem 5.1 of [10, III]). We state their result as

LEMMA 3.1. *If s_1, s_2, $s_3 = s_1 + \theta(s_2 - s_1)$ are integers, then*

$$W^{s_3 - 1/p, p}(\partial G) = [W^{s_1 - 1/p, p}(\partial G), W^{s_2 - 1/p, p}(\partial G); \delta(\theta)] .$$

We now define related spaces $\tilde{W}^{s,p}(\partial G)$ in the following way. For i an integer we set

$$\tilde{W}^{i-1/p,p}(\partial G) = W^{i-1/p,p}(\partial G) .$$

For s not an integer, we let i be the integer such that $i<s<i+1$. We then set

$$\tilde{W}^{s-1/p,p}(\partial G) = [W^{i-1/p,p}(\partial G), W^{i+1-1/p,p}(\partial G); \delta(s-i)] . \tag{3.4}$$

LEMMA 3.2. *For all real s*

$$[\tilde{W}^{s,p}(\partial G)]' = \tilde{W}^{-s,p'}(\partial G) .$$

PROOF. When $s+1/p$ is an integer, the space $\tilde{W}^{s,p}(\partial G)=W^{s,p}(\partial G)$ is equivalent to the "trace space" of Lions–Magenes [10], and the lemma follows from Proposition 2.10 of [10, III]. Otherwise let i be the integer such that $i<s+1/p<i+1$. Then by Lemma 2.3 of Part I, the dual space of $\tilde{W}^{s,p}(\partial G)$ is

$$\begin{aligned}
&[W^{i-1/p,p}(\partial G), W^{i+1-1/p,p}(\partial G); \delta(s-i+1/p)]' \\
&\qquad = [W^{-i+1/p,p'}(\partial G), W^{-i-1+1/p,p'}(\partial G); \delta(s-i+1/p)] \\
&\qquad = [W^{-i-1/p',p'}(\partial G), W^{-i+1-1/p',p'}(\partial G); \delta(i-s+1/p')] = \tilde{W}^{-s,p'}(\partial G)
\end{aligned}$$

and the lemma is proved.

COROLLARY 3.3. *The spaces $\tilde{W}^{s,p}(\partial G)$ are reflexive.*

LEMMA 3.3. *For any real s_1, s_2, $s_3=s_1+\theta(s_2-s_1)$*

$$\tilde{W}^{s_3,p}(\partial G) = [\tilde{W}^{s_1,p}(\partial G), \tilde{W}^{s_2,p}(\partial G); \delta(\theta)] .$$

PROOF. Set $X_k=W^{k-1/p,p}(\partial G)$, $k=0, \pm 1, \pm 2, \ldots$. By Corollary 3.3, the X_k are reflexive. By Lemma 3.1 the hypothesis (3.2) of Theorem 3.3 holds. The definition (3.4) shows us that our result is merely the conclusion of Theorem 3.3.

By Lemma 3.3, Theorem 3.1 will follow immediately from

THEOREM 3.4. *For each real s*

$$\tilde{W}^{s,p}(\partial G) = W^{s,p}(\partial G) . \tag{3.5}$$

PROOF. For s in the interval I there is nothing to prove (cf. (2.3)). We claim that it suffices to prove (3.5) for $s>1-1/p$. For then we shall know that $W^{s,p}(\partial G)=\tilde{W}^{s,p}(\partial G)$ is reflexive for such s (Corollary 3.3). Moreover, one easily checks that $W^{-s,p'}(\partial G)$ forms a complete set of bounded linear functionals on $W^{s,p}(\partial G)$. This means (a) that $W^{-s,p'}(\partial G)\subseteq [W^{s,p}(\partial G)]'$ and (b) if $\varphi \in W^{s,p}(\partial G)$ and $\langle\varphi,\psi\rangle=0$ for all $\psi \in W^{-s,p'}(\partial G)$, then $\varphi=0$. Both (a) and (b) follow from (2.2). Since $W^{-s,p'}(\partial G)$ is a

Banach space with the same norm as $[W^{s,p}(\partial G)]'$, it forms a closed subspace of $[W^{s,p}(\partial G)]'$. We now invoke the theorem that a complete set of linear functionals over a reflexive Banach space is dense. Thus $W^{-s,p'}(\partial G)$ is both dense and closed in $[W^{s,p}(\partial G)]'$ giving

$$W^{-s,p'}(\partial G) = [W^{s,p}(\partial G)]' . \tag{3.6}$$

Once (3.6) is proved we have

$$W^{-s,p'}(\partial G) = [W^{s,p}(\partial G)]' = [\tilde{W}^{s,p}(\partial G)]' = \tilde{W}^{-s,p'}(\partial G)$$

and (3.5) is proved for negative s as well. It therefore remains to prove the theorem for $s > 1-1/p$. This was essentially done in [16, Lemma 4.6]. We repeat the simple proof for the convenience of the reader. Let i be the integer such that $i < s+1/p < i+1$ (of course, there is nothing to prove when $s+1/p$ is an integer). By the definition (2.1) of the norms we know that the restriction operator γ_0 from $\bar{G}$ to ∂G is a bounded linear mapping of $H^{i,p}(G)$ into $W^{i-1/p,p}(\partial G)$ and from $H^{i+1,p}(G)$ into $W^{i+1-1/p,p}(\partial G)$. Hence by Lemma 2.1 and 2.4 of Part I, γ_0 is a bounded linear mapping of $H^{s+1/p,p}(G)$ into $\tilde{W}^{s,p}(\partial G)$. Thus $W^{s,p}(\partial G) \subseteq \tilde{W}^{s,p}(\partial G)$ with continuous injection. We prove the opposite inclusion as follows. For $\varphi \in C^\infty(\partial G)$ let $E\varphi$ denote the harmonic function in G which equals φ on ∂G. Then one sees that E can be extended to be a bounded linear mapping of $W^{i-1/p,p}(\mathrm{G})$ to $H^{i,p}(G)$ and from $W^{i+1-1/p,p}(\partial G)$ to $H^{i+1,p}(G)$ (cf., e.g., Theorem 6.1 of [16]). Hence it is a bounded mapping of $\tilde{W}^{s,p}(\partial G)$ into $H^{s+1/p,p}(G)$ giving $W^{s,p}(\partial G) \supseteq \tilde{W}^{s,p}(\partial G)$ and the proof is complete.

