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ON THE SOLUTION OF THE COMMUTATION RELATION $PQ-QP=-iI$

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1. Introduction and theorem.

The problem we shall consider in this note originates in quantum physics. If we—for simplicity—take a mechanical system with one degree of freedom, then quantum mechanics motivates the study of a pair of linear operators P and Q defined on a vector space $\mathscr{W}$ with a scalar product, such that P and Q are symmetric with respect to the scalar product and satisfy the commutation relation

$$PQ-QP = -iI \tag{1.1}$$

where I denotes the identity operator in $\mathscr{W}$.

Since a scalar product is assumed to exist in $\mathscr{W}$ it is natural to suppose $\mathscr{W}$ to be a Hilbert space. In this case Wielandt [11] has proved that at least one of the operators P or Q must be unbounded and hence only densely defined in $\mathscr{W}$. It follows that (1.1) cannot be satisfied in a literal sense.

By examples it is easy to see that the problem does not have a unique solution. Take $\mathscr{W}$ as $L^2(I)$, where I is an interval on the real axis, and take P as $-id/dt$, Q as multiplication by t. Then the assumptions are satisfied, and we see that if I is a bounded interval then Q is a bounded operator while if I is the whole real axis then Q is unbounded. Hence we have solutions to our problem which are not unitarily equivalent.

To assure uniqueness further assumptions on the operators P and Q are therefore needed, and the problem of posing such assumptions has been extensively studied by several authors (see [1], [2], [3], [5], [7], [8], [9]). The main difficulty is that the operators are only densely defined and it is therefore not easy to justify the necessary functional calculus. One can say that the Hilbert space in a certain sense is too large.

On the other hand one also wants to solve operator equations such as $P\varphi = 0$, and this equation has no non-trivial solutions in the Hilbert space. So, in another sense the Hilbert space is too small.

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One way to circumvent these difficulties is to use the theory of topological vector spaces since, loosely speaking, this theory allows us to use a space $\mathscr{W}$ smaller than the Hilbert space and with a larger dual space.

DEFINITION 1. By Schwartz' space $(\mathscr{S})$ we mean the space of those infinitely often differentiable, complex valued functions φ defined on the real axis for which

$$\sup |t^{\alpha}\varphi^{(\beta)}(t)| < \infty, \qquad \alpha,\beta = 0,1,\ldots . \tag{1.2}$$

The space $(\mathscr{S})$ is topologized by the norms (1.2). In $(\mathscr{S})$ we define a scalar product by

$$(\varphi,\psi) = \int_{-\infty}^{\infty} \varphi(t)\,\bar{\psi}(t)\,dt$$

and operators p and q by

$$(p\varphi)(t) = -i\varphi'(t), \qquad (q\varphi)(t) = t\varphi(t) .$$

Obviously the space $(\mathscr{S})$ with the operators p and q is a solution to our problem. In [6] the following theorem was proved. Let $\mathscr{S}^{?}$ be a vector space with a scalar product $(\cdot,\cdot)$ and let P and Q be linear operators in $\mathscr{S}^{?}$, symmetric with respect to $(\cdot,\cdot)$ and satisfying (1.1). If $\mathscr{S}^{?}$ is complete in the topology determined by all semi-norms $(K\varphi,K\varphi)^{\frac{1}{2}}$, where $K \in R$, the algebra generated by P and Q, and if there exists an element $\psi_0 \in \mathscr{S}^{?}$, such that $P\psi_0 = iQ\psi_0$ and such that the set $\{R\psi_0\}$ is dense in $\mathscr{S}^{?}$, then $\mathscr{S}^{?}$ is a copy of $(\mathscr{S})$ in the precise sense explained in theorem 1 below.

Here we shall show that these assumptions can be weakened considerably. We shall prove:

THEOREM 1. *Let $\mathscr{W} \neq \{0\}$ be a vector space over the complex field satisfying the following conditions:*

1° *There is defined a scalar product $(\cdot,\cdot)$ on $\mathscr{W}$ with corresponding norm $\|\cdot\|$.*

2° *There are defined two linear operators P and Q mapping $\mathscr{W}$ into $\mathscr{W}$ such that*

$$(P\varphi,\psi) = (\varphi,P\psi), \qquad (Q\varphi,\psi) = (\varphi,Q\psi)$$

for all φ, $\psi \in \mathscr{W}$, and such that

$$PQ-QP = -iI ,$$

where I is the identity operator in $\mathscr{W}$.

