MATH. SCAND. 13 (1963), 75–89

UNBOUNDED SOBOLEFF REGIONS

ROLF ANDERSSON

1. Introduction.

Let A be an open set in R^n, $n>1$, and let

$$|u|_q = \left(\int_A |u(x)|^q dx\right)^{1/q}, \qquad 1 \leqq q \leqq +\infty ,$$

be the norm in $L^q(A)$ with respect to Lebesgue measure dx. A distribution u in A, such that its gradient ∇u belongs to $L^p(A)$, is called a Beppo Levi function of type p. The space of all such functions equipped with the seminorm

$$|\nabla u|_p = \sum_{i=1}^{n} |\partial u/\partial x_i|_p$$

will be denoted by $BL^p(A)$. If A is connected and Λ is the space of constant functions in A, the quotient $\dot{B}L^p(A) = BL^p(A)/\Lambda$ is a Banach space (Deny et Lions [4]).

We say that an open connected region A is a Soboleff region of type (p,q) if

$$(1) \qquad u \in BL^p(A) \ \Rightarrow \ (u+c) \in L^q(A)$$

for some constant c. The class of such regions will be denoted by S^{pq}. If A has finite measure, (1) is equivalent to

$$u \in BL^p(A) \ \Rightarrow \ u \in L^q(A) .$$

Hence, if one of the regions A and B has finite measure and if both A and B belong to S^{pq} and the intersection $A \cap B$ is not empty then $A \cup B \in S^{pq}$. According to a classical result of Soboleff, S^{pq} contains all bounded regions with a sufficiently regular boundary, provided

$$1/p \leqq 1/q + 1/n, \qquad (1/p, 1/q) \neq (1/n, 0) .$$

If this condition does not hold, S^{pq} is empty.

We shall prove that S^{pq} contains unbounded regions with finite measure only if

Received November 6, 1962.

$$(2) \qquad 1/p \leqq 1/q, \qquad (1/p, 1/q) \neq (0,0)\,.$$

(Theorem 4). When it exists an unbounded region in S^{pq} this region has to be rather small at infinity. (Theorem 5).

In section 5 we construct unbounded regions with finite measure contained in S^{pq} for any (p,q) satisfying (2).

We show in section 6 that S^{pq} contains regions with infinite measure if and only if

$$1/p = 1/q + 1/n, \qquad (1/p, 1/q) \neq (1/n, 0)\,.$$

The entire space is an example of a region in this class.

Finally we give a inclusion property for the Soboleff spaces S^{pq}.

The subject of this paper was suggested to me by professor Lars Gårding. I wish to thank him for his continuing interest and valuable advice.

2. General conditions.

Our first theorem gives a necessary and sufficient condition for $A \in S^{pq}$, if A is an open connected region of finite measure. This theorem has been established and proved by Deny and Lions [4], but we state it again here for completeness and reference. To formulate the theorem we introduce a new class of functions,

$$T^{pq}(A) = BL^p(A) \cap L^q(A)\,.$$

This is a Banach space in the norm

$$|u|_{pq} = |\nabla u|_p + |u|_q, \qquad u \in T^{pq}(A)\,.$$

Theorem 1. *If A is an open connected region of finite measure, then $A \in S^{pq}$ if and only if there exists a constant K such that*

$$(3) \qquad \inf_{c=\text{const.}} |u+c|_q \leqq K|\nabla u|_p \quad \textit{for all} \quad u \in T^{pq}(A)\,.$$

Proof. (i) The condition is necessary. Denote by Λ the constant functions on A. Consider the quotient space $\dot{T}^{pq}(A) = T^{pq}(A)/\Lambda$. This is a Banach space in the quotient norm

$$|\dot{u}|_{pq} = \inf_{c=\text{const.}} |u+c|_{pq} = \inf_c |u+c|_q + |\nabla u|_p\,.$$

Let Γ be the identical mapping

$$\dot{u} \to \dot{u} \quad \text{from } \dot{T}^{pq}(A) \quad \text{into } \dot{B}L^p(A)\,.$$

This mapping is linear, continuous, one–one and, if $A \in S^{pq}$, it is also onto. From a theorem by Banach (see Bourbaki [2, p. 34]) it then follows that Γ is an isomorphism and hence we have the desired inequality.