Proof of Theorem 3.2. If $u \in X_{\theta\theta_0}$, then there is a $f(z) \in \mathscr{H}(X_0,X_1)$ such that $f(\theta\theta_0)=u$ (cf. Section 2 of Part I). We claim that for each real y_0, $f(\theta_0+iy_0) \in X_{\theta_0}$. For set $g(z)=f(z-iy_0)$. Then $g(iy) \in X_0$ and $g(1+iy) \in X_1$. Hence

$$f(\theta_0+iy_0) = g(\theta_0) \in X_{\theta_0} .$$

Next set $h(z)=f(\theta_0 z)$. Then $h(iy) \in X_0$ and $h(1+iy)=f(\theta_0+i\theta_0 y) \in X_{\theta_0}$. Hence $h(z) \in \mathscr{H}(X_0,X_{\theta_0})$ and since $u=f(\theta\theta_0)=h(\theta)$, we have

$$u \in [X_0,X_{\theta_0};\delta(\theta)] .$$

Proof of Corollary 3.1. By Lemma 2.3 of Part I

$$X_\theta' = [X_0',X_1';\delta(\theta)] .$$

Hence by Theorem 3.2

$$X'_{\theta\theta_0} \subseteq [X'_0,X'_{\theta_0};\delta(\theta)] .$$

Another application of Lemma 2.3 of Part I gives

$$[X_0, X_{\theta_0}; \delta(\theta)] \subseteqq X_{\theta\theta_0} .$$

We now combine this with Theorem 3.2. to complete the proof.

4. The spaces $H^{s,p}(E^k)$ and $W^{s,p}(E^k)$.

Let E^k be k dimensional Euclidean space. The space $H^{s,p}(E^k)$ is defined in the same manner as $H^{s,p}(G)$ for a bounded domain G in E^k. For s a non-negative integer it is defined as the completion of $C_0^\infty(E^k)$ with respect to the norm

$$\|u\|_{s,p} = \left(\int_{E^k} \sum_{|\mu| \leqq s} |D^\mu u|^p \, dx \right)^{1/p} .$$

For i an integer $\geqq 0$ and $i < s < i+1$, $H^{s,p}(E^k)$ is given by

$$H^{s,p}(E^k) = [H^{i,p}(E^k), H^{i+1,p}(E^k); \delta(s-i)] ,$$

and for negative s it is

$$H^{s,p}(E^k) = [H^{-s,p'}(E^k)]' . \tag{4.1}$$

An equivalent method of defining $H^{s,p}(E^k)$ is by means of Fourier transforms (cf. Calderón [6] and Lions–Magenes [10, III]). Let $\mathscr{F}$ denote the k dimensional Fourier transform. For any real number t the operator J^t is defined by

$$\mathscr{F} J^t u = (1 + |\xi'|^2)^{-t/2} \mathscr{F} u ,$$

where $|\xi'|^2 = \xi_1^2 + \ldots + \xi_k^2$. Then $H^{s,p}(E^k)$ is the space of distributions u such that $J^{-s}u$ is in $L^p(E^k)$. For the proof of the equivalence of the two definitions see, e.g., Lions [9]. From the second definition we have immediately

LEMMA 4.1. *J^t is an isomorphism between $H^{s,p}(E^k)$ and $H^{s+t,p}(E^k)$.*

We now consider the spaces $W^{s,p}(E^k)$. They are defined as follows. We consider E^k as the hyperplane $x_{k+1} = 0$ in E^{k+1} and as the boundary of the halfspace $x_{k+1} > 0$ (denoted by E_+^{k+1}). For $s \geqq 1 - 1/p$ the functions in $W^{s,p}(E^k)$ are the restrictions to E^k of functions in $H^{s+1/p,p}(E_+^{k+1})$. The norm in $W^{s,p}(E^k)$ is

$$\langle \varphi \rangle_{s,p}^{E^k} = \text{g.l.b.} \|u\|_{s+1/p,p}^{E_+{}^{k+1}} , \tag{4.2}$$

where the g.l.b. is taken over all $u \in H^{s+1/p,p}(E_+^{k+1})$ which equal φ on E^k. From the well known inequality

$$\|\varphi\|_{0,p}^{E^k} \leqq \text{const.}\,\|u\|_{1,p}^{E_+^{k+1}}$$

we see that (4.2) is actually a norm. For $s \leqq -1/p$ we define $W^{s,p}(E^k)$ as the completion of $C_0^\infty(E^k)$ with respect to the norm

$$\langle\varphi\rangle_{s,p}^{E^k} = \underset{\psi\in C_0^\infty(E^k)}{\text{l.u.b.}} \frac{\langle\varphi,\psi\rangle^{E^k}}{\langle\psi\rangle_{-s,p'}^{E^k}}.$$

Finally, for $s \in I$ we set

$$W^{s,p}(E^k) = [W^{-1/p,p}(E^k), W^{1-1/p,p}(E^k); \delta(s+1/p)]\,.$$

The following lemma is due to Lions–Magenes [10, III].