3° *The operators P and Q have the property that*

$$(P \pm iI)\mathscr{W} = (Q \pm iI)\mathscr{W} = \mathscr{W} .$$

4° The space $\mathscr{W}$ is complete in the topology T determined by all semi-norms of the form $\|K\cdot\|$, $K \in R$, the algebra generated by P and Q.

5° If $\mathscr{W}_1 \neq \{0\}$ is a subspace of $\mathscr{W}$ closed in the topology T and invariant under the operators P, Q, $(P\pm iI)^{-1}$ and $(Q\pm iI)^{-1}$, then $\mathscr{W}_1=\mathscr{W}$. (The operators all exist by the symmetry of P and Q and by 3°.)

Then there exists a one-to-one, bicontinuous, linear mapping J of $\mathscr{W}$ onto $(\mathscr{S})$, such that J preserves the scalar product and such that

$$JP = pJ, \qquad JQ = qJ .$$

We remark that by embedding $\mathscr{W}$ in a Hilbert space $\mathfrak{H}$ we can of course get a theorem about operators P and Q in $\mathfrak{H}$. This theorem is somewhat weaker than the corresponding theorem proved by Foias, Gehér and Sz.-Nagy [2].

2. Proof of the theorem.

Let $\mathscr{W}$ be a vector space satisfying the conditions stated in theorem 1. The topology T of $\mathscr{W}$ is determined by a countably infinite set of semi-norms e.g. by all semi-norms of the form

$$\|P^\alpha Q^\beta \cdot\|, \qquad \alpha,\beta=0,1,\ldots, \tag{2.1}$$

or by all semi-norms of the form

$$\|Q^\alpha P^\beta \cdot\|, \qquad \alpha,\beta=0,1,\ldots. \tag{2.2}$$

The topological vector space $\mathscr{W}$ is therefore a complete, metrizable space.

Lemma 1. *If $t \neq 0$ is real and if*

$$(P-itI)\mathscr{W} = (Q-itI)\mathscr{W} = \mathscr{W}, \tag{2.3}$$

then $(P-itI)^{-1}$ and $(Q-itI)^{-1}$ exist and are continuous in $\mathscr{W}$.

Furthermore, if

$$F(x) = (x-is)^m(x-it)^{-n}, \qquad m,n=0,1,\ldots,$$

then

$$P^\alpha F(Q)-F(Q)P^\alpha = \sum_{j=1}^{\alpha}(-i)^j\binom{\alpha}{j}F^{(j)}(Q)P^{\alpha-j}, \tag{2.4}$$

$$Q^\alpha F(P)-F(P)Q^\alpha = \sum_{j=1}^{\alpha} i^j\binom{\alpha}{j}F^{(j)}(P)Q^{\alpha-j}. \tag{2.5}$$

Proof. The existence of the operators mentioned follows from (2.3) and the symmetry of P and Q. The formulas (2.4) and (2.5) follow for $n=0$ by induction from condition 2° in theorem 1 and for $n>0$ by first putting e.g.

$$\varphi = (Q-itI)^n\psi$$

and then using this to calculate the left hand side of (2.4).

From this the continuity follows immediately using the fact that e.g.

$$\|(P-itI)^{-1}\varphi\| \leqq |t|^{-1}\|\varphi\| . \tag{2.6}$$

LEMMA 2. *Condition* (2.3) *is satisfied for all real* $t \neq 0$ *and hence* $(P-itI)^{-1}$ *and* $(Q-itI)^{-1}$ *exist and are continuous in* $\mathscr{W}$.

PROOF. If (2.3) is satisfied for t, then the series

$$\sum_{n=0}^{\infty} i^n (s-t)^n (P-itI)^{-n-1}\varphi \tag{2.7}$$

converges in the topology T for all real s for which $|s-t| < |t|$ and for all $\varphi \in \mathscr{W}$.

This follows from (2.5) and (2.6) since by (2.5) we only need to prove that (2.7) converges in norm and that any series of the form

$$\sum_{n=0}^{\infty} (s-t)^n n(n+1)\ldots(n+j-1)(P-itI)^{-n-j-1}\psi$$

converges in norm for all $\psi \in \mathscr{W}$ and all fixed j. But this follows from (2.6).