(ii) The condition is sufficient. Suppose that the inequality (3) is valid and consider the identical mapping Γ from $\dot{T}^{pq}(A)$ into $\dot{B}L^p(A)$. We have to prove that Γ is onto. As the inequality (3) is valid, the image of $\dot{T}^{pq}(A)$ is closed in $\dot{B}L^p(A)$. Thus it is sufficient to show that $\dot{T}^{pq}(A)$, considered as a subspace of $\dot{B}L^p(A)$, is dense in $BL^p(A)$. Let $u \in \dot{B}L^p(A)$. It is then sufficient to prove there exist $u_k \in T^{pq}(A)$ such that

$$|\nabla(u-u_k)|_p \to 0 \quad \text{when } k \to +\infty \, .$$

It is no restriction to assume that u is real. Set

$$u_k(x) = \begin{cases} k & \text{when } u(x) > k \, , \\ u(x) & \text{-} \quad |u(x)| < k \, , \\ -k & \text{-} \quad u(x) < -k \, . \end{cases}$$

Since $m(A) < +\infty$, $u_k \in T^{pq}(A)$. Further by construction $u_k \to u$ in $BL^p(A)$ and this completes the proof.

REMARK 1. We can replace the inequality (3) by

$$|u|_q \leqq K(|\nabla u|_p + |L(u)|) \, , \tag{4}$$

where L is a continuous, linear functional on the space $T^{pq}(A)$ with $L(1) \neq 0$. That (4) implies (3) is obvious. For the proof that (4) is necessary for $A \in S^{pq}$ we refer to Björup [1]. His proof is for the case $(p,q) = (2,2)$, but the general case can be proved using the same reasoning.

REMARK 2. Let A' be an open subset of A with the following property: there exists a function f with bounded gradient which is equal 1 on A' and vanishes outside some open set A'' belonging both to S^{pq} and to S^{pp} such that $A' \subset A'' \subset A$. Then $A \in S^{pq}$, if and only if

$$|u|_q \leqq K|\nabla u|_p \tag{4'}$$

for all $u \in T^{pq}(A)$ which vanish in A'. In fact, if $u \in BL^p(A)$ then

$$\nabla(fu) = (\nabla f u + f \nabla u) \in L^p(A'')$$

$\big(u \in L^p(A'')$ as $\nabla u \in L^p(A'')\big)$, and hence $fu \in L^q(A'')$. But then we also have $fu \in L^q(A)$. Now $(u - fu)$ vanishes in A' and it follows from (4′), as in the proof of the theorem, that $(u - fu) \in L^q(A)$. Thus

$$u = \big(fu + (u - fu)\big) \in L^q(A)$$

and we have shown the sufficiency of the condition. The proof of necessity is trivial.

REMARK 3. The property $A \in S^{pq}$ is invariant under Lipschitz transformations, i.e. one–one mappings F such that F and F^{-1} have bounded gradients.

Some of the classes S^{pq} are empty. We have the following theorem due to Soboleff [7].

THEOREM 2. *If S^{pq} is not empty, then*

$$1/p \leqq 1/q+1/n, \qquad (1/p,1/q) \neq (1/n,0)\,. \tag{5}$$

PROOF. Let $A \in S^{pq}$ and assume that A contains the origin. Put

$$u(x) = |x|^{-\lambda/p+1}h(x)\,,$$

where h is infinitely differentiable with compact support and equal to 1 in a neighbourhood of the origin. If $0<\lambda<n$, then $\nabla u \in L^p(A)$. Since $A \in S^{pq}$, $u \in L^q(A)$ by definition. But then it is necessary that

$$(\lambda/p-1)q < n \quad \text{for} \quad 0<\lambda<n\,.$$

Thus

$$(n/p-1)q \leqq n\,,$$

and the first part of the theorem is established.

To prove the second part we observe that $\nabla v \in L^n(A)$ where (with h as above)

$$v(x) = |\log|x||^{\lambda}\, h(x), \qquad 0<\lambda<1-1/n\,.$$

But $v \notin L^\infty(A)$ and hence $S^{n\infty}$ is empty. We shall see in next section that for bounded regions A with sufficiently regular boundary the conditions (5) are also sufficient for $A \in S^{pq}$.