LEMMA 4.2. *For arbitrary integers i and j, J^j is an isomorphism between $W^{i-1/p,p}(E^k)$ and $W^{i+j-1/p,p}(E^k)$.*

The proof of Lemma 4.2. relies upon the fact that for l an integer

$$W^{l-1/p,p}(E^k) = T(p,0;\, H^{l,p}(E^k), H^{l-1,p}(E^k))\,, \tag{4.3}$$

where $T(p,\alpha;\, X_0,X_1)$ represents "trace" interpolation between Banach spaces X_0 and X_1 (cf. [10, III, Proposition 1.4]). A result similar to Lemma 2.1 of Part I holds for this method of interpolation (cf. Theorem 1.1 of [10, III]). Thus since J^j is an isomorphism between $H^{i,p}(E^k)$ and $H^{i+j,p}(E^k)$ and between $H^{i-1,p}(E^k)$ and $H^{i+j-1,p}(E^k)$ (Lemma 4.1), it is an isomorphism between $W^{i-1/p,p}(E^k)$ and $W^{i+j-1/p,p}(E^k)$.

LEMMA 4.3. *For $s+1/p$ an integer*

$$[W^{s,p}(E^k)]' = W^{-s,p'}(E^k)\,. \tag{4.4}$$

PROOF. For $s=i-1/p$, i an integer, we have by (4.3),

$$[W^{s,p}(E^k)]' = [T(p,0;\, H^{i,p}(E^k), H^{i-1,p}(E^k))]'\,.$$

Now by Theorem 1.1 of Chapeter II in [8],

$$\begin{aligned}[T(p,0;\, H^{i,p}(E^k), H^{i-1,p}(E^k))]' &= T(p',0; H^{1-i,p'}(E^k), H^{-i,p'}(E^k)) \\ &= W^{1-i-1/p',p'}(E^k) = W^{-s,p'}(E^k)\end{aligned}$$

and the lemma follows.

LEMMA 4.4. *For s_i real and $s_i+1/p \geqq 1$, $i=1,2,3$, $s_3=s_1+\theta(s_2-s_1)$,*

$$[W^{s_1,p}(E^k), W^{s_2,p}(E^k); \delta(\theta)] = W^{s_3,p}(E^k)\,. \tag{4.5}$$

PROOF. The proof is similar to that of Theorem 3.4 (cf. also Lemma 4.6 of [16]). The restriction mapping γ_0 of E_+^{k+1} to E^k is bounded from $H^{s_i+1/p,p}(E_+^{k+1})$ to $W^{s_i,p}(E^k)$, $i=1,2$. Hence, by Lemma 2.4 of Part I, it is bounded from $H^{s_3+1/p,p}(E_+^{k+1})$ to X, the space on the left in (4.5).

Thus $X \supseteqq W^{s_3,p}(E^k)$. Next, for $\varphi \in C_0^\infty(E^k)$ let $E\varphi$ denote the harmonic function in $L^p(E_+^{k+1})$ which equals φ on E^k. It is easily checked that E is a bounded mapping from $W^{s_i,p}(E^k)$ into $H^{s_i+1/p,p}(E_+^{k+1})$, $i=1,2$.

Hence it is a bounded mapping of X into $H^{s_3+1/p,p}(E_+^{k+1})$ showing that $X \subseteqq W^{s_3,p}(E^k)$ and the proof is complete.

LEMMA 4.5. *(4.5) holds when s_i+1/p is an integer, $i=1,2,3$.*

PROOF. Let j be an integer such that $s_i+j+1/p \geqq 1$. By Lemma 4.2, J^j is an isomorphism between $W^{s_i,p}(E^k)$ and $W^{s_i+j,p}(E^k)$, $i=1,2$. Hence it is an isomorphism between X and

$$[W^{s_1+j,p}(E^k), W^{s_2+j,p}(E^k); \delta(\theta)] = W^{s_3+j,p}(E^k)$$

(Lemma 4.4). Hence, by Lemma 4.2, the identity mapping (which is $J^{-j}J^j$) is an isomorphism between X and $W^{s_3,p}(E^k)$ and the proof is complete.

LEMMA 4.6. *(4.4) holds for arbitrary real s.*

PROOF. We first note that it follows from Lemma 2.3 of Part I and Lemmas 4.3 and 4.4 that $W^{s,p}(E^k)$ is reflexive for $s>0$. One easily checks that $W^{-s,p'}(E^k)$ forms a complete closed set of bounded linear functionals on $W^{s,p}(E^k)$ and the assertion follows (cf. the proof of Theorem 3.4).

LEMMA 4.7. *(4.5) holds for s_i real, $i=1,2,3$.*

PROOF. All we need note is that the hypotheses of Theorem 3.3 are now known to be fulfilled. This follows from another application of Lemma 2.3 of Part I to Lemmas 4.4 and 4.6 together with Lemma 4.5.

THEOREM 4.1. *For each integer i and each real s, J^i is an isomorphism between $W^{s,p}(E^k)$ and $W^{s+i,p}(E^k)$.*

PROOF. Let j be the integer such that $j \leqq s+1/p < j+1$. Then by Lemma 4.2 J^i is an isomorphism between $W^{j-1/p,p}(E^k)$ and $W^{j+i-1/p,p}(E^k)$ and between $W^{j+1-1/p,p}(E^k)$ and $W^{j+i+1-1/p,p}(E^k)$. We now apply Lemma 4.7 to complete the proof.

The next important result was proved by Calderón [6].

LEMMA 4.8. *For $s \geqq 1-1/p$ and $1<p\leqq 2$*

$$W^{s,p}(E^k) \leqq H^{s,p}(E^k) \tag{4.6}$$

with continuous injection.

With the aid of Theorem 4.1 we extend this to

THEOREM 4.2. (4.6) *holds for* $1<p\leqq 2$ *and all real* s.

PROOF. Suppose $u \in W^{s,p}(E^k)$ for $s<1-1/p$. Let i be an integer such that $s+i \geqq 1-1/p$. By Theorem 4.1 J^i is an isomorphism between $W^{s,p}(E^k)$ and $W^{s+i,p}(E^k)$. Thus by Lemma 4.8 $J^i u \in H^{s+i,p}(E^k)$. Hence, by Lemma 4.1, $u=J^{-i}J^i u \in H^{s,p}(E^k)$.