If ψ is the sum of (2.7), then we have

$$\varphi = (P-isI)\psi ,$$

and hence (2.3) is satisfied with $t=s$.

Using condition 3° we get the lemma for $0<t<2$ and for $-2<t<0$, and then by "induction" for all $t \neq 0$.

Let $\mathfrak{H}$ denote the completion of $\mathscr{W}$ with respect to the scalar product; then $\mathfrak{H}$ is a Hilbert space, and by conditions 2° and 3° it follows that P and Q are essentially self-adjoint operators in $\mathfrak{H}$, i.e. that the closures $\bar{P}$ and $\bar{Q}$ are self-adjoint operators in $\mathfrak{H}$. Let $E_P(x)$ and $E_Q(x)$ denote the resolutions of the identity connected with $\bar{P}$ and $\bar{Q}$. Then we define the operators

$$U(s) = \exp is\bar{P} = \int_{-\infty}^{\infty} e^{isx}\, dE_P(x), \qquad s \text{ real} , \tag{2.8}$$

$$V(t) = \exp it\bar{Q} = \int_{-\infty}^{\infty} e^{itx}\, dE_Q(x), \qquad t \text{ real} . \tag{2.9}$$

It is well known that e.g. $U(s)$ form a strongly continuous group of unitary operators in $\mathfrak{H}$ and that an element f is in the domain of $\bar{P}$ if and only if

$$\frac{1}{is}(U(s)-I)f$$

has a limit as s tends to zero and that the limit is $\bar{P}f$.

Lemma 3. *For all real s and t the following statements hold:*

$$U(s)\mathscr{W} = V(t)\mathscr{W} = \mathscr{W} \tag{2.10}$$

and $U(s)$, $V(t)$ are continuous operators in $\mathscr{W}$,

$$U(s)V(t) = e^{ist}V(t)U(s); \tag{2.11}$$

when s tends to zero, then for all $\varphi \in \mathscr{W}$

$$\frac{1}{is}(U(s)-I)\varphi \to P\varphi,$$

$$\frac{1}{is}(V(s)-I)\varphi \to Q\varphi,$$

in the topology T.

Proof. The operators

$$F_n(s,P) = \left(I+\frac{is}{2n}P\right)^n\left(I-\frac{is}{2n}P\right)^{-n}$$

exist as continuous operators in $\mathscr{W}$ for all real s and all $n=0,1,\ldots$. We prove that for all $\varphi \in \mathscr{W}$ the sequence $\{F_n(s,P)\varphi\}$ converges in the topology T to $U(s)\varphi$. To this end we first have to prove that

$$\{Q^\alpha P^\beta F_n(s,P)\varphi\}$$

(cf. (2.2)) is a Cauchy sequence in norm. Using (2.5), a straightforward calculation shows that it is enough to prove that

$$\left\{\left(I+\frac{is}{2n}P\right)^{n-j}\left(I-\frac{is}{2n}P\right)^{-n-k}\psi\right\}$$

is a Cauchy sequence in norm for all $\psi \in \mathscr{W}$ and all fixed j and k. But this follows from the formula

$$\left\|U(s)\psi-\left(I+\frac{is}{2n}P\right)^{n-j}\left(I-\frac{is}{2n}P\right)^{-n-k}\psi\right\|^2$$
$$= \int_{-\infty}^{\infty}\left|e^{isx}-\left(1+\frac{is}{2n}x\right)^{n-j}\left(1-\frac{is}{2n}x\right)^{-n-k}\right|^2 d\|E_P(x)\psi\|^2$$

since this expression tends to zero as n becomes infinite. Hence the sequence $\{F_n(s,P)\varphi\}$ converges in the topology T, and since it converges in norm to $U(s)\varphi$, we have $U(s)\varphi \in \mathscr{W}$. The inverse $U(-s)$ also maps $\mathscr{W}$ into $\mathscr{W}$, and thus $U(s)$ maps $\mathscr{W}$ onto $\mathscr{W}$.