3. Bounded regions.

We say that a region A has a regular boundary (in the sense of Soboleff) if every boundary point of A is the vertex of a cone C contained in A which is the image of some circular cone

$$C_0\colon\ x_2^2+x_3^2+\ldots+x_n^2 < bx_1^2, \qquad 0<x_1<a$$

under an orthogonal transformation.

The following theorem is due to Soboleff; for the proof we refer to Deny et Lions [4] and Soboleff [7].

THEOREM 3. *Any bounded open connected region with a regular boundary belongs to S^{pq} provided*

$$1/p \leqq 1/q+1/n \quad \textit{and} \quad (1/p,1/q) \neq (1/n,0)\,.$$

REMARK. The study of bounded regions in S^{pq} with a non-regular boundary seems to be an open field. It is a well-known fact that not all bounded regions have the property stated in theorem 3. An example of a bounded region which does not belong to S^{22} is given in Courant–Hilbert [3, p. 521].

4. Unbounded regions of finite measure. Necessary conditions.

If we require S^{pq} to contain unbounded regions theorem 3 has to be sharpened.

THEOREM 4. *If S^{pq} contains an unbounded region A of finite measure, then*

$$1/p \leqq 1/q \quad \textit{and} \quad (1/p, 1/q) \neq (0,0)\,.$$

PROOF. The last statement is trivial. In fact, $|x|$ has a bounded gradient in R^n but tends to $+\infty$ at infinity. Let $\varphi(x) > 0$, $x \neq 0$, be a continuously differentiable function, homogeneous of degree one. Further, let

$$u(x) = \psi(\varphi(x))$$

where $\psi(s) = \psi_{t,\delta}(s)$ vanishes for $s < t-\delta$, $\delta > 0$, is equal one for $s > t$ and increases linearly from $t-\delta$ to t. Since u is bounded, $u \in L^q(A)$, and since ∇u has compact support, $\nabla u \in L^p(A)$. Hence $u \in T^{pq}(A)$. Since $A \in S^{pq}$, remark 1 of theorem 1 now gives

$$|u|_q \leqq K\big(|\nabla u|_p + |L(u)|\big)\,.$$

Choose an L with compact support. Then $L(u) = 0$, if $t-\delta$ is sufficiently large. Thus the above inequality gives

$$\Big(\int_A dx\Big)^{1/q} \leqq K\delta^{-1}\Bigg(\int_{\substack{A\\ t-\delta<\varphi(x)<t}} dx\Bigg)^{1/p}.$$

Put (for the notation see e.g. [5, p. 35])

$$\varphi_A(t) = \int_A \delta\big(t-\varphi(x)\big)\,dx\,, \tag{6}$$

where δ is the Dirac function. In this notation we can write the above inequality as

$$\Big(\int_t^\infty \varphi_A(s)\,ds\Big)^{1/q} \leqq K\cdot\delta^{-1}\Big(\int_{t-\delta}^t \varphi_A(s)\,ds\Big)^{1/p}. \tag{7}$$

Choose s_i such that

$$\int_{s_i}^{\infty} \varphi_A(s)\,ds = 2^{-i}$$

and set $\delta_i = s_i - s_{i-1}$. If we choose $t = s_i$ and $\delta = \delta_i$ in inequality (7) we get

$$2^{-i/q} \leqq K \cdot \delta_i^{-1} \cdot 2^{-1/p} .$$

Thus

$$\delta_i \leqq K \cdot (2^{1/p-1/q})^{-i} .$$

If $p < q$ the series $\sum^{\infty} \delta_i$ is convergent, which contradicts the assumption that A is unbounded. Hence $q \leqq p$ and the theorem is proved.

We shall need the following lemma.