THEOREM 4.3. *If* $2 \leqq p < \infty$, *then* $H^{s,p}(E^k) \subseteqq W^{s,p}(E^k)$ *for all real* s *with continuous injection.*

PROOF. By Theorem 4.2 we have $W^{-s,p'}(E^k) \subseteqq H^{-s,p'}(E^k)$. Since $[W^{s,p}(E^k)]' = W^{-s,p'}(E^k)$ (Lemma 4.6) and $[H^{s,p}(E^k)]' = H^{-s,p'}(E^k)$ (by (4.1)), the result follows by duality.

Subsequent to our investigations, we were informed that both Calderón and E. M. Stein had obtained our Theorems 4.2 and 4.3. (Cf. announcement by Stein [18].) Moreover, Stein characterized the spaces $W^{s,p}(E^k)$ analytically for $s>0$. For $0<s<2$ $W^{s,p}(E^k)$ consists of those functions f in $L^p(E^k)$ for which

$$\int_{E^k}\int_{E^k} \frac{|f(x-y)+f(x+y)-2f(x)|^p}{|y|^{k+sp}}\,dx\,dy$$

is finite. For other positive s, the function f is in $W^{s,p}(E^k)$ if f is in $L^p(E^k)$ and its first derivatives are in $W^{s-1,p}(E^k)$. By a result of Lions [7], $W^{s,p}(E^k)$ is equivalent to the "trace space" of Lions–Magenes [10, III] for all values of s except for s an integer and $p \neq 2$. (It had previously been known only that they are equivalent for $s-1/p$ an integer. This fact was exploited in the present paper (cf. Section 3).) As a result it follows that the norm of $W^{s,p}(E^k)$ satisfies (2.1) for all positive s.

5. The spaces $H^{s,p}(\partial G)$.

In this section we give the proof of Theorem 2.4. It depends upon some properties of the spaces $H^{s,p}(\partial G)$ which we shall develop here. The following lemma will be useful.

LEMMA 5.1. *For each real* $t>0$ *there is a linear mapping* c_t *of* $L^p(G)$ *into* $L^p(E^n)$ *such that* $\sigma_t u = u$ *in* G *and*

$$\|\sigma_t u\|_{s,p}^{E^n} \leqq \text{const.}\, \|u\|_{s,p}^{G} \leqq \text{const.}\, \|\sigma_t u\|_{s,p}^{E^n} \tag{5.1}$$

for each real s *in the interval* $0 \leqq s \leqq t$.

It clearly suffices to construct the mapping σ_t for t an integer. We employ the construction due to Lions (cf. [10, III, Lemma 2.1]). For s

an integer, the first inequality of (5.1) follows from Lemma 2.1 of [10, III]. The second is trivial. For s not an integer, both inequalities follow by interpolation.

We now consider a result similar to Lemma 14.2 of [3]. Let G be a domain in E^n with boundary ∂G of class C^∞. Let N be n dimensional neighborhood of a boundary point x_0. We know that there is a C^∞ homeomorphism T of $\bar{N}$ into $\bar{S}^n$ which maps $N \cap \partial G$ into E^{n-1} (i.e., into the hyperplane $x_n = 0$.) By modifying N slightly, we may assume that T maps $\bar{N}$ onto $\bar{S}^n$. Let N_0 be a neighborhood of x_0 such that $\bar{N}_0 \subset N$. For any function ϱ defined on $N \cap \bar{G}$ we set

$$\begin{aligned} \tau\varrho(x) &= \varrho(T^{-1}x), \qquad & x \in T(N \cap \bar{G}) \\ &= 0, & x \notin T(N \cap \bar{G}) \end{aligned}$$

For a function η defined on $T(N \cap \bar{G})$ we set

$$\begin{aligned} \tau^{-1}\eta(y) &= \eta(Ty), \qquad & y \in N \cap \bar{G} \\ &= 0, & y \notin N \cap \bar{G} \end{aligned}$$

The restriction of τ to functions defined on $N \cap \partial G$ is denoted by the same symbol.

LEMMA 5.2. *For each $s \notin I$ there is a constant C_s such that*

$$\langle\varphi\rangle_{s,p}^{\partial G} \leqq C_s \langle\sigma_s \tau\varphi\rangle_{s,p}^{E^{n-1}} \tag{5.2}$$

for all $\varphi \in C_0^\infty(N_0 \cap \partial G)$, where σ_s is the mapping for S^n of Lemma 5.1.

PROOF. Consider first the case $s \geqq 1 - 1/p$. Let v be a function in $H^{s+1/p,p}(E_+^n)$ which equals $\sigma_s\tau\varphi$ on E^{n-1} and such that

$$\|v\|_{s+1/p,p}^{E_+^n} \leqq 2\langle\sigma_s\tau\varphi\rangle_{s,p}^{E^{n-1}} .$$

Let $\zeta \in C_0^\infty(S^n)$ be such that $\zeta \equiv 1$ on $T(N_0)$ and set $w = \tau^{-1}\zeta v$. One easily checks that

$$\|w\|_{s+1/p,p}^{G} \leqq \text{const.}\, \|v\|_{s,p}^{E_+^n}$$

when $s + 1/p$ is an integer. When $s + 1/p$ is not an integer it follows by interpolation. Since $w = \varphi$ on ∂G we have

$$\langle\varphi\rangle_{s,p}^{\partial G} \leqq \text{const.} \langle\sigma_s\tau\varphi\rangle_{s,p}^{E^{n-1}}$$

and the lemma is proved for $s \geqq 1 - 1/p$. For $s \leqq -1/p$ we note that

$$\langle\varphi\rangle_{s,p}^{\partial G} = \underset{\psi \in C^\infty(\partial G)}{\text{l.u.b.}} \frac{\langle\varphi, \psi\rangle^{\partial G}}{\langle\psi\rangle_{-s,p'}^{\partial G}} .$$

