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THE DEPTH AND LS CATEGORY OF A TOPOLOGICAL SPACE

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Abstract

The depth of an augmented ring $\varepsilon\colon A \to \Bbbk$ is the least p, or ∞, such that

$$\mathrm{Ext}_A^p(\Bbbk, A) \neq 0.$$

When X is a simply connected finite type CW complex, $H_*(\Omega X; \mathbb{Q})$ is a Hopf algebra and the universal enveloping algebra of the Lie algebra L_X of primitive elements. It is known that depth $H_*(\Omega X; \mathbb{Q}) \leq \mathrm{cat}\, X$, the Lusternik-Schnirelmann category of X.

For any connected CW complex we construct a completion $\widehat{H}(\Omega X)$ of $H_*(\Omega X; \mathbb{Q})$ as a complete Hopf algebra with primitive sub Lie algebra L_X, and define depth X to be the least p or ∞ such that

$$\mathrm{Ext}_{UL_X}^p(\mathbb{Q}, \widehat{H}(\Omega X)) \neq 0.$$

Theorem: for any connected CW complex, depth $X \leq \mathrm{cat}\, X$.

The Lusternik-Schnirelmann category of a topological space X is the least number m such that X can be covered by $(m+1)$ open sets, each contractible in X. On the other hand, if $\varepsilon\colon A \to \mathbb{Q}$ is an augmented algebra then the *depth* of A is the least integer p such that $\mathrm{Ext}_A^p(\mathbb{Q}, A) \neq 0$, where $\mathbb{Q}$ is an A-module via ε. When X is a simply-connected CW complex with finite rational Betti numbers, the principal theorem of [3] asserts that

$$\mathrm{depth}\, H_*(\Omega X; \mathbb{Q}) \leq \mathrm{cat}\, X.$$

This result remains true [4, Chap. 35] when $\mathbb{Q}$ is replaced by any field $\Bbbk$. Additionally, an extension of the rational result to some non-simply connected spaces is established in [5].

Our objective here is to introduce a new invariant *depth* X, defined via a completion $\widehat{H}(\Omega X)$ of $H_*(\Omega X; \mathbb{Q})$. Here $\widehat{H}(\Omega X)$ is a complete Hopf algebra with primitive sub Lie algebra L_X and

$$\mathrm{depth}\, X = \text{ least } p \text{ (or } \infty\text{) such that } \mathrm{Ext}_{UL_X}^p(\mathbb{Q}, \widehat{H}(\Omega X)) \neq 0.$$

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Then, in our main theorem, we establish the inequality

$$\text{depth}\, X \leq \text{cat}\, X$$

for all connected CW complexes X.

Here we first outline the construction of $\widehat{H}(\Omega X)$, with the details and proofs provided in Section 1, and Section 2. In Section 3 we interpret $\widehat{H}(\Omega X)$ in terms of Sullivan models, and use this to establish the main theorem.

The completion $\widehat{H}(\Omega X)$ is constructed by considering homotopy classes of maps,

$$f_\alpha : X \longrightarrow Y_\alpha,$$

where Y_α is a nilpotent CW complex for which $H_1(Y_\alpha)$ and $\pi_{\geq 2}(Y_\alpha)$ are finite-dimensional rational vector spaces. Spaces Y_α satisfying this condition are called F-spaces. For such spaces Y_α, let $I_{Y_\alpha} \subset H_*(\Omega Y_\alpha; \mathbb{Q})$ be the augmentation ideal and set

$$\widehat{H}(\Omega Y_\alpha) = \varprojlim_n H_*(\Omega Y_\alpha; \mathbb{Q})/I^n_{Y_\alpha};$$

this is the classical completion of $H_*(\Omega Y_\alpha; \mathbb{Q})$. As observed in Proposition 3.2 below, it follows from the work of Quillen [8] that

$$\widehat{H}(\Omega Y_\alpha) = \widehat{UL_{Y_\alpha}},$$

where L_{Y_α} is the primitive sub Lie algebra of the complete Hopf algebra $\widehat{H}(\Omega Y_\alpha)$.

We then restrict attention to those $f_\alpha : X \to Y_\alpha$ which satisfy the following property: if f_α factors up to homotopy as

$$X \xrightarrow{f_\beta} Y_\beta \xrightarrow{g_{\alpha\beta}} Y_\alpha,$$

where Y_β is also an F-space, then $\pi_*(g_{\alpha\beta})$ is surjective.

For such maps $\text{Im}\, \pi_*(f_\alpha)$ is maximal in $\pi_*(Y_\alpha)$. In particular, if $f_\alpha : X \to Y_\alpha$ satisfies $H_1(f_\alpha; \mathbb{Q})$ and $\pi_{\geq 2}(f_\alpha) \otimes \mathbb{Q}$ are surjective, then f_α satisfies this condition. If f_α, f_β both satisfy this condition we set $f_\alpha \leq f_\beta$. It follows from Proposition 1.6 that this makes the set of based homotopy classes $[f_\alpha]$ into an inverse system and that $\widehat{H}(\Omega g_{\alpha\beta})$ is independent of the choice of $g_{\alpha\beta}$. Thus the collection $\widehat{H}(\Omega Y_\alpha)$, indexed by the $[f_\alpha]$ is also an inverse system, and we set

$$\widehat{H}(\Omega X) := \varprojlim_\alpha \widehat{H}(\Omega Y_\alpha).$$

This is a complete Hopf algebra depending functorially on X.

Now it follows from the construction that the maps f_α induce morphisms

$$H_*(\Omega X;\mathbb{Q}) \longrightarrow \varprojlim_n H_*(\Omega X;\mathbb{Q})/I_X^n \longrightarrow \varprojlim_\alpha \widehat{H}(\Omega Y_\alpha) = \widehat{H}(\Omega X),$$

which exhibits $\widehat{H}(\Omega X)$ as a completion of $H_*(\Omega X;\mathbb{Q})$. Moreover, when X is simply connected, an early result of Milnor-Moore-Cartan-Serre ([4]) identifies the Hopf algebra $H_*(\Omega X;\mathbb{Q})$ as the universal enveloping algebra of the graded Lie algebra $L(X) = \pi_*(\Omega X)\otimes\mathbb{Q}$.

In this case our construction defines a morphism $L(X)\to L_X$ of graded Lie algebras, but unless X has finite rational Betti numbers this map may not be an isomorphism. However, when X has finite rational Betti numbers then

$$H_*(\Omega X;\mathbb{Q}) \xrightarrow{\cong} \widehat{H}(\Omega X) \qquad \text{and} \qquad L(X) \xrightarrow{\cong} L_X,$$

so that our result reduces to the original one in [4]. In general, the possible connections even in the simply-connected case between depth $H_*(\Omega X;\mathbb{Q})$, and depth X and cat X remain an open question.

