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FACTORIZATION OF BANACH SPACES[1]

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Introduction.

This paper treats the factorization of Banach spaces as l-products (like l_1) and as m-products (like l_∞). The central results are, first, that the subspaces of a given space which are l-factors (or those which are m-factors) form a Boolean algebra; so for finite-dimensional spaces there is a unique factorization into prime factors. Second, with a well known exception in dimension 2, a non-trivial l-product cannot be a non-trivial m-product. Thus for finite-dimensional spaces there is a unique decomposition as an l-product of m-products of ... (and so on, alternating) except for the ambiguities concerning 1-dimensional factors. In general, the Boolean algebra of l-factors is complete. Splitting it into atomic and atomless parts, every Banach space is represented as the l-product of all its prime l-factors and a remainder. On the other hand, simple examples show that the algebra of m-factors need not be complete, and even if it is complete atomic, the space may be a proper subspace of the m-product of its m-prime factors.

There is a whole continuum of product constructions, like l_p, and one may conjecture that similar results hold for $p \neq 2$. Probably nothing in this paper could be of any use in proving such results.

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1. Pure factorization.

For any family of Banach spaces $\{E_\alpha\}$, the *l-product* $\vee E_\alpha$ is the set of all choice functions $\{x_\alpha\}$, with $x_\alpha \in E_\alpha$, such that $\|x_\alpha\|$ is zero except for countably many indices and the non-zero norms form a convergent series. The norm $\|\{x_\alpha\}\|$ is defined to be $\sum \|x_\alpha\|$. One verifies easily that $\vee E_\alpha$ is a Banach space E, and that the external definition of the l-product corresponds to the following internal characterization. E is (isomorphic with) the l-product of its subspaces E_α if the E_α are (finitely) linearly

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independent and the unit ball of E is the smallest closed convex set containing the unit balls of all E_α. (For a general reference, see [1].)

The *m-product* $\times E_\alpha$ is the set of all choice functions $\{x_\alpha\}$ such that $\|x_\alpha\|$ is bounded, with the norm $\|\{x_\alpha\}\|$ defined as $\sup\|x_\alpha\|$. As the subspaces E_α do not generate a dense subspace of $\times E_\alpha$, internal characterization is not so simple. Note that both $\vee$ and $\times$ are unrestrictedly associative (and commutative); so at any rate the subspaces E_α are m-factors. ($\times E_\alpha = E_\beta \times F$, where $F = \times\{E_\alpha : \alpha \neq \beta\}$.) Then E is the l-product of a set of l-factors whenever they are linearly independent and generate a dense subspace; E is the m-product of a set of m-factors E_α if they are linearly independent and the projections $E \to E_\alpha$ are the coordinates of a one-to-one mapping of E onto $\times E_\alpha$. This awkward characterization is not more awkward than it seems, for the projections $E \to E_\alpha$ are unique; the complementary m-factor F is the set of all x such that $y \in E_\alpha$ and $\|y\| \leqq \|x\|$ imply $\|x+y\| = \|x\|$.

For finite products, the conjugate space of $\vee E_i$ is the m-product of the conjugate spaces, $\times E_i^*$; and the conjugate space of $\times E_i$ is $\vee E_i^*$. In any case, the extreme points of the unit ball of $\vee E_\alpha$ are exactly the extreme points of the unit balls of the subspaces E_α. The extreme points of the unit ball of $\times E_\alpha$ are those $\{x_\alpha\}$ such that every x_α is an extreme point of the unit ball of E_α. By the Krein–Milman Theorem, the unit ball of any conjugate space is the closed convex hull of its extreme points.

LEMMA. *Any two factorizations of a conjugate Banach space as an l-product, $\vee F_\alpha = \vee G_\beta$, have a common refinement.*

PROOF. The subspaces $F_\alpha \cap G_\beta$ are linearly independent. As their unit balls include every extreme point of the unit ball of the given space, it is the smallest closed convex set containing them.