Lemma 1. *If* $g \in L^r(1, +\infty)$, $1 < r < +\infty$, *then*

$$f(t) = \int_t^{\infty} \big(g(s)/s\big)\,ds \in L^r(1, +\infty) .$$

Proof. It is no restriction to suppose that $g(s) \geqq 0$. It is then sufficient to show that the function

$$F(a) = \int_1^a \big(f(t)\big)^r dt, \qquad 1 < a < +\infty ,$$

is bounded. Hölder's inequality gives

$$f(t) = \int_t^{\infty} \big(g(s)/s\big)ds \leqq \left(\int_t^{\infty} \big(g(s)\big)^r ds\right)^{1/r} \left(\int_t^{\infty} s^{-r/(r-1)} ds\right)^{1-1/r}$$

$$\leqq K t^{-1/r} \left(\int_1^{\infty} g(s)^r ds\right)^{1/r} .$$

Hence $f(t)^r t$ is bounded. Applying Hölder's inequality once again, we obtain by partial integration

$$\int_1^a f(t)^r dt = [t f(t)^r]_1^a + r \cdot \int_1^a f(t)^{r-1} g(t)\,dt$$

$$\leqq K \int_1^{\infty} g(s)^r ds + r \cdot \left(\int_1^a f(s)^r ds\right)^{1-1/r} \left(\int_1^{\infty} g(s)^r ds\right)^{1/r} .$$

It follows from this inequality that $F(a)$ is bounded which completes the proof.

Let φ and φ_A be as in the proof of theorem 4. If we assume that φ_A is decreasing, we get the following necessary condition for A to be of type (p,q).

THEOREM 5. *If A is an unbounded region of finite measure, if $\varphi_A(t)$ is decreasing (for large t), and if $A \in S^{pq}$, $(p,q) \neq (+\infty,1)$, then*

$$\varphi_A(t)^{-1/r'} \int_t^\infty \varphi_A(s)\,ds \in L^r(\ ,+\infty)\ , \tag{8}$$

where

$$1/r = 1/q - 1/p; \quad 1/r + 1/r' = 1\ .$$

PROOF. Assume $q<p$ and $(p,q) \neq (+\infty,1)$. We first prove

$$\int_1^\infty t^r \varphi_A(t)\,dt < +\infty\ .$$

Let $l_0=0$ and $l_i=\sum_{k=0}^{i-1}(q/p)^k$ for $i=1,2,\ldots$. Then $ql_i \to r$, when $i \to +\infty$. Put

$$u_i(x) = |x|^{l_i}, \qquad i = 0,1,2,\ldots\ .$$

Then

$$|\nabla u_{i+1}|_p \leqq n l_{i+1} \| |x|^{l_{i+1}-1}|_p \leqq n r q^{-1} (|u_i|_q)^{q/p}\ . \tag{9}$$

In fact,

$$l_{i+1}-1 = (q \cdot l_i)/p, \qquad q \cdot l_{i+1} \leqq r\ .$$

Now by hypothesis, $m(A) < +\infty$ and $A \in S^{pq}$. Hence by (9)

$$u_i \in L^q(A) \Rightarrow \nabla u_{i+1} \in L^p(A) \Rightarrow u_{i+1} \in L^q(A)\ .$$

Since $u_0 \in L^q(A)$, this shows that

$$u_i \in L^q(A), \qquad \nabla u_i \in L^p(A)\ ,$$

that is

$$u_i \in T^{pq}(A) \qquad \text{for all } i\ .$$

Remark 1 of theorem 1 now gives

$$|u_i|_q \leqq K \cdot (|\nabla u_i|_p + |L(u_i)|)\ .$$

Since $|\nabla u_i|_p \geqq K_1 > 0$ for some constant K_1 and all $i>0$, and as we can choose an L such that the sequence $\{L(u_i)\}_{i=1}^\infty$ is bounded, we have for some K

$$|u_i|_q \leqq K \cdot |\nabla u_i|_p \qquad \text{for all } i > 0\ .$$

It now follows from inequality (9) that

$$|u_i|_q \leqq K(|u_{i-1}|_q)^{q/p}\ .$$

We get by induction

$$|u_i|_q \leqq K^{l_i}|u_0|^{(q/p)^i} .$$

Hence the sequence $\{|u_i|_q\}_{i=1}^{\infty}$ is bounded and as $u_i(x)^q \to |x|^r$ at every point it follows that $|x|^r \in L(A)$. As $\varphi(x)$ is continuous and homogeneous of degree one, there exists a constant c such that $\varphi(x) \leqq c|x|$. From this we conclude that

$$\int_0^\infty t^r \varphi_A(t)\,dt < +\infty . \tag{10}$$

Since φ_A is decreasing we have

$$\int_t^\infty \varphi_A(s)\,ds \leqq \varphi_A(t)^{1/r'} \int_t^\infty \varphi_A(s)^{1/r}\,ds .$$

From (10) we obtain that

$$\int_t^\infty \varphi_A(s)^{1/r}\,ds = \int_t \big(g(s)/s\big)\,ds ,$$

where $g \in L^r(\ ,+\infty)$. The theorem now follows from lemma 1 when $1 < r < +\infty$.