Whereas the definitions of UL_X and $\widehat{H}(\Omega X)$ rely on the work of Quillen, the proof of the main theorem relies on the minimal models of Sullivan ([9], [5]). This ([5, Chapter 1]) assigns to each path-connected CW complex, X, a commutative differential graded algebra (cdga for short), $(A_{PL}(X), d)$, a quasi-isomorphism from a minimal Sullivan algebra,

$$m\colon (\wedge V, d) \xrightarrow{\simeq} (A_{PL}(X), d),$$

a spatial realization $|\wedge V, d|$, and a natural homotopy class of maps

$$\overline{m}\colon X \longrightarrow |\wedge V, d|.$$

An early result in rational homotopy, following a suggestion of Jean-Michel Lemaire, is the introduction in [1] of an invariant $\mathrm{cat}(\wedge V, d)$ and the proof that $\mathrm{cat}(\wedge V, d) \le \mathrm{cat}\, X$. Given this, the proof of the main theorem has two parts. First, associated with $(\wedge V, d)$ is a graded Lie algebra $L_V \cong s^{-1}\,\mathrm{Hom}(V,\mathbb{Q})$ and an invariant Sdepth L_V defined via the acyclic closure of $(\wedge V, d)$. The first part of the proof is given in [2], where we show that

$$\mathrm{Sdepth}\, L_V \le \mathrm{cat}(\wedge V, d).$$

The second part of the proof, which we provide here, is the equality

$$\mathrm{depth}\, X = \mathrm{Sdepth}\, L_V.$$

It depends in part on an isomorphism $L_V \cong L_X$, which gives a topological interpretation of the Lie algebra L_V.

1. *F*-maps and their Sullivan representatives

Throughout this paper, all spaces, cdga's, maps, morphisms and homotopies are based. The lower central series of a group G is denoted by

$$G = G^1 \supset G^2 \supset \cdots,$$

and a morphism $\sigma\colon G \to H$ of groups induces morphisms $\sigma(k)\colon G^k/G^{k+1} \to H^k/H^{k+1}$.

DEFINITION 1.1. An *F-space* is a connected CW complex Y satisfying:

(i) for $k \geq 2$, $\pi_k(Y)$ is a rational vector space, and $\sum_{k\geq 2} \dim \pi_k(Y) < \infty$,

(ii) $H_1(Y)$ is a finite-dimensional rational vector space,

(iii) $\pi_1(Y)$ is nilpotent and acts nilpotently in each $\pi_k(Y)$, $k \geq 2$.

DEFINITION 1.2. An *F-map* is a map $f\colon X \to Y$ from a connected CW complex to an F-space.

LEMMA 1.3.

(i) *If Y is an F-space then each $\pi_1^k(Y)/\pi_1^{k+1}(Y)$ is a finite-dimensional rational vector space and $\pi_1^k(Y) = 0$ for some k.*

(ii) *If $g\colon Y' \to Y$ is a map between F-spaces, then $\pi_*(g)$ is surjective if and only if $H_1(g)$ and $\pi_{n\geq 2}(g)$ are surjective.*

PROOF. Since $H_1(Y) = \pi_1(Y)/[\pi_1(Y), \pi_1(Y)]$ this is automatic for $k = 1$. Moreover, an identity of Hall ([6, Theorem 5.3]) shows that the commutator map $a, b \mapsto [a, b]$ induces a surjection

$$\pi_1(Y)/[\pi_1(Y), \pi_1(Y)] \times \pi_1^k(Y)/\pi_1^{k+1}(Y) \longrightarrow \pi_1^{k+1}(Y)/\pi_1^{k+2}(Y)$$

of abelian groups. Thus (i) follows by induction on k.

The same argument establishes (ii).

If Y is an F-space then

$$\sum_k \dim \pi_1^k(Y)/\pi_1^{k+1}(Y) + \sum_{k\geq 2} \dim \pi_k(Y)$$

is the *length* of Y. Thus if length $Y = r$ then Y has the homotopy type of a finite Postnikov tower

$$Y \sim P_r \longrightarrow P_{r-1} \longrightarrow \cdots \longrightarrow P_i \xrightarrow{\rho_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_0 = \mathrm{pt}$$

in which each ρ_i is a principal $K(\mathbb{Q}, n_i)$-fibration.

On the other hand, associated with any minimal Sullivan algebra $(\wedge V, d)$ is the surjection $\wedge^+ V \to \wedge^+ V/\wedge^{\geq 2} V$, which we identify as a linear map,

$$\zeta\colon \wedge^+ V \longrightarrow V,$$

satisfying $\zeta \circ d = 0$. A morphism $\varphi\colon (\wedge V, d) \to (\wedge Z, d)$ of minimal Sullivan algebras induces the linear map

$$Q(\varphi)\colon V \longrightarrow Z$$

defined by $Q(\varphi)\zeta = \zeta \circ \varphi$, and $Q(\varphi)$ depends only on the homotopy class of φ. Evidently, if $\psi\colon (\wedge W, d) \to (\wedge V, d)$ is also a morphism then $Q(\varphi \circ \psi) = Q(\varphi) \circ Q(\psi)$. Finally, associated with $(\wedge V, d)$ is the CW complex $|\wedge V, d|$ together with a natural morphism

$$\lambda\colon (\wedge V, d) \longrightarrow A_{PL}(|\wedge V, d|).$$

Now suppose $m\colon (\wedge V, d) \xrightarrow{\simeq} A_{PL}(X)$ is a minimal Sullivan model of a connected CW complex. Then m determines a homotopy class of maps

$$\overline{m}\colon X \longrightarrow |\wedge V, d|$$

satisfying $m \sim A_{PL}(\overline{m}) \circ \lambda$. It also determines maps

$$p_X\colon \pi_n(X) \longrightarrow \mathrm{Hom}(V^n, \mathbb{Q}), \qquad n \geq 1,$$

which are linear for $n \geq 2$ and are defined as follows: identify S^n as the quotient $\Delta^n/\partial\Delta^n$, equipped with the standard orientation, and with fundamental class $[S^n] \in H_n(S^n; \mathbb{Z})$. If $\sigma \in \pi_n(X)$ is represented by $g\colon S^n \to X$, compose a Sullivan representative of g with the natural surjection from the minimal model of S^n to $H^*(S^n; \mathbb{Q})$ to obtain a morphism

$$\gamma\colon (\wedge V, d) \longrightarrow H^*(S^n; \mathbb{Q}).$$

This, restricted to $\wedge^+ V$, factors over ζ to define $\overline{\gamma}\colon V \to H^*(S^n; \mathbb{Q})$, and p_X is defined by

$$\langle v, p_X\sigma\rangle = \langle \overline{\gamma} v, [S^n]\rangle.$$

For simplicity, we will write

$$\langle v, \sigma\rangle := \langle v, p_X\sigma\rangle.$$

Now suppose $f\colon X \to Y$ is a map between connected CW complexes with Sullivan models $(\wedge V, d)$ and $(\wedge W, d)$. If $\varphi\colon (\wedge W, d) \to (\wedge V, d)$ is a

Sullivan representative for f, then the homotopy class of φ only depends on the homotopy class of f and the diagrams

$$\begin{array}{ccc} X & \longrightarrow & |\wedge V, d| \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle |\varphi|} \\ Y & \longrightarrow & |\wedge W, d| \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \pi_*(X) & \xrightarrow{p_X} & \mathrm{Hom}(V, \mathbb{Q}) \\ {\scriptstyle \pi_*(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(Q(\varphi),\mathbb{Q})} \\ \pi_*(Y) & \xrightarrow[p_Y]{} & \mathrm{Hom}(W, \mathbb{Q}) \end{array} \tag{1}$$

are respectively homotopy commutative and commutative.