But for $\varphi \in C_0^\infty(N_0 \cap \partial G)$

$$\langle \varphi, \psi \rangle^{\partial G} = \langle \varphi, \eta\psi \rangle^{\partial G} = \langle \tau\varphi, h\tau\eta\psi \rangle^{E^{n-1}} = \langle \sigma_s \tau\varphi, h\tau\eta\psi \rangle^{E^{n-1}},$$

where η is a function in $C_0^\infty(N \cap \partial G)$ which is identically one on $N_0 \cap \partial G$ and h is the Jacobian of the mapping of $N \cap \partial G$ onto S^{n-1}. Thus

$$|\langle \varphi, \psi \rangle^{\partial G}| \leqq \langle \sigma_s \tau\varphi \rangle_{s,p}^{E^{n-1}} \langle h\tau\eta\psi \rangle_{-s,p'}^{E^{n-1}} \leqq \text{const.} \langle \sigma_s t\varphi \rangle_{s,p}^{E^{n-1}} \langle \tau\eta\psi \rangle_{-s,p'}^{E^{n-1}} .$$

If we can prove that for each $t > 0$ and $p > 1$ and each neighborhood N' such that $\bar{N}' \subset N$

$$\langle \tau\varrho \rangle_{t,p}^{E^{n-1}} \leqq \text{const.} \langle \varrho \rangle_{t,p}^{\partial G} \tag{5.3}$$

for all $\varrho \in C_0^\infty(N' \cap \partial G)$, then it will follow that

$$\langle \tau\eta\psi \rangle_{-s,p'}^{E^{n-1}} \leqq \text{const.} \langle \psi \rangle_{-s,p'}^{\partial G}$$

and the lemma will be proved. Thus it remains to prove (5.3). The reasoning is very similar to the above. We let $w \in H^{t+1/p,p}(G)$ be such that $w = \varrho$ on ∂G and

$$\|w\|_{t+1/p,p}^{G} \leqq 2\langle \varrho \rangle_{t,p}^{\partial G} .$$

Let $\xi \in C_0^\infty(N)$ be such that $\xi \equiv 1$ in N' and set $u = \tau\xi w$. Then

$$\|u\|_{t+1/p,p}^{E_+^n} \leqq \text{const.} \|w\|_{t+1/p,p}^{G} .$$

We now note that $u = \tau\varrho$ on E^{n-1} and hence

$$\langle \tau\varrho \rangle_{t,p}^{E^{n-1}} \leqq \|u\|_{t+1/p,p}^{E_+^n} .$$

Combining the last three inequalities we obtain (5.3) and the proof is complete.

LEMMA 5.3. *The space $H^{s,p}(\partial G)$ consists of those distributions u on ∂G such that $\tau_\nu \zeta_\nu u \in H^{s,p}(S^{n-1})$ for each ν.*

PROOF. Clearly, if $u \in H^{s,p}(\partial G)$, then $\tau_\nu \zeta_\nu u \in H^{s,p}(S^{n-1})$ for each ν. Conversely, if this holds, then there is a sequence $w_\nu^{(i)}$ of functions in $C^\infty(S^{n-1})$ which converge to $\tau_\nu \zeta_\nu u$ in $H^{s,p}(S^{n-1})$. We set $w^{(i)} = \sum_k \zeta_k \tau_k^{-1} w_k^{(i)}$. Then $w^i \in C^\infty(\partial G)$ and we claim that it converges to u in $H^{s,p}(\partial G)$. In fact

$$\|\tau_\nu \zeta_\nu (\zeta_k \tau_k^{-1} w_k^{(i)} - \zeta_k^2 u)\|_{s,p}^{S^{n-1}} = \|\tau_\nu \zeta_\nu \zeta_k \tau_k^{-1} (w_k^{(i)} - \tau_k \zeta_k u)\|_{s,p}^{S^{n-1}}$$

which tends to zero as $i \to \infty$. Hence $\zeta_k \tau_k^{-1} w_k^{(i)}$ converges to $\zeta_k^2 u$ in $H^{s,p}(\partial G)$. Hence $w^{(i)}$ converges to u and the proof is complete.

THEOREM 5.1.

$$[H^{s,p}(\partial G)]' = H^{-s,p'}(\partial G) .$$

PROOF. If f and g are in $C^\infty(\partial G)$, then

$$\langle f,g\rangle^{\partial G} = \langle \sum \zeta_\nu^2 f,g\rangle^{\partial G} = \sum \langle \zeta_\nu f,\zeta_\nu g\rangle^{\partial G} = \sum \langle \tau_\nu\zeta_\nu f, h_\nu \tau_\nu \zeta_\nu g\rangle^{S^{n-1}} ,$$

where the ζ_ν and τ_ν are defined as in Section 2 and h_ν is the Jacobian of the transformation of $\bar{N}_\nu \cap \partial G$ onto $\bar{S}^{n-1}$. Thus

$$|\langle f,g\rangle^{\partial G}| \leqq \text{const.} \|f\|^{\partial G}_{s,p} \|g\|^{\partial G}_{-s,p'} .$$

This shows that $H^{-s,p}(\partial G) \subseteqq [H^{s,p}(\partial G)]'$. Conversely, let F be a bounded linear functional on $H^{s,p}(\partial G)$. Then by the Hahn–Banach theorem there are functions $f_\nu \in H^{-s,p'}(S^{n-1})$ such that

$$Fg = \sum \langle \tau_\nu \zeta_\nu g, f_\nu\rangle^{S^{n-1}} = \sum \langle \zeta_\nu g, \tau_\nu^{-1} h_\nu^{-1} f_\nu\rangle^{\partial G} = \langle g, \sum \zeta_\nu \tau_\nu^{-1} h_\nu^{-1} f_\nu\rangle^{\partial G} .$$

Since $f = \sum \zeta_\nu \tau_\nu^{-1} h_\nu^{-1} f_\nu$ is in $H^{-s,p'}(\partial G)$ by Lemma 5.3, we see that $[H^{s,p}(\partial G)]' \subseteqq H^{-s,p'}(\partial G)$ and the proof is complete.