If $F(x)=(1-isx)^m(1-itx)^{-n}$, then

$$\begin{aligned} F(P)V(t') &= V(t')F(P+t'I), \\ F(Q)U(s') &= U(s')F(Q-s'I). \end{aligned} \tag{2.12}$$

For $n=0$ this is a consequence of the continuity of $F(P)$ and lemma 1. For $n>0$ we first remark that

$$\mathscr{W} = V(-t')(I-isP)V(t')\mathscr{W} = \big(I-is(P+t'I)\big)\mathscr{W}. \tag{2.13}$$

Thus $\big(I-is(P+t'I)\big)^{-1}$ and also $F(P+t'I)$ exist as continuous operators in $\mathscr{W}$. Then (2.12) follows easily. Now the continuity of $U(s)$ and $V(s)$ is obvious.

By (2.13) and trivial calculations we see that the pair $P'=P+tI$ and Q satisfies all the conditions of theorem 1 and hence for all $\varphi \in \mathscr{W}$

$$F_n(s,P+tI)\varphi \to e^{its}U(s)\varphi$$

in the topology T. From this (2.11) follows using (2.12) and the continuity of $V(t)$.

The last part of the lemma follows from (2.12), (2.4), and the remarks preceding lemma 3.

Remark. Since $\mathscr{W}$ is dense in $\mathfrak{H}$, the relation (2.11) is valid when we consider $U(s)$ and $V(t)$ as operators in $\mathfrak{H}$. Therefore we have a representation of the commutation relation in $\mathfrak{H}$ in the sense of Weyl [10]. In [7] von Neumann found all representations of the commutation relation in the sense of Weyl. We shall follow von Neumann's exposition in some detail.

If $a(s,t)$ is Lebesgue integrable in the plane, then

$$A = \iint a(s,t)\exp(-\tfrac{1}{2}ist)\, U(s)V(t)\, ds dt \tag{2.14}$$

is a bounded operator in the Hilbert space $\mathfrak{H}$ defined by the formula

$$(Af,g) = \iint a(s,t)\exp(-\tfrac{1}{2}ist)\big(U(s)V(t)f,g\big)\, ds dt.$$

Here $A=0$ if and only if $a(s,t)=0$ almost everywhere, and if $a(s,t)$ is real, then A is self-adjoint. (For the details we refer the reader to [7].)

LEMMA 4. *If $a(s,t)$ is continuous and if*

$$\sup_{s,t} |s^\alpha t^\beta a(s,t)| < \infty, \qquad \alpha,\beta = 0,1,\ldots\,,$$

then the operator A defined by (2.14) maps $\mathscr{W}$ into $\mathscr{W}$.

PROOF. Let K be any closed unit square in the plane with sides parallel to the coordinate axes. We divide K into squares of side length 2^{-n}. Let (s_j,t_j), $j=1,\ldots,2^{2n}$, be the midpoints of these squares.

If $b(s,t)$ is continuous in K, then it follows from the strong continuity of $U(s)$ and $V(t)$ that

$$B(K,n)\varphi = \sum_{j=1}^{2^{2n}} 2^{-2n} b(s_j,t_j) U(s_j) V(t_j)\varphi$$

converges in norm to $B(K)\varphi$, say, as n tends to infinity. In view of (2.12) it is obvious that it also converges in the topology T. A short calculation shows furthermore that we have an estimate of the form

$$\|P^\alpha Q^\beta B(K)\varphi\| \leqq \varepsilon(K) \iint_K (1+s^2+t^2)^{-2}\, ds\, dt\,,$$

where

$$\varepsilon(K) = \sum_{\gamma,\delta} \sup_K |p_{\gamma\delta}(s,t) b(s,t)|\, \|P^\gamma Q^\delta \varphi\|\,,$$

the sum being finite and the $p_{\gamma\delta}$ polynomials.

If $b(s,t)$ satisfies the conditions of the lemma, then $\varepsilon(K)$ tends to zero if K is "moved out to infinity". If we divide the plane into unit squares $K_1,K_2,\ldots$, then $\varepsilon(K_n)$ tends to zero as n tends to infinity, and therefore, for all $\varphi \in \mathscr{W}$, the series

$$B\varphi = \sum_{n=1}^{\infty} B(K_n)\varphi$$

converges in the topology T and hence determines an operator B mapping $\mathscr{W}$ into $\mathscr{W}$.

Let A' be determined in this way from

$$b(s,t) = a(s,t)\exp(-\tfrac{1}{2}ist)\,.$$

Then by construction $(A'\varphi,\psi) = (A\varphi,\psi)$ for all $\varphi,\psi \in \mathscr{W}$ such that $A = A'$. This proves the lemma.