In particular a minimal Sullivan algebra $(\wedge W, d)$ is the Sullivan model of an F-space Y if and only if $\dim W < \infty$. In this case it follows from [5] that the maps $p_Y : \pi_*(Y) \to \mathrm{Hom}(W, \mathbb{Q})$ are bijections and that the map $Y \to |\wedge W, d|$ is a homotopy equivalence. In particular, we may and do restrict attention to F-spaces of the form $|\wedge W, d|$ with model morphism the canonical morphism $(\wedge W, d) \to A_{PL}(|\wedge W, d|)$.

PROPOSITION 1.4. *Suppose* $(\wedge V, d)$ *and* $(\wedge W, d)$ *are respectively the minimal models of a connected CW complex* X *and an* F*-space* Y.

(i) *The correspondences*

$$\varphi \mapsto |\varphi| \circ \overline{m} \qquad \textit{and} \qquad f \mapsto \textit{a Sullivan representative } \varphi$$

define inverse bijections between homotopy classes of morphisms $\varphi : (\wedge W, d) \to (\wedge V, d)$ *and of maps* $f : X \to Y$.

(ii) *If* X *is also an* F*-space and* φ *is a Sullivan representative of* $f : X \to Y$ *then* $\pi_*(f)$ *is surjective if and only if* $Q(\varphi)$ *is injective.*

PROOF. (i) As observed above we may assume $Y = |\wedge W, d|$. Then, in view of (1), $f \sim |\varphi| \circ \overline{m}$ where φ is a Sullivan representative of f. On the other hand, it follows from Proposition 1.15 in [5] that any morphism $\psi : (\wedge W, d) \to (\wedge V, d)$ is a Sullivan representative of $|\psi| \circ \overline{m}$.

(ii) In this case the commutative diagram

$$\begin{array}{ccc} \pi_*(X) & \xrightarrow{\cong} & \mathrm{Hom}(V, \mathbb{Q}) \\ {\scriptstyle \pi_*(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(Q(\varphi),\mathbb{Q})} \\ \pi_*(Y) & \xrightarrow{\cong} & \mathrm{Hom}(W, \mathbb{Q}) \end{array}$$

shows that $\pi_*(f)$ is surjective if and only if $\mathrm{Hom}(Q(\varphi), \mathbb{Q})$ is surjective. But this is equivalent to $Q(\varphi)$ is injective.

DEFINITION 1.5. An F-map $f : X \to Y$ from a connected CW complex is F*-surjective* if, whenever f factors as the composite

$$f : X \xrightarrow{f'} Y' \xrightarrow{g} Y$$

of an F-map f' and a map g, then $\pi_*(g)$ is surjective.

PROPOSITION 1.6. *Let X be a connected CW complex. Then*

(i) *an F-map $f\colon X \to Y$ is F-surjective if and only if a Sullivan representative, φ, for f satisfies $Q(\varphi)$ is injective,*

(ii) *any F-map $f\colon X \to Y$ factors up to homotopy as*

$$f\colon X \xrightarrow{f'} Y' \xrightarrow{g} Y$$

in which f' is F-surjective

(iii) *if $f_\alpha, f_\beta\colon X \to Y_\alpha, Y_\beta$ are F-surjective then there is a third F-surjection $f_\gamma\colon X \to Y_\gamma$ and a homotopy commutative diagram*

$$\begin{array}{ccccc} & & Y_\beta & & \\ & \overset{f_\beta}{\nearrow} & & \overset{g_\beta}{\nwarrow} & \\ X & & \xrightarrow{\quad f_\gamma \quad} & & Y_\gamma \\ & \underset{f_\alpha}{\searrow} & & \underset{g_\alpha}{\swarrow} & \\ & & Y_\alpha & & \end{array}$$

(iv) *if*

$$\begin{array}{ccc} & \overset{f_\beta}{\nearrow} & Y_\beta \\ X & & \downarrow{\scriptstyle g_{\alpha\beta}} \\ & \underset{f_\alpha}{\searrow} & Y_\alpha \end{array}$$

is a homotopy commutative diagram in which f_α and f_β are F-surjections then $\pi_*(g_{\alpha\beta})$ is surjective, and independent of the choice of $g_{\alpha\beta}$.

PROOF. (i) Suppose $\varphi\colon (\wedge W, d) \to (\wedge V, d)$ is a Sullivan representative for f, so that we may assume $f = |\varphi| \circ \overline{m}$. If f factorizes as

$$X \xrightarrow{f'} Y' \xrightarrow{g'} Y$$

and φ' and ψ are Sullivan representatives for f' and g, then $Q(\varphi) = Q(\varphi') \circ Q(\psi)$. If $Q(\varphi)$ is injective so is $Q(\psi)$ and Proposition 1.4 asserts that $\pi_*(g)$ is surjective. Thus f is F-surjective.

Conversely, suppose f is F-surjective. For some finite dimensional subspace $Z \subset V$ with $\wedge Z$ preserved by d we can decompose φ as

$$(\wedge W, d) \xrightarrow{\psi} (\wedge Z, d) \xrightarrow{\chi} (\wedge V, d).$$

Now $f \sim |\psi| \circ (|\chi| \circ \overline{m})$ and therefore $\pi_*(|\psi|)$ is surjective, which implies $Q(\psi)$ is injective. Since $Q(\chi)$ is injective by construction, $Q(\varphi)$ must be injective.

(ii) As in the proof of (i), factor a Sullivan representative of f as $\varphi = \chi \circ \psi$ with χ extending the inclusion of a finite dimensional subspace $Z \subset V$. Thus $|\psi|$ is an F-map and by (i), $|\chi| \circ \overline{m}$ is F-surjective since $Q(\chi)$ is injective. But $f \sim |\psi| \circ (|\chi| \circ \overline{m})$. Thus we have a decomposition of f as $g \circ f'$ in which f' is F-surjective.