THEOREM 1. *The l-factors of any Banach space form a Boolean algebra (ordered by inclusion); and the m-factors form a Boolean algebra.*

PROOF. Any two finite m-factorizations of E, say $\times F_i = \times G_j$, have a common refinement; for the factorizations of E^* as $\vee F_i^* = \vee G_j^*$ have a common refinement which induces a common refinement in $E^{**} = \times F_i^{**} = \times G_j^{**}$, and the natural embedding of E in E^{**} gives the refinement in E. By a similar argument, any two finite l-factorizations of E have a common refinement. This shows that the l-factors and the m-factors form lattices with 0 and 1; and each factor has at least one complement, the complementary factor. If $E = F_1 \times F_2 = G_1 \times G_2$, where G_2 is another complement of F_1, one computes $F_1 = G_1$ from $F_1 = (F_1 \cap G_1) \times (F_1 \cap G_2)$, and then $F_2 = G_2$ similarly. Hence complements are unique and the m-factors form a Boolean algebra. Similarly, the l-factors form a Boolean algebra.

THEOREM 2. *The Boolean algebra of l-factors of a Banach space is complete. In particular, the l-product of all prime l-factors is an l-factor.*

REMARK. One can define l- and m-products for arbitrary metric spaces. The factors can be identified, not as subspaces, but as equivalence relations. It is not very difficult to prove that the l-factors of any (complete) metric space form a (complete) Boolean algebra. Even for a space of four points, the m-factors need not form a distributive lattice. A serious study of this decomposition theory should go much further, and I have not attempted it.

PROOF OF THEOREM 2. Let $\{F_\alpha\}$ be a disjoint set of l-factors of E, that is, $F_\alpha \cap F_\beta = 0$ for $\alpha \neq \beta$. Let $\{p_\alpha\}$ be the associated projections, $p_\alpha : E \to F_\alpha$. As the norm on a finite l-product is the sum norm, hence for each x in E the finite partial sums of $\sum p_\alpha(x)$ have increasing norms bounded above by $\|x\|$, and they converge to a limit $p(x)$. Since $p_\alpha(p(x)) = p_\alpha(x)$, $p^2 = p$; and the linear operator p projects E upon a subspace F. Also $1-p$ projects E upon a subspace G. By continuity, $\|x\| = \|p(x)\| + \|x - p(x)\|$; so F is an l-factor, an upper bound of $\{F_\alpha\}$. It is the supremum since the F_α generate a dense subspace of F.

For l-factorizations, the algebra of l-factors tells the complete story; representations of E as an l-product correspond precisely to Boolean partitions, i.e. disjoint sets of factors having supremum E. For m-factorizations, first, the algebra is not complete, e.g. for the space of all convergent sequences. The subspace of sequences converging to zero is worse, in that its algebra is complete atomic but has partitions corresponding to no m-factorization. One can associate "subdirect" factorizations to all partitions; also, one easily proves the following:

Any two factorizations of a Banach space as an m-product have a common refinement.

2. Mixed factorization.

LEMMA. *An m-product $F \times G$ cannot be an l-product if the unit ball of F has an extreme point and the unit ball of G has two linearly independent extreme points.*

Of course this lemma will imply, by passing to the conjugate space:

THEOREM 3. *A Banach space of dimension different from 2 cannot be both an l-product and an m-product.*

PROOF OF LEMMA. Let p be an extreme point of the unit ball A of F; let q, r be linearly independent extreme points of the unit ball B of G. Then $p \pm q$ and $p \pm r$ are extreme in the unit ball of $E = F \times G$. If $E = J \vee K$, these four points must be in the unit balls C of J, D of K. But $p+q$ and $p-q$ cannot be in the same one. If they were, that unit ball would contain p and q. Observe that the boundary of the unit ball of E is the geometric join of the boundaries S of C, T of D; that is, the union of all line segments joining points of S to points of T. Suppose p and q lie in S. The set $p+B$ is in the boundary of the unit ball, and it is a union of line segments passing through p. Every such line segment with an interior point in S lies in S; so S contains $p+B$. Similarly S contains $A+q$. Thus J contains $A+B$, and $K=0$. We conclude that the points $p \pm q$ are one in J, one in K, and so are $p \pm r$. By choice of notation we may suppose $p+q$ and $p+r$ are the ones in J. But

$$q-r = (p+q)-(p+r) = (p-r)-(p-q);$$

J and K are not linearly independent. The contradiction completes the proof.

REFERENCE

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