Still following von Neumann we define A by (2.14) using the function

$$a(s,t) = (2\pi)^{-1}\exp\bigl(-\tfrac{1}{4}(s^2+t^2)\bigr)\,.$$

Then A is non-zero, self-adjoint and maps $\mathscr{W}$ into $\mathscr{W}$. Furthermore, by [7] we have

(2.15) $$A \exp(-\tfrac{1}{2}ist)\, U(s)\, V(t)\, A \;=\; \exp\big(-\tfrac{1}{4}(s^2+t^2)\big)\, A\,,$$

and hence A is an orthogonal projection in $\mathfrak{H}$. Let $\mathfrak{H}_A$ denote the range of A in $\mathfrak{H}$. Then $\mathfrak{H}_A \cap \mathscr{W}$ is dense in $\mathfrak{H}_A$, since $f \in \mathfrak{H}_A$, f orthogonal to $\mathfrak{H}_A \cap \mathscr{W}$ imply that

$$0 \;=\; (f, A\varphi) \;=\; (f, \varphi)$$

for all $\varphi \in \mathscr{W}$. Thus $\mathfrak{H}_A \cap \mathscr{W} \neq \{0\}$.

Let $\varphi \in \mathfrak{H}_A \cap \mathscr{W}$. Then using (2.15) and

$$(P-iQ)\varphi \;=\; \lim_{s\to 0}\left[\frac{1}{is}(U(s)-I)\varphi - \frac{1}{s}(V(s)-I)\varphi\right]$$

one finds that

(2.16) $$P\varphi \;=\; iQ\varphi\,.$$

We have proved

LEMMA 5. *There exists a non-zero element $\varphi \in \mathscr{W}$ such that $P\varphi = iQ\varphi$.*

Let ψ_0 be a normalized solution of (2.16), and let the operators B and B^* be defined by

$$B \;=\; \frac{1}{2^{\frac{1}{2}}}(P-iQ), \quad B^* \;=\; \frac{1}{2^{\frac{1}{2}}}(P+iQ)\,.$$

Then

(2.17) $$BB^* - B^*B \;=\; I\,.$$

From (2.17) it follows that the sequence

(2.18) $$\psi_n \;=\; \frac{1}{(n!)^{\frac{1}{2}}} B^{*n}\psi_0, \qquad n \;=\; 0,1,\ldots\,,$$

is orthonormal in $\mathscr{W}$. Furthermore we have

(2.19) $$B^*\psi_n \;=\; (n+1)^{\frac{1}{2}}\psi_{n+1}\,,$$

(2.20) $$B\psi_n \;=\; n^{\frac{1}{2}}\psi_{n-1}\,,$$

the latter expression being interpreted as zero if $n=0$.

Let $\mathscr{W}_0$ denote the smallest closed subspace of $\mathscr{W}$ containing all ψ_n and let $\tilde{\mathscr{W}}_0$ denote the linear span of the ψ_n's. From (2.19) and (2.20) it follows that $\tilde{\mathscr{W}}_0$ and $\mathscr{W}_0$ are invariant under B and B^* and hence under P and Q. If $\varphi \in \tilde{\mathscr{W}}_0$, then it can be written uniquely as

(2.21) $$\varphi \;=\; \sum_{n=0}^{\infty} x_n \psi_n\,,$$

where the sum is only formally infinite. If K is any polynomial in P and Q, then it can be written as a polynomial in B and B^*. Using this remark, (2.19) and (2.20), it is easily seen that if we topologize $\tilde{\mathscr{W}}_0$ by the norms

$$\|\varphi\|_r^2 = \big(\varphi,(BB^*)^r\varphi\big) = \sum_{n=0}^{\infty}(n+1)^r|x_n|^2, \qquad r = 0,1,\ldots, \tag{2.22}$$

then this topology in $\tilde{\mathscr{W}}_0$ coincides with the topology induced by the topology T of $\mathscr{W}$.

Since $\mathscr{W}_0$ is the topological closure of $\tilde{\mathscr{W}}_0$, we get that $\mathscr{W}_0$ is the space of all those $\varphi \in \mathscr{W}$ which can be written as an infinite series (2.21) where all the norms defined by (2.22) are finite. The series (2.21) then converges to φ in the topology T.