It remains to prove the theorem when $p = q = +\infty$. If we choose $\delta = 1$ in the inequality (7) we get

$$\int_{t+1}^\infty \varphi_A(s)\,ds \leqq K \int_t^{t+1} \varphi_A(s)\,ds .$$

Thus, if

$$\int_t^{t+1} \varphi_A(s)\,ds \leqq K\varphi_A(t) \quad \text{for some } K ,$$

which of course is the case if $\varphi_A(t)$ is decreasing, we obtain

$$\int_t^\infty \varphi_A(s)\,ds \leqq K\varphi(t) \qquad \text{for sufficiently large } t .$$

Hence the theorem is also proved for the case $p = q \neq +\infty$, that is, when $r = +\infty$.

REMARK 1. In the case $p = q$ we need not assume that φ_A is decreasing. It follows from the proof that it is sufficient to suppose that there exists a constant K such that

$$\int_t^{t+1} \varphi_A(s)\,ds \leqq K\cdot\varphi(t) \qquad \text{for sufficiently large } t\,.$$

REMARK 2. It is obvious that if $A \in S^{\infty 1}$ we have

$$\int_A |x|\,dx < +\infty\,.$$

5. Unbounded regions of finite measure. A sufficient condition.

In this section we prove that the necessary condition of theorem 5 for $A \in S^{pq}$ is also sufficient if we restrict ourselves to a special type of unbounded regions of finite measure. For the proof we need three lemmas.

We assume in this section that $q \leqq p$ and $q < +\infty$. If $p = q = +\infty$ there exists of course no unbounded region in S^{pq}. We define r and r' by

$$1/r = 1/q - 1/p; \qquad 1/r + 1/r' = 1\,.$$

Let g be a positive, summable function. Further, let $L_g{}^q(A)$ be all functions u in A such that $ug^{1/q} \in L^q(A)$. We denote by $T_g{}^{pq}(A)$ all functions u such that $u \in L_g{}^q(A)$ and $\nabla u \in L_g{}^p(A)$. We now have

LEMMA 2. *Let g be a positive function such that*

$$f(t) = g(t)^{-1/r'} \int_t^\infty g(s)\,ds \in L^r(0, +\infty)\,.$$

Then

$$\left(\int_0^\infty |u|^q g(t)\,dt\right)^{1/q} \leqq K\left(\int_0^\infty (du/dt)^p g(t)\,dt\right)^{1/p}$$

for some constant K and all functions $u \in T_g{}^{pq}(0, +\infty)$ which vanish in a neighbourhood of the origin.

PROOF. It is sufficient to prove the lemma for functions u with compact supports in $(0, +\infty)$. In fact, these functions constitute a dense subspace of the functions in $T_g{}^{pq}(0, +\infty)$ which vanish in a neighbourhood of the origin.

Further, it is no restriction to suppose that u is real. Since u has compact support we obtain by partial integration

$$\int_0^\infty |u|^q g(s)\,ds \leqq q\int_0^\infty \left(|u|^{q-1}|du/dt| \int_t^\infty g(s)\,ds\right)dt$$

$$\leqq q\int_0^\infty |u|^{q-1} g(t)^{(q-1)/q} |du/dt|\, g(t)^{1/p} f(t)\,dt\,.$$

The last inequality follows from the hypothesis on g. If we apply Hölder's inequality we get

$$\int_0^\infty |u|^q g(t)\,dt \leqq q\left(\int_0^\infty |u|^q g(t)\,dt\right)^{1-1/q}\left(\int_0^\infty |du/dt|^p g(t)\,dt\right)^{1/p}\left(\int_0^\infty f(t)^r\,dt\right)^{1/r}.$$

Hence,

$$\left(\int_0^\infty |u|^q g(t)\,dt\right)^{1/q} \leqq K\left(\int_0^\infty |du/dt|^p g(t)\,dt\right)^{1/p}$$

and the lemma is proved.