THEOREM 5.2. *For any real* s_1, s_2

$$H^{s_3,p}(\partial G) = [H^{s_1,p}(\partial G), H^{s_2,p}(\partial G); \delta(\theta)] , \tag{5.4}$$

where $s_3 = s_1 + \theta(s_2 - s_1)$.

PROOF. Let X denote the space on the right hand side of (5.4). Define $\pi_\nu u$ to be $\tau_\nu \zeta_\nu u$ in S^{n-1} and zero in $E^{n-1} - S^{n-1}$. Then one easily checks that π_ν is a bounded linear mapping from $H^{s,p}(\partial G)$ to $H^{s,p}(E^{n-1})$ for each real s. Hence it is bounded from X to $H^{s_3,p}(E^{n-1})$. Thus if $u \in X$, then $\tau_\nu \zeta_\nu \in H^{s_3,p}(S^{n-1})$ for each ν and hence $u \in H^{s_3,p}(\partial G)$ by Lemma 5.3. Hence $X \subseteqq H^{s_3,p}(\partial G)$. By Lemma 2.3 of Part I, the same reasoning gives $X' \subseteqq H^{-s_3,p'}(\partial G)$. Thus $X \supseteqq H^{s_3,p}(\partial G)$ and the proof is complete.

We can now give the

PROOF OF THEOREM 2.4. We assume $p \geqq 2$ and prove the first relationship (2.12). By Lemmas 3.3 and 5.3, the relation (2.13) follows by duality. Let g be any function in $C^\infty(\partial G)$. As remarked previously, we assume that $T_\nu(N_\nu) = S^n$. Thus for $s \notin I$

$$\begin{aligned}\langle g\rangle^{\partial G}_{s,p} = \langle \sum \zeta_\nu^2 g\rangle^{\partial G}_{s,p} &\leqq \sum \langle \zeta_\nu^2 g\rangle^{\partial G}_{s,p} \\ &\leqq C_s \sum \langle \sigma_s \tau_\nu \zeta_\nu^2 g\rangle^{E^{n-1}}_{s,p} \\ &\leqq \text{const.} \sum \|\tau_\nu \zeta_\nu^2 g\|^{S^{n-1}}_{s,p} \\ &= \text{const.} \|(\tau_\nu \zeta_\nu)(\tau_\nu \zeta_\nu g)\|^{S^{n-1}}_{s,p} \leqq \text{const.} \|g\|^{\partial G}_{s,p} ,\end{aligned}$$

where we have employed Lemma 5.1 and Theorem 4.3. Thus for $s \notin I$ the identity mapping is continuous from $H^{s,p}(\partial G)$ into $W^{s,p}(\partial G)$. By interpolation it is continuous as well from $H^{s,p}(\partial G)$ to $W^{s,p}(\partial G)$, $s \in I$

(Theorem 3.1 and 5.2). Hence $H^{s,p}(\partial G) \subseteqq W^{s,p}(\partial G)$ for $p \geqq 2$ and the theorem is proved.

PROOF OF LEMMA 2.1. We first note that $\varphi \in C_0^\infty(E^{n-1})$ and $\langle\varphi\rangle_{s,p}^{E^{n-1}} = 0$, $s \geqq 1 - 1/p$ implies $\varphi = 0$. For then the $L^p(E^k)$ norm of φ is 0 (cf. Section 4). If $s \leqq -1/p$, then $\langle\varphi\rangle_{s,p}^{E^{n-1}} = 0$ implies $\langle\varphi,\psi\rangle^{E^{n-1}} = 0$ for all $\psi \in C_0^\infty(E^{n-1})^{s,p}$. This again gives $\varphi = 0$. Next, if $\varphi \in C^\infty(\partial G)$ and $\langle\varphi\rangle_{s,p}^{\partial G} = 0$, $s \notin I$, then by (5.3) we have $\tau_\nu\varphi = 0$ for each ν. Hence $\varphi = 0$ and the lemma is proved.

6. The use of regularity.

In proving the theorems of Section 2 we shall make use of the following regularity theorem.

LEMMA 6.1. *Under the hypotheses of Section 2, if f is a distribution and $(f, Au) = 0$ for all $u \in V$, then $f \in C^\infty(\bar{G})$.*

The proof of Lemma 6.1 may be found in [1,13]).

We now choose normal operators $B_{r+1}, \ldots, B_m$ so that the orders m_j of the operators $B_1, \ldots, B_m$ are distinct and $< m$.

LEMMA 6.2. *Under the same hypotheses there are normal operators $B_1', \ldots, B_m'$ such that*

$$(6.1) \qquad (Au, v) - (u, A'v) = \sum_{j=1}^{m} \langle B_j u, B'_{m-j+1} v\rangle$$

for all $u, v \in C^\infty(\bar{G})$, where A' is the formal adjoint of A. The order of B'_{m-j+1} is $m - m_j - 1$.

For a proof of Lemma 6.2 see, e.g., [4]. From it we see easily that V' consists of those $v \in C^\infty(\bar{G})$ which satisfy $B_j'v = 0$ on ∂G for $1 \leqq j \leqq r$. Hence the function f of Lemma 6.1 is in V'.

COROLLARY 6.1. *If F, G are distributions such that for some $j \leqq r$*

$$(6.2) \qquad (F, Au) + \langle G, B_j u\rangle = 0$$

for all $u \in C^\infty(\bar{G})$ which satisfy

$$(6.3) \qquad B_i u = 0 \quad \text{on} \quad \partial G, \qquad 1 \leqq i \leqq r,\ i \neq j,$$

then $F \in V'$, $A'F = 0$ and

$$B'_{m-j+1}F = -G \quad on \quad \partial G.$$

PROOF. By (6.2) and (6.3), $(F, Au) = 0$ for all $u \in V$. Hence by Lemma 6.1, $F \in C^\infty(\bar{G})$. Applying (6.1) we have

$$-\langle B_j u, G\rangle - (u, A'F) = \langle B_j u, B_{m-j+1}F\rangle + \sum_{i=r+1}^{m} \langle B_i u, B'_{m-i+1}F\rangle$$

from which the conclusions immediately follow.