(iii) Because of (ii) the map $(f_\alpha, f_\beta)\colon X \to Y_\alpha \times Y_\beta$ factors as

$$X \xrightarrow{f_\gamma} Y \xrightarrow{!(g_\alpha, g_\beta)} Y_\alpha \times Y_\beta$$

in which f_γ is F-surjective.

(iv) This follows because Sullivan representatives $\varphi_\alpha, \varphi_\beta$ for f_α, f_β satisfy $Q(\varphi_\alpha)$ and $Q(\varphi_\beta)$ are injective, and because if $\varphi_{\alpha\beta}$ is a Sullivan representative of $g_{\alpha\beta}$ then $Q(\varphi_{\alpha\beta})$ is independent of the choice of $\varphi_{\alpha\beta}$.

Proposition 1.7. *Suppose $f\colon X \to Y$ is an F-map.*

(i) *If $H_1(f; \mathbb{Q})$ and $\pi_k(f) \otimes \mathbb{Q}$, $k \geq 2$, are surjective then f is F-surjective.*

(ii) *If the natural maps*

$$p_X \otimes \mathbb{Q}\colon \pi_k(X) \otimes \mathbb{Q} \longrightarrow \mathrm{Hom}(V^k, \mathbb{Q}), \qquad k \geq 2,$$

are surjective then f is F-surjective if and only if $H_1(f; \mathbb{Q})$ and $\pi_k(f) \otimes \mathbb{Q}$, $k \geq 2$, are surjective.

Proof. Let $\varphi\colon (\wedge W, d) \to (\wedge V, d)$ be a Sullivan representative of f. Then $Q(\varphi) = \varphi\colon W^1 \to V^1$. Denote by ψ the restriction of φ to $(\wedge W^1, d)$. Since $\dim W < \infty$ an easy induction shows that ψ is injective if and only if $H^1(\psi)$ is injective. But

$$H^1(\psi) = H^1(\varphi) = H^1(f; \mathbb{Q}),$$

since φ is a Sullivan representative of f. But $H^1(f; \mathbb{Q})$ is the dual of $H_1(f; \mathbb{Q})$, and the dual of a linear map is injective if and only if the linear map is surjective. This establishes

$$Q(\varphi)|_{W^1} \text{ is injective } \iff H_1(f; \mathbb{Q}) \text{ is surjective.}$$

On the other hand, for $k \geq 2$ we have a commutative diagram

$$\begin{array}{ccc} \pi_k(X) \otimes \mathbb{Q} & \longrightarrow & \mathrm{Hom}(V^k, \mathbb{Q}) \\ {\scriptstyle \pi_k(f)\otimes\mathbb{Q}} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(Q(\varphi),\mathbb{Q})} \\ \pi_k(Y) & \xrightarrow{\cong} & \mathrm{Hom}(W^k, \mathbb{Q}) \end{array}$$

Thus if $\pi_k(f)\otimes\mathbb{Q}$ is surjective, then $\mathrm{Hom}(Q(\varphi),\mathbb{Q})$ is surjective and $Q(\varphi)$ is injective. In the reverse direction, if $\pi_k(X)\otimes\mathbb{Q}\to\mathrm{Hom}(V^k,\mathbb{Q})$ is surjective and $Q(\varphi)$ is injective then $\mathrm{Hom}(Q(\varphi),\mathbb{Q})$ is surjective and $\pi_k(f)\otimes\mathbb{Q}$ must be surjective.

EXAMPLE 1.8.

1. The F-map $f\colon (S^3\times S^3)\vee S^5\to S^6$ defined as the smash product on $S^3\times S^3$ and the trivial map on S^5 is not F-surjective because it factorizes through $S^3\times S^3$.
2. The conditions of Proposition 1.7 are not the only examples of F-surjective maps. In fact, let $\omega\in H^6((S^3)^\infty;\mathbb{Q})$ be an indecomposable element. Then the associated map $f\colon (S^3)^\infty\to K(Q,6)$ is trivial in homotopy but is F-surjective, since if φ is a Sullivan representative of f then $Q(\varphi)$ is injective.
3. Sullivan spaces. A *Sullivan space* [5, Chap. 7] is a connected CW complex X such that in particular its minimal Sullivan model $(\wedge V,d)$ satisfies $p_X\otimes\mathbb{Q}\colon \pi_{\geq 2}(X)\otimes\mathbb{Q}\xrightarrow{\cong}\mathrm{Hom}(V^{\geq 2},\mathbb{Q})$. Thus if $f\colon X\to Y$ is an F-map from a Sullivan space then f is F-surjective if and only if $H_1(f;\mathbb{Q})$ and $\pi_{\geq 2}(f)\otimes\mathbb{Q}$ are surjective.
4. Spaces with Sullivan minimal models of the form $(\wedge V^1,d)$. For these spaces it is trivially true that an F-map f is F-surjective if and only if $H^1(f;\mathbb{Q})$ is injective. A number of examples of such spaces are provided in [5, Chap. 8].

2. Construction of $\widehat{H}(\Omega X)$ and the definition of depth X

Denote by $\mathcal{S}=\{\alpha\}$ the set of homotopy classes of F-surjective maps $f_\alpha\colon X\to Y_\alpha$ from a connected CW complex X. Then set

$$f_\beta\geq f_\alpha \iff f_\alpha\sim g_{\alpha\beta}\circ f_\beta$$

for some map $g_{\alpha\beta}\colon Y_\beta\to Y_\alpha$. It follows from Proposition 1.6 that this makes this set of homotopy classes into an inverse system. Moreover, since $\pi_*(g_{\alpha\beta})=\eta_{\alpha\beta}$ is independent of the choice of $g_{\alpha\beta}$ it follows that

$$\{\pi_*(Y_\alpha),\eta_{\alpha\beta}\}_{\alpha\in\mathcal{S}}$$

is an inverse system of groups.

Recall now the structure of $H_*(\Omega X;\mathbb{Q})$ when X is a CW complex with fundamental group $G=\pi_1(X)$. Let $\widetilde{X}$ be the universal cover of X and for $g\in G$ denote by $(\Omega X)_g$ the component of ΩX of the loops representing g. Then $\Omega X=\coprod_{g\in G}(\Omega X)_g$ and $(\Omega X)_e=\Omega\widetilde{X}$. Finally let $\gamma\colon G\to\Omega X$ be a choice of representing elements. For $\omega \in (\Omega X)_e$ and $g \in G$ we define ω^g to the the composition of loops: $\omega^g = \gamma(g)^{-1} \cdot \omega \cdot \gamma(g)$. Then the morphism

$$\varphi\colon G \times (\Omega X)_e \longrightarrow \Omega X$$

defined by $\varphi(g, \omega) = \gamma(g) \cdot \omega$ induces an isomorphism of algebras

$$\mathbb{Q}[G] \otimes H_*(\Omega \widetilde{X}; \mathbb{Q}) \longrightarrow H_*(\Omega X; \mathbb{Q}),$$

where the multiplication on the left is given by

$$(g, \alpha) \cdot (g', \alpha') = (gg', \alpha^{g'} \cdot \alpha').$$

This isomorphism is independent of the choice of γ, and the action of G, $\alpha \mapsto \alpha^{g'}$, is induced by the conjugation $\omega \mapsto \omega^{g'}$.