We say that $A \in S_g^{pq}$ if $\nabla u \in L_g^p(A)$ implies that $(u+c) \in L_g^q(A)$ for some constant c. In this notation we can write lemma 3 as follows.

Lemma 3. *Let* $g(x_1)$ *be a positive, summable function and*

$$B = \{x \mid x_1 > 0,\ x_2^2 + \ldots + x_n^2 < 1\}.$$

Then $B \in S_g^{pq}$ *if*

$$g(t)^{-1/r'}\int_t^\infty g(s)\,ds \in L^r(0,+\infty).$$

Proof. Theorem 1 with its remarks is valid for these generalised Soboleff regions. It follows from remark 2 that it is sufficient to show that there exists a constant K such that

$$|g^{1/q}u|_q \leqq K|g^{1/p}\nabla u|_p \tag{11}$$

for all $u \in T_g^{pq}(B)$ which vanish for e.g. $0 < x_1 < 1$. For these functions we obtain by lemma 2

$$\left(\int_0^\infty |u(x_1)|^q g(x_1)\,dx_1\right)^{1/q} \leqq K\left(\int_0^\infty |(\partial u/\partial x_1)x_1|^p g(x_1)\,dx_1\right)^{1/p}$$

almost everywhere in $S = \{x \mid x_1 = 0, x_2^2 + \ldots + x_n^2 < 1\}$. If we integrate this inequality over S we get

$$\int_S dx \int_0^\infty |u|^q g(x_1)\,dx_1 \leqq K\int_S dx\left(\int_0^\infty |(\partial u/\partial x_1)x_1|^p g(x_1)\,dx_1\right)^{q/p}$$

$$\leqq K\left(\int_S dx\left(\int_0^\infty |\partial u/\partial x_1|^p g(x_1)\,dx_1\right)\right)^{q/p}.$$

Hence (11) is valid, which proves the lemma.

Let f be the one–one mapping from A onto A' which is given by

$$x_i' = f_i(x_1, \ldots, x_n), \qquad i = 1, 2 \ldots, n ,$$

where the f_i are continuously differentiable. Put

$$f_{ik} = \partial f_i / \partial x_k, \qquad k = 1, 2, \ldots, n .$$

We write $|\nabla| \geqq |\nabla'|$ if

$$\sum_{k=1}^{n} \left| \sum_{i=1}^{n} a_i f_{ik} \right| \geqq \sum_{i=1}^{n} |a_i|$$

for all complex numbers a_i.

Let g and g' be two positive, summable functions in A and A' respectively. We denote by dx/dx' the Jacobian of the mapping f and write $|dx/dx'|g \sim g'$ if there exist two constants $0 < c_1 \leqq c_2$ such that

$$c_1 g'(x) \leqq |dx/dx'| g(x) \leqq c_2 g'(x) .$$

In this notation we have the following lemma.

Lemma 4. *If there exists a continuously differentiable one–one mapping from A onto A' such that $c|\nabla| \geqq |\nabla'|$ for some constant c and such that $|dx/dx'|g \sim g'$, where g and g' are two positive summable functions, then*

$$A' \in S_{g'}^{pq} \Rightarrow A \in S_g^{pq} .$$

Proof. The proof is obvious. In fact, the inequality

$$\inf_c |(u+c) g^{1/q}|_q \leqq K |g^{1/p} \nabla u|_p, \qquad u \in T_{g'}^{pq}(A') ,$$

follows immediately from

$$\inf_c |(u+c) g'^{1/q}|_q \leqq K |g'^{1/p} \nabla u|_p, \qquad u \in T_g^{pq}(A) ,$$

and the lemma now follows from theorem 1.

We can now state our main theorem, which follows imediately from lemmas 3 and 4.