Proof of Theorem 2.1. For s an integer, (2.6) is Theorem 6.1 of [16]. We prove it for other s by interpolation. Let N denote the set of those $u \in V$ such that $Au=0$. We know that N is finite dimensional (cf. e.g., [3]). The same is true of N', the set of $v \in V'$ which satisfy $A'v=0$. Thus by Rellich's lemma, for s an integer,

$$\|u\|_{s,p} \leqq \text{const.} \left(|Au|'_{s-m,p} + \sum_{j=1}^{r} \langle B_j u\rangle_{s-m_j-1/p,p}\right) \tag{6.4}$$

for all $u \in C^\infty(\bar{G})/N$. Consider the set of functions $\Phi=\{f,g_1,\ldots,g_r\}$, $f \in C^\infty(\bar{G})$, $g_j \in C^\infty(\partial G)$ for which there is a solution of

$$Au = f \quad \text{in} \quad G\,,$$
$$B_j u = g_j \quad \text{on} \quad \partial G, \qquad 1 \leqq j \leqq r\,.$$

Set $u=T\Phi$, where u is the unique solution in $C^\infty(\bar{G})/N$. Then by (6.4) T can be extended to be a bounded linear mapping of a subset of

$$M_s = V'^{s-m,p}(G) \times \prod_{j=1}^{r} W^{s-m_j-1/p,p}(\partial G)$$

into $H^{s,p}(G)$. (The definition of $V'^{s,p}(G)$ is given in Section 2. If $s \geqq m$, $V'^{s,p}(G)$ should be replaced by $H^{s-m,p}(G)$ in M_s). Now the domain of T is closed by (6.4). Moreover, it has finite codimension. For if $F,G_1,\ldots,G_r$ are distributions such that

$$(F,Au) + \sum_{j=1}^{r} \langle G_j, B_j u\rangle = 0$$

for all $u \in C^\infty(\bar{G})$, then by Corollary 6.1 we have $F \in N'$ and

$$B_j'F = -G_{m-j+1}, \qquad r<j \leqq m\,.$$

Thus $\{F,B_1,\ldots,B_r\}$ belongs to a finite dimensional set. Hence T may easily be extended to be bounded in the whole of M_s

$$\|T\Phi\|_{s,p} \leqq \text{const.}\,\|\Phi\|_{M_s}, \qquad s \text{ an integer}\,.$$

We now interpolate between consecutive integers. For the spaces $H^{s,p}(G)$ we apply Lemma 2.4 of Part I. For the spaces $V'^{s-m,p}(G)$ we apply Theorem 4.1 of Part I when $s<m$. For $s \geqq m, V'^{s-m,p}(G)$ is replaced by $H^{s-m,p}(G)$. Finally for the spaces $W^{s-m_i-1/p,p}(\partial G)$ we apply Theorem 3.1. Thus (6.4) holds for all real s. The extension of (6.4) to (2.6) is

elementary and is left to the reader (cf. the proof of Theorem 6.2 of [16]). This completes the proof.

PROOF OF THEOREM 2.2. By (6.4)

$$\|u\|_{m-s,p'} \leqq \text{const.}\,(|Au|'_{-s,p'} + \langle B_1 u\rangle_{m-s-m_1-1/p',p'}) \tag{6.5}$$

for all $u \in C^\infty(\bar{G})/N$ which satisfy (2.8). Set

$$F\{Au, B_1 u\} = (f, Au) + \langle g, B_1 u\rangle$$

for such u. By (2.7) and (6.5) F is a bounded linear functional on a subspace of

$$V^{-s,p'}(G) \times W^{m-s-m_1-1/p',p'}(\partial G)\ .$$

Extending to the whole space we see that there is an $f_0 \in V'^{s,p}(G)$ and a $g_0 \in W^{s+m_1-m+1/p',p}(\partial G)$ satisfying

$$|f_0|'_{s,p} + \langle g_0\rangle_{s+m_1-m+1/p',p} \leqq \text{const.}\, c_0 \tag{6.6}$$

and

$$F\{Au, B_1 u\} = (f_0, Au) + \langle g_0, B_1 u\rangle$$

for such u (cf. Lemma 3.3). Thus

$$(f - f_0, Au) + \langle g - g_0, B_1 u\rangle = 0$$

for such u. Applying Corollary 6.1, we have

$$f - f_0 \in N' \quad \text{and} \quad g_0 - g = B_m'(f - f_0) \in C^\infty(\partial G)\ .$$

Hence f and g belong to the same spaces as f_0 and g_0, respectively. The estimate (2.9) comes from the estimate (6.6) coupled with the fact that $f - f_0$ and $g - g_0$ belong to finite dimensional spaces (cf. above).