Therefore,

$$H_*(\Omega Y_\alpha; \mathbb{Q}) = \mathbb{Q}[\pi_1 Y_\alpha] \otimes H_*(\Omega \widetilde{Y_\alpha}) = \mathbb{Q}[\pi_1 Y_\alpha] \otimes U E_\alpha,$$

where $E_\alpha = \pi_*(\widetilde{Y_\alpha})$ with the Samelson Lie bracket ([7]). In particular, the morphism $H_*(\Omega g_{\alpha\beta})$ is determined by $\pi_*(g_{\alpha\beta})$ and so is independent of $g_{\alpha\beta}$. This makes $\{H_*(\Omega Y_\alpha; \mathbb{Q})\}$ into an inverse system.

On the other hand, the $H_*(\Omega Y_\alpha; \mathbb{Q})$ are naturally augmented Hopf algebras with augmentation ideals I_α and, the collection

$$\widehat{H_*}(\Omega Y_\alpha; \mathbb{Q}) := \varprojlim_n H_*(\Omega Y_\alpha; \mathbb{Q})/I_\alpha^n$$

is an inverse system of complete Hopf algebras ([5], [8]). We define

$$\widehat{H}(\Omega X) = \varprojlim_\alpha \widehat{H}_*(\Omega Y_\alpha; \mathbb{Q}).$$

Now set

$$\widehat{H}(\Omega X) \mathbin{\widehat{\otimes}} \widehat{H}(\Omega X) = \varprojlim_\alpha \widehat{H}(\Omega Y_\alpha; \mathbb{Q}) \mathbin{\widehat{\otimes}} \widehat{H}(\Omega Y_\alpha; \mathbb{Q}).$$

Then $\widehat{H}(\Omega X)$ is a complete Hopf algebra with diagonal $\Delta = \varprojlim_\alpha \Delta_\alpha$.

LEMMA 2.1. *The primitive sub Lie algebra L_X of $\widehat{H}(\Omega X)$ satisfies*

$$L_X = \varprojlim_\alpha L_{Y_\alpha},$$

and the inclusion $L_X \hookrightarrow \widehat{H}(\Omega X)$ extends to an inclusion

$$j\colon UL_X \hookrightarrow \widehat{H}(\Omega X)$$

of graded algebras.

PROOF. The fact that j exists is immediate because L_X is a sub Lie algebra of $\widehat{H}(\Omega X)$. The fact that the extension is injective follows from the Poincaré-Birkhoff-Witt Theorem ([4, Theorem 21.1]) and the fact that $(j \otimes j) \circ \Delta_{UL_X} = \Delta \circ j$.

Finally the restriction maps $\widehat{H}(\Omega X) \to \widehat{H}(\Omega Y_\alpha; \mathbb{Q})$ necessarily send L_X to L_α and so define a morphism $\sigma \colon L_X \to \varprojlim_\alpha L_{Y_\alpha}$. The commutative diagram

$$\begin{array}{ccc} \widehat{H}(\Omega X) & \xrightarrow{=} & \varprojlim_\alpha \widehat{H}(\Omega Y_\alpha; \mathbb{Q}) \\ \uparrow & & \uparrow \\ L_X & \xrightarrow{\sigma} & \varprojlim_\alpha L_{Y_\alpha} \end{array}$$

shows that σ is injective. The inverse limit of injections is injective, and so $\varprojlim_\alpha L_{Y_\alpha} \subset \varprojlim_\alpha \widehat{H}(\Omega Y_\alpha; \mathbb{Q})$. But if $\Phi \in \widehat{H}(\Omega X)$ corresponds to an element of $\varprojlim_\alpha L_{Y_\alpha}$, then $\Delta\Phi - \Phi \otimes 1 - 1 \otimes \Phi = 0$, and so $\Phi \in L_X$.

DEFINITION 2.2. The *depth* of a connected CW complex X is the least p, or ∞, such that

$$\mathrm{Ext}^p_{UL_X}(\mathbb{Q}, \widehat{H}(\Omega X)) \neq 0.$$

EXAMPLE 2.3.

1. Let X be the wedge of infinitely many spheres S^3,

$$X = \bigvee_{k \geq 1} S^3_k.$$

   Then $\pi_*(\Omega X) \otimes \mathbb{Q}$ is the free Lie algebra on infinitely many variables in degree 2. The loop space homology $H_*(\Omega X; \mathbb{Q}) = T(\oplus_k \mathbb{Q} a_k) = U\mathbb{L}$ is the tensor algebra on the a_k. Let α_j be the basis of $H_*(\Omega X)$ formed by the monomials in the a_i. Then $\widehat{H}(\Omega X)$ is the set of series $\sum_j \lambda_j \alpha_j$ with $\lambda_j \in \mathbb{Q}$, with usual multiplication. Remark that in this case the Lie algebra L_X is very big; for instance $(L_X)_2$ is the bidual of $\pi_3(X) \otimes \mathbb{Q}$.

2. Let X be the wedge $X = S^1 \vee S^2$. Then $\pi_2(X) \otimes \mathbb{Q}$ is countably infinite with a basis a_i, $i \in \mathbb{Z}$, and $\mathbb{Z} = \pi_1(X)$ acts on $\pi_2(X) \otimes \mathbb{Q}$ by translation: if t is the generator of $\mathbb{Z}$, then $t \cdot a_i = a_{i+1}$. Not $(L_X)_0$ is $\mathbb{Q}$, the Malcev completion of $\mathbb{Z}$, and $(L_X)_1 = \widehat{H}_1(\Omega X)$ is the Malcev completion of $\pi_2(X) \otimes \mathbb{Q}$ as a module over $\pi_1(X) \otimes \mathbb{Q}$,

$$(L_X)_1 = \lim_p \pi_2(X) \otimes \mathbb{Q}/I^p,$$

   where I^p denotes the submodule generated by the $(t - \mathrm{id})^p(a_i)$.