We now turn to the space $(\mathscr{S})$ of definition 1. It is easily verified that the topology of $(\mathscr{S})$ also can be determined by the semi-norms $\|k\cdot\|$, where $\|\cdot\|$ denotes the L^2-norm and where $k \in R$, the algebra generated by p and q.

Let $\tilde{\psi}_0$ be the function

$$\tilde{\psi}_0(t) = \pi^{-\frac14}\exp(-\tfrac12 t^2),$$

and let the operators b and b^* be defined by

$$b = \frac{1}{2^{\frac12}}(p-iq), \qquad b^* = \frac{1}{2^{\frac12}}(p+iq).$$

Then the functions

$$\tilde{\psi}_n = \frac{1}{(n!)^{\frac12}}\,b^{*n}\tilde{\psi}_0, \qquad n = 0,1,\ldots, \tag{2.23}$$

are the normalized Hermite functions. These functions are all in $(\mathscr{S})$ and they form a complete orthonormal system in L^2.

By exactly the same reasoning as above and since $(\mathscr{S})$ is a complete space, we see that $(\mathscr{S})$ contains all functions $\tilde{\varphi} \in L^2$ whose Fourier development with respect to the Hermite functions

$$\tilde{\varphi} = \sum_{n=0}^{\infty} x_n\tilde{\psi}_n \tag{2.24}$$

has the property that

$$\|\varphi\|_r^2 = \sum_{n=0}^{\infty}(n+1)^r|x_n|^2 < \infty \tag{2.25}$$

for all $r=0,1,\ldots$.

On the other hand, since $\tilde{\varphi} \in (\mathscr{S})$ implies that $(bb^*)^r\tilde{\varphi}$ exists and is in L^2, we have

$$\big((bb^*)^r\tilde{\varphi},\tilde{\psi}_n\big) = \big(\tilde{\varphi},(bb^*)^r\tilde{\psi}_n\big) = (n+1)^r x_n,$$

and we see that the Fourier development (2.24) of $\tilde{\varphi}$ satisfies (2.25).

The following lemma is now obvious.

LEMMA 6. *There exists a one-to-one, bicontinuous, linear mapping J of the subspace $\mathscr{W}_0$ of $\mathscr{W}$ onto $(\mathscr{S})$, such that J preserves the scalar product, $J\psi_0 = \tilde{\psi}_0$, $JP = pJ$ and $JQ = qJ$.*

To prove theorem 1 we now only have to prove that $(\mathscr{S})$ and hence $\mathscr{W}_0$ satisfies the conditions of theorem 1. But this is almost obvious since 1°, 2°, 3° and 4° are satisfied for $(\mathscr{S})$ and we have proved that any space satisfying these conditions contains a copy of $(\mathscr{S})$. Note that until now we have not used 5° at all.

This finishes the proof of theorem 1.

3. Concluding remarks.

For a mechanical system with a finite number f of degrees of freedom the situation is completely analogous to the case studied above. Here one is led to study f pairs of operators $P_1, Q_1, \ldots, P_f, Q_f$ defined on a vector space $\mathscr{W}_f$ with a scalar product, such that the operators are symmetric with respect to the scalar product and satisfy the relations

$$P_m Q_n - Q_n P_m = -i\delta_{mn} I ,$$
$$P_m P_n = P_n P_m, \qquad Q_m Q_n = Q_n Q_m .$$

In this situation the obvious generalization of theorem 1 is true, and the proof is also the obvious generalization of the proof given above.

For the case $f = \infty$ — the case pertinent for quantum field theory — the situation is radically changed. For, although we can prove (by the same reasoning as given in the proof of theorem 1) that for any finite n the system of equations

$$P_m \varphi = iQ_m \varphi, \qquad m = 1, \ldots, n ,$$

has a non-trivial solution, we cannot prove that this holds for the infinite system of equations. Indeed by use of the results of Gårding and Wightman [4] one can construct counterexamples.

Assuming as an extra axiom the existence of a solution of the infinite system of equations, one can again show existence and uniqueness of space and operators. Sadly enough it turns out that the resulting space is singularly uninteresting — it does not have the simplicity of the Hilbert space and on the other hand it is still so large that its dual space is of no use. Therefore we shall not give this analysis.

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