Theorem 6. *Let A be an unbounded open connected region of finite measure and assume that there exists a one–one, continuously differentiable mapping f from A onto*

$$B = \{x \mid x_1 > 0, x_2^2 + \ldots + x_n^2 < 1\}$$

such that $c|\nabla| \geqq |\nabla'|$ for some constant c. Further, assume that there exists a continuously differentiable function $\varphi_A(x) > 0$, $x \neq 0$, homogeneous of degree one, such that $|dx/dx'| \sim \varphi_A(x)$ and

$$f(\{x \mid x \in A, \varphi(x)=t\}) = B \cap \{x \mid x_1=t\} .$$

Then $A \in S^{pq}$ *if*

$$\varphi_A(t)^{-1/r'} \int_t^{\infty} \varphi_A(s)ds \in L^r(\, , +\infty) .$$

THEOREM 7. *Let* $A=\{x \mid x_1>0, x_2^2+\ldots+x_n^2<g(x_1)\}$, *where* g *is a positive, decreasing and continuously differentiable function. Then* $A \in S^{p,q}$ *if*

$$g(t)^{-(n-1)/r'} \int_t^{\infty} g(s)^{n-1}ds \in L^r(\, , +\infty) .$$

PROOF. Denote by A' the image of A under the transformation

$$\begin{cases} x_1' = x_1-(x_2^2+\ldots+x_n^2)^{1/2} , \\ x_i' = x_i, \qquad i = 2,3\ldots,n . \end{cases}$$

Let $A^*=A' \cap \{x \mid x_1>0\}$. Then A^* is of the same type as A and the corresponding g^* is a positive, decreasing and continuously differentiable function which satisfies the condition of the theorem. Further, the derivative of g^* is bounded by 1. It is easy to see that $A \in S^{pq}$ if and only if $A^* \in S^{pq}$. In fact, the transformation is a Lipschitz mapping. Hence, it is no restriction to assume that the derivative of g is bounded.

Consider now the mapping from A onto B which is given by

$$\begin{cases} x_1' = x_1 \\ x_i' = \big(1/g(x_1)\big)x_i, \qquad i = 2,3\ldots,n . \end{cases}$$

It is possible to choose a continuously differentiable function $\varphi(x)>0$, $x \neq 0$, homogeneous of degree one, such that for $t>1$

$$A \cap \{x \mid x_1=t\} = A \cap \{x \mid \varphi(x)=t\} .$$

To prove that $A \in S^{pq}$ it is sufficient to show that f and φ have the properties in theorem 6. From the condition imposed on g in the hypothesis, the nature of φ and the fact that

$$dx/dx' = g(x_1)^{n-1} = \varphi_A(x) \quad \text{for } x_1 > 1$$

we see that it only remains to show that there exists a constant c such that $c|\nabla| \geqq |\nabla'|$, that is,

$$c\sum_{k=1}^{n} \left| \sum_{i=1}^{n} a_i f_{ik} \right| = c\left(\left| a_1-(g'(x_1)/g(x_1)) \sum_{i=2}^{n} x_i a_i \right| + \sum_{i=2}^{n} |a_i/g(x_1)| \right) \geqq \sum_{i=1}^{n} |a_i| .$$

for all complex numbers a_i. This is obvious, since g', g and x_i, $i=2,3\ldots,n$, are bounded. Thus f and φ satisfy all the conditions of theorem 6, and hence we have $A \in S^{pq}$.

REMARK. The proof is also valid if g is continuously differentiable except at a countable set of points having no finite point of accumulation.

COROLLARY. *Let* $g(t)=t^{-a}$. *Then* $A \in S^{pq}$ *if and only if*

$$(n-1)\cdot a > r .$$

This follows from theorem 5 and 7 after some calculation. Further, if $g(t)=e^{-t}$ then the corresponding $A \in S^{pq}$ provided $q \leqq p$ and $q < +\infty$.

6. Regions of infinite measure.

Let A be an open connected region of infinite measure. In this case theorem 2 can be sharpened.

THEOREM 8. *If* A *is of infinite measure and* $A \in S^{pq}$, *then*

$$1/p = 1/q + 1/n \quad \text{and} \quad (p,q) \neq (n, +\infty) .$$

PROOF. It follows from theorem 2 that it is sufficient to prove

$$1/p \geqq 1/q + 1/n .$$

Put

$$k = \sup\left\{\alpha \,\middle|\, \int_A |x|^\alpha h(x)\,dx < +\infty\right\}.$$

where h is infinitely differentiable, equal to 0 in a neighbourhood of the origin and equal to 1 for $|x|>1$. We have that

$$\nabla\big(|x|^{\lambda/p+1}h(x)\big) \in L^p(A) \qquad \text{for } \lambda < k .$$