3. The loop space homology of an F-space, Y

Here we identify $\widehat{H}_*(\Omega Y;\mathbb{Q})$ in terms of a Sullivan minimal model $(\wedge W, d)$ for Y. For this recall that the homotopy Lie algebra, L, for $(\wedge W, d)$ is defined [5, Chap. 2] by $L_k = \mathrm{Hom}(W^{k+1}, \mathbb{Q})$ with Lie bracket determined by the quadratic part of the differential. Recall as well [5, Chap. 3] that (as with any minimal Sullivan algebra) $(\wedge W, d)$ extends to an acyclic closure $(\wedge W \otimes \wedge U_W, d)$ with homology just $\mathbb{Q}$, and where the quotient differential in $\wedge U_W = \mathbb{Q} \otimes_{\wedge W} (\wedge W \otimes \wedge U_W)$ is zero. Moreover, according to [5, §6.1], $\wedge U_W$ is equipped with a diagonal which makes it into a graded Hopf algebra. Finally, we recall [5, Chap. 3] the commutative diagram

$$\begin{array}{ccccc} (\wedge W, d) & \longrightarrow & (\wedge W \otimes \wedge U_W, d) & \longrightarrow & (\wedge U_W, 0) \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{m_\Omega} \\ A_{PL}(Y) & \longrightarrow & A_{PL}(PY) & \longrightarrow & A_{PL}(\Omega Y) \end{array}$$

Proposition 3.1. *With the notation above, if Y is simply connected there is a natural isomorphism of Hopf algebras*

$$UL \xrightarrow{\cong} H_*(\Omega Y;\mathbb{Q})$$

restricting to an isomorphism

$$L \xrightarrow{\cong} L_Y$$

of graded Lie algebras.

Proof. Since $H^*(Y;\mathbb{Q})$ has finite type, Theorem 5.1 in [5] asserts that m_Ω induces an isomorphism

$$\wedge U_W \xrightarrow{\cong} H^*(\Omega Y;\mathbb{Q}).$$

Corollary 6.2 in [5] then asserts that the dual isomorphism,

$$\mathrm{Hom}(\wedge U_W, \mathbb{Q}) \xleftarrow{\cong} H_*(\Omega Y;\mathbb{Q})$$

is an isomorphism of graded Hopf algebras. On the other hand, Theorem 6.2 and Proposition 6.3 in [5] provide a natural isomorphism

$$UL \xrightarrow{\cong} \mathrm{Hom}(\wedge U_W, \mathbb{Q})$$

of graded Hopf algebras. Since L and L_Y are respectively the primitive sub Lie algebras of UL_W and $H_*(\Omega Y;\mathbb{Q})$ it follows that the composite isomorphism

$$UL \cong H(\Omega Y;\mathbb{Q})$$

restricts to an isomorphism $L \xrightarrow{\cong} L_Y$ of graded Lie algebras.

On the other hand, because Y is an F-space, the natural map $Y \to |\wedge W, d|$ is a homotopy equivalence [5, Theorem 1.4]. We use this to identify $\pi_1(Y) = \pi_1(|\wedge W, d|)$. Thus Theorem 2.4 in [5] produces a natural isomorphism of groups,

$$\pi_1(Y) \xrightarrow{\cong} G_L,$$

where G_L is the group of group-like elements in the complete Hopf algebra $\widehat{UL_0}$.

PROPOSITION 3.2. *The inclusion of G_L extends uniquely to an isomorphism*

$$\widehat{\mathbb{Q}[G_L]} \longrightarrow \widehat{UL_0}$$

of complete Hopf algebras.

PROOF. Because exp and log are inverse bijections between L and G_L, it follows that G_L is nilpotent and each G_L^k/G_L^{k+1} is a rational vector space. Thus it follows from [8] that the completion, $\widehat{\mathbb{Q}[G_L]}$ satisfies

$$\widehat{\mathbb{Q}[G_L]} = \widehat{UP},$$

where $P \subset \widehat{\mathbb{Q}[G_L]}$ is the primitive sub Lie algebra. In fact the inclusion of P extends uniquely to a morphism

$$p\colon \widehat{UP} \longrightarrow \widehat{\mathbb{Q}[G]}.$$

By [8, Appendix A, Corollary 3.9], the linear map $P \to \widehat{UP}$ is an isomorphism onto the primitive sub Lie algebra of $\widehat{UP}$. By [8, Appendix A, Corollary 2.18], p is thus an isomorphism.

On the other hand, the inclusion $G_L \to \widehat{UL_0}$ extends to a morphism $\widehat{\mathbb{Q}[G_L]} \to \widehat{UL_0}$ which sends G_L to itself. Moreover, by [8, Appendix A, Corollary 3.7] G_L is the group of group-like elements. Thus we obtain a morphism

$$\widehat{UP} \longrightarrow \widehat{UL_0}$$

which is the identity on the group of group-like elements. Since exp and log define inverse bijections $P \xrightarrow{\cong} G_L$ and $L_0 \xrightarrow{\cong} G_L$ and so the morphism above restricts to an isomorphism $P \xrightarrow{\cong} L_0$. Thus altogether we obtain the natural isomorphism

$$\widehat{\mathbb{Q}[G_L]} \xrightarrow{\cong} \widehat{UL_0}$$

extending the identity in G_L.

We can now prove

PROPOSITION 3.3. *Suppose L is the homotopy Lie algebra of the minimal Sullivan model, $(\wedge W, d)$, of an F-space Y. Then with the notation above there is a natural isomorphism of complete Hopf algebras,*

$$\widehat{UL} \xrightarrow{\cong} \widehat{H}_*(\Omega Y;\mathbb{Q}),$$

restricting to an isomorphism

$$L \xrightarrow{\cong} L_Y$$

of graded Lie algebras.

PROOF. First recall the isomorphism of Hopf algebras

$$H_*(\Omega Y;\mathbb{Q}) = \mathbb{Q}[\pi_1 Y] \otimes H_*(\Omega\widetilde{Y};\mathbb{Q})$$

in which the product uses the action of $\pi_1(Y)$ on $H_*(\Omega\widetilde{Y})$. Passing to completions gives

$$\widehat{H}_*(\Omega Y;\mathbb{Q}) = \widehat{\mathbb{Q}[\pi_1 Y]} \otimes H_*(\widetilde{\Omega} Y) = \widehat{UL_0} \otimes UL_{\geq 1} = \widehat{UL}.$$

It follows from [5, Theorem 2.5] that the middle identification is an isomorphism of graded Hopf algebras, and this is trivially true for the other two. Thus this isomorphism $H_*(\Omega Y;\mathbb{Q}) \cong \widehat{UL}$ restricts to an isomorphism $L \xrightarrow{\cong} L_Y$, because [5, Prop. 2.3] L is the primitive subspace of $\widehat{UL}$.