PROOF OF THEOREM 2.3. We first note that the result is clearly seen to be true for $s \geqq m$. In fact, the right hand side of (2.10) is greater than const. $\|u\|_{s,p}$ (Theorem 2.1) which in turn is greater than the left hand side (by (2.1)). Next we show that it is also true for $s \leqq 0$. For by Lemma 6.2

$$\begin{aligned} &|(u, A'v) + \sum_{j=1}^{r} \langle B_{m-j+1}u, B_j'v\rangle| \\ &\quad\leqq |(Au, v)| + \sum_{j=1}^{r} |\langle B_j u, B'_{m-j+1}v\rangle| \\ &\quad\leqq \|Au\|_{s-m,p}\|v\|_{m-s,p'} + \sum_{j=1}^{r} \langle B_j u\rangle_{s-m_j-1/p,p}\langle B'_{m-j+1}v\rangle_{m_j-s+1/p,p'} \\ &\quad\leqq \text{const.}\,\|v\|_{m-s,p'}(\|Au\|_{s-m,p} + \sum_{j=1}^{r} \langle B_j u\rangle_{s-m_j-1/p,p}) \end{aligned}$$

where we have employed (2.1) and made use of the fact that the order of B'_{m-j+1} is $m-m_j-1$. We now apply Theorem 2.2 to the operators $A', B_1', \ldots, B_r'$ (which satisfy the same hypotheses as $A, B_1, \ldots, B_r$, cf. [13]) to conclude that

$$|u|'_{s,p} + \sum_{j=r+1}^{m} \langle B_j u\rangle_{s-m_j-1/p,p} \leqq \text{const.}\,(\|Au\|_{s-m,p} + \sum_{j=1}^{r} \langle B_j u\rangle_{s-m_j-1/p,p} + \|u\|_{s-m,p})\ ,$$

since $s-m+(m-m_j-1)+1 = s-m_j$. Next we observe that we can add

$$\sum_{j=1}^{r} \langle B_j u\rangle_{s-m_j-1/p,p}$$

to the left hand side by adjusting the constant on the right hand side. In addition, for each l, $0 \leqq l < m$,

$$\partial^l u/\partial n^l = \sum \Lambda_{lj} B_j\ ,$$

where Λ_{lj} is an operator involving only tangential derivatives and is of order $\leqq l-m_j$, and summation is taken only over those j for which $m_j \leqq l$ (cf. [4]). Thus

$$\langle \partial^l u/\partial n^l\rangle_{s-l-1/p,p} \leqq \sum \langle \Lambda_{lj} B_j u\rangle_{s-l-1/p,p} \leqq \text{const.} \sum \langle B_j u\rangle_{s-m_j-1/p,p}\ .$$

Finally, to prove (2.10) for $0 \leqq s \leqq m$ we eliminate the term $\|u\|_{s-m,p}$ by means of Rellich's lemma with the resulting inequality holding for $u \in C^\infty(\bar{G})/N$. Then we apply interpolation as before to obtain the complete result.

In proving Theorem 2.6 and 2.7 we make use of the following regularity results similar to Lemma 6.1. For a proof cf., e.g., [14].

LEMMA 6.3. *If f is a distribution and*

$$(f, Au) = 0$$

for all $u \in V \cap C_0^\infty(G \cup \Gamma)$, then $\zeta f \in C^\infty(\bar{G})$ for each $\zeta \in C_0^\infty(G \cup \Gamma)$.

PROOF OF THEOREM 2.6. By Theorem 2.1 and Rellich's lemma

$$\|u\|_{m-s,p'} \leqq \text{const.}\,|Au|'_{-s,p'}$$

for $u \in V \cap C_0^\infty(G \cup \Gamma)/N$. Hence

$$|(f, Au)| \leqq \text{const.}\,|Au|'_{-s,p'} \leqq \text{const.}\,\|Au\|_{-s,p}$$

for such u. Setting $F(Au) = (Au, f)$, we see that F is a bounded linear functional on a subspace of $H^{-s,p'}(G)$. Thus there is an $f_0 \in H^{s,p}(G)$ such that

$$F(Au) = (Au, f_0)$$

for such u. Substracting, we see that

$$(f-f_0, Au) = 0$$

for all $u \in V \cap C_0^\infty(G \cup \Gamma)$ from which we conclude via Lemma 6.3 that $\zeta(f-f_0) \in C^\infty(\bar{G})$ for each $\zeta \in C^\infty(G \cup \Gamma)$. Hence $\zeta f = \zeta f_0 + \zeta(f-f_0) \in H^{s,p}(G)$ and the proof is complete.

PROOF OF THEOREM 2.7. By Theorem 2.1 and Rellich's lemma we have

$$|(f, Au) + \langle g, B_1 u\rangle| \leqq \text{const.}\,(\|Au\|_{-s\ p'} + \langle B_1 u\rangle_{m-s-m_1-1/p', p'})$$

for all $u \in C_0^\infty(G \cup \Gamma)/N$ satisfying (2.16). By the same reasoning as above there is an $f_0 \in H^{s,p}(G)$ and a $g_0 \in W^{s-m+m_1+1-1/p,\, p}(\partial G)$ such that

$$(f-f_0, Au) + \langle g-g_0, B_1 u\rangle = 0$$

for all $u \in C_0^\infty(G \cup \Gamma)$ which satisfy (2.16). Let G_0 be any subdomain of G such that $\bar{G}_0 \subset G \cup \Gamma$. By Lemma 6.3. $\zeta(f-f_0) \in C^\infty(\bar{G})$ for each $\zeta \in C_0^\infty(G \cup \Gamma)$. Let ζ be such that it equals one on $\bar{G}_0$. Then

$$\big(\zeta(f-f_0), Au\big) + \langle \zeta(g-g_0), B_1 u\rangle = 0$$

for all $u \in C_0^\infty(G_0 \cup \Gamma)$ satisfying (2.16). By Lemma 6.2

$$\big(A'\zeta(f-f_0), u\big) + \langle B_m'\zeta(f-f_0), B_1 u\rangle + \sum_{j=r+1}^{m} \langle B'_{m-j+1}\zeta(f-f_0), B_j u\rangle + \\ + \langle \zeta(g-g_0), B_1 u\rangle = 0$$

from which one concludes, among other things, that

$$\zeta(g_0 - g) = B_m'\zeta(f-f_0)\,.$$

Since $\zeta \equiv 1$ in $\bar{G}_0$, we have $g - g_0 \in C^\infty(\bar{G}_0 \cap \Gamma)$. Since G_0 was arbitrary, the theorem follows.

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INSTITUTE OF MATHEMATICAL SCIENCES,
NEW YORK UNIVERSITY