Hence

$$|x|^{\lambda/p+1}h(x) \in L^q(A) ,$$

since $A \in S^{pq}$. But then we must have

$$(\lambda/p+1)q \leqq k \qquad \text{for } \lambda < k .$$

Hence

$$(k/p+1)q \leqq k .$$

We see from this that $k \neq 0$, thus $k<0$ by the definition of k, as $m(A)= +\infty$. The inequality now becomes

$$1/p \geqq 1/q - 1/k .$$

But since $0 > k \geqq -n$, we finally obtain

$$1/p \geqq 1/q + 1/n ,$$

which completes the proof.

From the proof we get $k = -n$. One can also derive that

$$\int_A |x|^{-n} h(x)\, dx = +\infty .$$

If p and q are as stated, then there exist regions of infinite measure which belong to S^{pq}. The whole space R^n is such a space (see Schwartz [6, p. 40]).

7. Inclusion properties.

PROPOSITION. *If*

$$1/q - 1/p \geqq 1/q_0 - 1/p_0 > -1/n, \qquad q \neq +\infty \qquad \text{and } p \geqq p_0 \text{ or } q \geqq q_0$$

then

$$S^{p_0 q_0} \subset S^{pq} .$$

PROOF. If $A \in S^{pq}$ we get from theorem 8 and the above inequality that $m(A) < +\infty$. Hence, $L^q(A) \subset L^r(A)$ if $r \leqq q$ (use Hölder's inequality). Now it follows from the definition of S^{pq} that $S^{pr} \subset S^{pq}$ if $q \leqq r$ and that $S^{rq} \subset S^{pq}$ if $r \leqq p$. We see from this that it is sufficient to prove the proposition for

$$p \geqq p_0,\ q \geqq q_0,\ q < +\infty \quad \text{and} \quad 1/q - 1/p = 1/q_0 - 1/p_0 .$$

Let now $A \in S^{p_0 q_0}$. Consider all $u \in T^{pq}(A)$ which vanish in

$$A' = \{x \mid |x - x_0| < \delta,\ \delta > 0,\ x_0 \in A\} .$$

If δ is sufficiently small we get that A fullfills the conditions in remark 2 of theorem 1. We assume that u is real. Then $v = |u|^{q/q_0} \in L^{q_0}(A)$. Further, $\nabla v \in L^{p_0}(A)$ since

$$|\nabla v| \leqq q/q_0 |u|^{(q/q_0 - 1)} |\nabla u| \in L^{p_0}(A)$$

and

$$1/p_0 = 1/p - 1/q + 1/q_0 .$$

Since $A \in S^{p_0 q_0}$ and all $v \in T^{p_0 q_0}(A)$ and vanish in A' we obtain from (4') that

$$|v|_{q_0} \leqq K |\nabla v|_{p_0} .$$

By Hölder's inequality we get from this

$$(|u|_q)^{q/q_0} \leqq K |u|_q^{q/q_0 - 1} |\nabla u|_p .$$

Hence,

$$|u|_q \leqq K|\nabla u|_p$$

for all $u \in T^{pq}(A)$ which vanish in A' and the proposition now follows from remark 2 of theorem 1.

REFERENCES

1. K. Björup, *On inequalities of Poincaré's type*, Math. Scand. 8 (1960), 157–160.
2. N. Bourbaki, *Espaces vectoriels topologiques*, Ch. I (Act. Sci. Ind. 1189), Paris, 1953.
3. R. Courant und D. Hilbert, *Methoden der mathematischen Physik* II, Berlin 1937.
4. J. Deny et J. L. Lions, *Les espace du type de Beppo Levi*, Ann. Inst. Fourier Grenoble 5 (1955), 305–370.
5. L. Gårding and J. L. Lions, *Functional analysis*, Nuovo Cimento, Supplemento a (10) 14 (1959), 9–66.
6. L. Schwartz, *Théorie des distributions* II (Act. Sci. Ind. 1122), Paris, 1950.
7. S. Soboleff, *Sur un théorème d'analyse fonctionnelle*, Mat. Sbornik (46) 4 (1938), 471–496.

UNIVERSITY OF LUND, SWEDEN