4. The main theorem

THEOREM 4.1. *Let X be a connected CW complex with minimal Sullivan model $(\wedge V, d)$. Then L_X is the homotopy Lie algebra L_V of $(\wedge V, d)$ and*

$$\operatorname{depth} X \leq \operatorname{cat} X.$$

First recall that the homotopy classes of F-maps $f_\alpha\colon X \to Y_\alpha$ are in bijection with the homotopy classes of morphisms

$$\varphi_\alpha\colon (\wedge W_\alpha, d) \longrightarrow (\wedge V, d)$$

to the minimal Sullivan model of W, where $\dim W_\alpha < \infty$. Moreover, f_α is F-surjective if and only if $Q(\varphi_\alpha)$ is injective. In particular, the inverse system of the main theorem is isomorphic to the inverse system of homotopy classes of such morphisms, with

$$\varphi_\beta \geq \varphi_\alpha \Longleftrightarrow \varphi_\alpha \sim \varphi_\beta \circ \varphi_{\alpha\beta}$$

for some $\varphi_{\alpha\beta}\colon (\wedge W_\alpha, d) \longrightarrow (\wedge W_\beta, d)$.

Now any morphism of minimal Sullivan algebras $\varphi\colon (\wedge W, d) \to (\wedge V, d)$ with $\dim W < \infty$ satisfies $\varphi(\wedge W) \subset \wedge S$ for some finite dimensional subspace $S \subset V$ with $\wedge S$ preserved by d. It follows that the homotopy classes of inclusions $\varphi_\alpha\colon (\wedge W_\alpha, d) \to (\wedge V, d)$ extending the inclusion of a subspace $W_\alpha \subset V$ form a cofinal set

$$\mathscr{J} = \{\alpha\}$$

in our inverse system.

Now observe that in $\mathscr{J}$,

$$\beta \geq \alpha \Longleftrightarrow W_\alpha \subset W_\beta,$$

since in this case the map $\mathbb{Q}(\varphi_{\alpha\beta})$ is just the inclusion $W_\alpha \hookrightarrow W_\beta$. (This follows at once from the fact that $Q(\varphi_\alpha)$ and $Q(\varphi_\beta)$ are just the inclusions of W_α and W_β in V.)

Denote by L_α the homotopy Lie algebra of $(\wedge W_\alpha, d)$ and by L_V the homotopy Lie algebra of $(\wedge V, d)$. Then

$$sL_V = \mathrm{Hom}(V, \mathbb{Q}) = \varprojlim_{\alpha \in \mathscr{J}} \mathrm{Hom}(W_\alpha, \mathbb{Q}) = \varprojlim_{\alpha \in \mathscr{J}} sL_\alpha.$$

It follows from the definition of the Lie bracket [5, Chap 2] that this defines an isomorphism

$$L_V \xrightarrow{\cong} \varprojlim_{\alpha} L_\alpha$$

of graded Lie algebras. Moreover, the surjections $L_V \to L_\alpha$ induce morphisms $UL_V \to \widehat{UL_\alpha}$, which define a morphism

$$UL_V \longrightarrow \varprojlim_{\alpha} \widehat{UL_\alpha}.$$

Proposition 4.2. *Let X be a connected CW complex. Then there are natural isomorphisms*

$$L_V \xrightarrow{\cong} L_X \qquad \textit{and} \qquad \varprojlim_{\alpha} \widehat{UL_\alpha} \xrightarrow{\cong} \widehat{H}(\Omega X)$$

which make the diagram

$$\begin{array}{ccc} \varprojlim_{\alpha} \widehat{UL_\alpha} & \xrightarrow{\cong} & \widehat{H}(\Omega X) \\ \uparrow & & \uparrow \\ UL_V & \xrightarrow{\cong} & UL_X \end{array}$$

commute.

PROOF. In view of Lemma 2.1, this is immediate from the isomorphisms $L_\alpha \xrightarrow{\cong} L_{Y_\alpha}$ and $\widehat{UL_\alpha} \xrightarrow{\cong} \widehat{H}(\Omega Y_\alpha; \mathbb{Q})$ of Proposition 3.3.

As described in the previous section, we have natural isomorphisms

$$\mathrm{Hom}(\wedge U_\alpha, \mathbb{Q}) \xrightarrow{\cong} \widehat{UL_\alpha}.$$

Moreover, it follows from [5, §6.2] that these isomorphisms convert right multiplication by L_α to the dual of the holonomy representation of L_α in $\wedge U_\alpha$. Thus we obtain

$$\begin{aligned} \mathrm{Ext}_{UL_X}(\mathbb{Q}, \widehat{H}(\Omega X)) &\cong \mathrm{Ext}_{UL_V}(\mathbb{Q}, \varprojlim_{\alpha \in \mathscr{J}} \widehat{UL_\alpha}) \\ &\cong \mathrm{Ext}_{UL_V}(\mathbb{Q}, \mathrm{Hom}(\varinjlim_{\alpha \in \mathscr{J}} \wedge U_\alpha, \mathbb{Q})) \\ &= \mathrm{Hom}(\mathrm{Tor}^{UL_V}(\mathbb{Q}, \varinjlim_{\alpha} \wedge U_\alpha), \mathbb{Q}). \end{aligned}$$

But if $(\wedge V \otimes \wedge U, d)$ is the acyclic closure of $(\wedge V, d)$ then

$$\wedge U = \varinjlim_{\alpha \in \mathscr{J}} \wedge U_\alpha$$

as L_V-modules, because $V = \varinjlim_{\alpha \in \mathscr{J}} W_\alpha$. This yields

$$\mathrm{Ext}^p_{UL_X}(\mathbb{Q}, \widehat{H}(\Omega X)) = \mathrm{Hom}(\mathrm{Tor}^{UL_V}_p(\mathbb{Q}, \wedge U), \mathbb{Q}).$$

Now by definition ([2]),

$$\mathrm{Sdepth}\, L_V = \text{ least } p \text{ (or } \infty) \text{ such that } \mathrm{Tor}^{UL_V}_p(\mathbb{Q}, \wedge U) \neq 0.$$

This establishes

$$\mathrm{depth}\, X = \mathrm{Sdepth}\, L_V.$$

By [2, Theorem C], $\mathrm{Sdepth}\, L_V \leq \mathrm{cat}(\wedge V, d) \leq \mathrm{cat}\, X$, and the main theorem is proved.

5. The morphism $\pi_*(\Omega X) \otimes \mathbb{Q} \to L_X$: examples

We begin with the Eilenberg-MacLane space $X = K(V, 3)$, where V is a rational vector space whose dimension is countably infinite. Denote by $x_1, x_2, \ldots$ a basis of V and by $(\wedge W, d)$ a minimal Sullivan model for X. Denote by $y_i \in \mathrm{Hom}(V, \mathbb{Q})$ the elements defined by $\langle y_i, x_j \rangle = \delta_{ij}$. The series

$$\omega = \sum_{i \geq 1} y_{2i-1} \wedge y_{2i}$$

is then a well-defined element in $\mathrm{Hom}(\wedge^2 V, \mathbb{Q})$.

LEMMA 5.1. $\omega \notin \wedge^2(\mathrm{Hom}(V, \mathbb{Q}))$.

PROOF. Denote by $V_m \subset V$ the subspace generated by $x_1, \ldots, x_m$. Then the restriction of ω^m to V_{2m} is

$$m!\, y_1 \wedge y_2 \wedge \ldots \wedge y_{2m}.$$

Therefore $\omega^m \neq 0$ for all m. This implies that $\omega \notin \wedge^2(\mathrm{Hom}(V, \mathbb{Q}))$ because if $\omega = \sum_{i=1}^r f_i \wedge g_i$ then $\omega^{r+1} = 0$.

PROPOSITION 5.2. *The minimal Sullivan model* $(\wedge W, d)$ *for* X *satisfies the following properties:*

$$H^*(X; \mathbb{Q}) = \mathrm{Hom}(\wedge V, \mathbb{Q}), \quad W^3 = \mathrm{Hom}(V, \mathbb{Q}), \\ W^4 = W^5 = 0, \quad \textit{and} \quad W^6 \neq 0.$$

PROOF. Let e be a base point in S^3 and let $\varphi_e\colon (S^3)^r \to (S^3)^{r+1}$ be the map defined by $\varphi_x(u_1, \ldots, u_r) = (u_1, \ldots, u_r, e)$. We form the space $(S^3)^\infty = \varinjlim_r (S^3)^r$. Since homology commutes with direct limits, $H_*((S^3)^\infty; \mathbb{Q}) = \wedge V$, and $H^*((S^3)^\infty; \mathbb{Q}) = \mathrm{Hom}(\wedge V, \mathbb{Q})$. On the other hand, since each map $S^q \to (S^3)^\infty$ factors through some $(S^3)^r$, we have $(S^3)^\infty = X$. By construction $W^3 = \mathrm{Hom}(V, \mathbb{Q})$, and by Lemma 5.1, $W^6 \neq 0$.

Denote by z an element of W^6 corresponding to the class ω. The rational homotopy Lie algebra $\pi_*(\Omega X) \otimes Q$ is isomorphic to $s^{-1}V$ and L_X is the dual of sW. Therefore the morphism $\pi_*(\Omega X) \otimes \mathbb{Q} \to L_X$ is injective but not surjective.

Now consider the map $g\colon X \to K(\mathbb{Q}, 6)$ associated to ω, and let

$$K(\mathbb{Q}, 5) \longrightarrow Y \longrightarrow X$$

be the pullback along g of the path space fibration on $K(\mathbb{Q}, 6)$. Since $\pi_q(X) \neq 0$ for $q \neq 3$, $\pi_5(Y) = \mathbb{Q}$, and $\pi_*(Y) = \pi_5(Y) \oplus \pi_*(X) = \pi_5(Y) \oplus \pi_3(X) = \pi_5(Y) \oplus V$.

Proposition 5.3. *The map $\pi_*(\Omega Y) \otimes \mathbb{Q} \to L_Y$ is neither injective or surjective.*

Proof. The relative Sullivan model for the path fibration has the form $(\wedge(c, u), d)$ with $du = c$ and a basis element $v \in \pi_4(\Omega K(\mathbb{Q}, 5))$ satisfies

$$\langle u, sv \rangle = 1.$$

The map $c \mapsto z$ gives a (non-minimal) Sullivan model $(\wedge W \otimes \wedge u, d)$ for Y, with $du = z$. Since z is a generator in $\wedge W$, the minimal Sullivan model of Y is $(\wedge W/(z), d)$. Thus $(L_Y)_2 = \mathrm{Hom}(W^3, \mathbb{Q})$ and the map $V \to \mathrm{Hom}(W^3, \mathbb{Q})$ is not surjective.

On the other hand $\pi_4(\Omega Y) \cong \pi_5(Y) = \mathbb{Q}$ and the image of this element is zero in L_Y, since $W^5 = 0$. This shows that $\pi_4(\Omega Y) \otimes \mathbb{Q} \to L_Y$ is not injective.

Proposition 5.4. *The Lie algebra $L = \pi_*(\Omega Y) \otimes \mathbb{Q}$ is the quotient of the free Lie algebra $\mathbb{L}$ on the elements $a_i = s^{-1}x_i$ by the ideal I generated by $\mathbb{L}^3$, the brackets $[a_i, a_j]$ for $|i - j| > 2$, and the elements $[a_{2i-1}, a_{2i}] - [a_1, a_2]$.*

Proof. Note first that $L^3 = 0$ for degree reasons. Then fix integers j and k and let $h\colon \mathbb{Q}^2 \to V$ be the injection of the sub vector space generated by x_j and x_k. The morphism h induces a map $h\colon K(\mathbb{Q}^2, 3) \to X$ and we denote by Z the pullback of Y over h:

$$\begin{array}{ccc}
K(\mathbb{Q}, 5) & \xrightarrow{\ =\ } & K(\mathbb{Q}, 5) \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\ \ell\ } & Y \\
\downarrow & & \downarrow \\
K(\mathbb{Q}^2, 3) & \xrightarrow{\ h\ } & X.
\end{array}$$

In the case $(j, k) = (2i - 1, 2i)$, the minimal Sullivan model of Z is given by $(\wedge(y_{2i-1}, y_{2i}, u), d)$ with $du = y_{2i-1}y_{2i}$. Then the relation between the quadratic part of the differential and the Lie bracket of L_Y [4, Prop. 23.2] gives

$$\langle du, sa_{2i-1}, sa_{2i} \rangle = -\langle u, s[a_{2i-1}, a_{2i}] \rangle.$$

Since Z is a nilpotent space with finite Betti numbers, in $\pi_*(\Omega Z)$ we have $v = -[a_{2i-1}, a_{2i}]$. By naturality, this is also true in $\pi_*(\Omega Y)$. Since the left hand side is independent of i, in $\pi_*(Y)$, we have $[a_{2i-1}, a_{2i}] = [a_1, a_2]$ for all i.

When $|j - k| > 2$, then $H^*(h)\omega = 0$ and therefore the minimal Sullivan model of Z is $(\wedge(y_j, y_k, u), d = 0)$. It follows in the same way that $[a_j, a_k] = 0$. This gives a surjection $\mathbb{L}/I \to \pi_*(\Omega Y) \otimes \mathbb{Q}$, and since $s^{-1}V \oplus v$ maps onto $\mathbb{L}/I$, this surjection is an isomorphism.

Finally consider the space T obtained from the cohomology class $\omega \cdot [y_1] \in H^9(X; \mathbb{Q})$,

$$K(\mathbb{Q}, 8) \longrightarrow T \longrightarrow X.$$

The minimal Sullivan model of T is $(\wedge W \otimes \wedge t, d)$ with $dt = zy_1$, so L_T is non abelian. On the other hand, for degree reasons, $\pi_*(\Omega T) \otimes \mathbb{Q}$ is an abelian Lie algebra.

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