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ON PERMUTATIONS INDUCED BY COMMUTING FUNCTIONS, AND AN EMBEDDING QUESTION

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1. Introduction.

Let $f \equiv f(x)$ and $g \equiv g(x)$ be two continuous and commuting functions (under substitution) each mapping the closed interval $[a,b]$ into itself. Isbell [2] has conjectured that f and g must have a common fixed point, or equivalently, that f and the composite function $h = fg \equiv f(g(x))$ must have a common fixed point. Except in special cases the conjecture has not been verified.

One interesting special case of the conjecture was investigated a number of years ago by Ritt [4]. He proved that if f and g are polynomials which do not belong to a certain class (f and g do not come from the multiplication theorems of e^z and $\cos z$, cf. [3] for definition), then (neglecting a linear transformation) they are both iterates of a third polynomial p. Thus, they would have as common fixed points the fixed points of p. Also in this case, f and g would be members of a semi-group of commuting functions formed from the iterates of p.

Among the commuting polynomials excluded from the Ritt theorem are the Tchebycheff polynomials defined by $T_n(x) = \cos(n \arccos x)$ for $-1 \leqq x \leqq 1$. It appears that even in this case the polynomials could be embedded in a one-parameter semi-group of commuting functions defined by

$$(1) \qquad f_t(x) = T_\alpha(x) = \cos(\alpha \arccos x), \qquad \alpha = \exp(t).$$

This suggests that one method for attacking the Isbell conjecture is to try to embed the commuting functions in a semi-group and thereby hope to prove the existence of the common fixed point.

Let us first observe that there is difficulty in defining a one-parameter

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semi-group of commuting functions by means of formula (1). Using half-angle formulas, the function $T_{3/2}$ can be shown to satisfy

$$[T_{3/2}(x)]^2 = \tfrac{1}{2}(4x^3-3x+1) \tag{2}$$

and that the commutativity relations

$$T_4(T_{3/2}(x)) = T_6(x), \qquad [T_{3/2}(T_4(x))]^2 = [T_6(x)]^2 \tag{3}$$

are valid. On the other hand, it is difficult to see how one would define the square root in (2) to determine $T_{3/2}$ so that $T_{3/2}$ and T_4 would commute. We shall show below that this difficulty is really fundamental, and that in fact we cannot embed the Tchebycheff polynomials in the semi-group, as suggested. We will give a simple necessary condition for when it is possible to embed f and g in a "nice" semi-group of commuting and continuous functions. This condition is not satisfied by the Tchebycheff polynomials.

The key to our result lies in an investigation of what f and g do to the fixed points of the composite function h. It is easily seen that the restrictions of f and g to the set H of fixed points of h are permutations of H which are inverses of each other. However, in case the set H is finite, and we shall assume this throughout, much more can be said. If we let $H=\{x_1,x_2,\ldots,x_n\}$ with $x_i<x_{i+1}$, then each of the intervals $[x_i,x_{i+1}]$ can be classified as an *up-interval* or *down-interval* according as $h(x)>x$ or $h(x)<x$, for $x_i<x<x_{i+1}$, respectively. The classification of the intervals between the successive points in H induces a classification of the points in H. The point x_i will be called:

(a) an *up-crossing* if $I_{i-1}=[x_{i-1},x_i]$ and $I_i=[x_i,x_{i+1}]$ are down and up-intervals, respectively,

(b) a *down-crossing* if I_{i-1} and I_i are up and down-intervals, respectively,

(c) a *touching* if I_{i-1} and I_i are intervals of the same type,

and

(d) the point x_1 is a down-crossing or touching according as I_1 is a down or up-interval, respectively,

(e) the point x_n is a down-crossing or touching according as I_{n-1} is an up or down-interval, respectively.

Now, let σ be either of the (inverse) permutations σ_f and σ_g of H induced by the functions f and g. For convenience, we shall write $x_{\sigma(i)}$ for $\sigma(x_i)$, that is, we write σ as a permutation of the index set $\{1,2,\ldots,n\}$ rather than the set H. In a recent paper by one of us [1], it was shown that σ must satisfy the following three properties.

(α) σ preserves each of the three classes of points.

(β) Suppose $\sigma(i)$ and $\sigma(i+1)$ are consecutive integers. Then, for $\sigma(i+1)=\sigma(i)+1$, the intervals $[x_i, x_{i+1}]$ and $[x_{\sigma(i)}, x_{\sigma(i+1)}]$ are of the same type. For $\sigma(i)=\sigma(i+1)+1$, the intervals $[x_i, x_{i+1}]$ and $[x_{\sigma(i+1)}, x_{\sigma(i)}]$ are of opposite type.

(γ) Suppose $\sigma(i)$ and $\sigma(i+1)$ are not consecutive integers. Then, for each $v=\sigma(j)$ between $\sigma(i)$ and $\sigma(i+1)$, $j=\sigma^{-1}(v)>i+1$ or $j=\sigma^{-1}(v)<i$ according as $[x_i, x_{i+1}]$ is an up or down-interval, respectively.

It is clear that for every $k \geqq 2$, the permutation σ^k will satisfy property (α). However, we cannot expect in general that σ^k, $k \geqq 2$, will satisfy properties (β) and (γ). If each of the permutations $\sigma, \sigma^2, \ldots, \sigma^{m-1}, \sigma^m = I$ (the identity) satisfy properties (α), (β) and (γ), then we shall say that σ *generates an admissible sequence of permutations of* H. We are now in a position to state a special case of one of our main theorems.

THEOREM 1$'$. *Let f and g be elements of a semi-group $\{f_t, t \in T\}$ of commuting and continuous functions on $[a,b]$ whose parameter set T consists of the non-negative elements of an additive subgroup of the real numbers. Let the set H of fixed points of the composite function $h=fg$ be finite, and let σ_f be the permutation of H induced by f. Then, σ_f generates an admissible sequence of permutations of H.*

Let us return for the moment to the Tchebycheff polynomials discussed above. If, for example, $f(x)=T_3(x)=4x^3-3x$ and $g(x)=T_2(x)=2x^2-1$ could be embedded in a semi-group of the type considered, then the permutation σ_f of the six fixed points of the composite function $h=T_6$ would generate an admissible sequence. A computation shows that $\sigma_f=(135)(24)(6)$ and $\sigma_f{}^2=(153)(2)(4)(6)$, and we observe that $\sigma_f{}^2$ does not satisfy property (γ). As the proof of our theorem will show, the permutation $\sigma_f{}^2$ would be the one associated with the hypothetical function $T_{3/2}$ mentioned in (2) and (3). Thus, no such function exists and there is no „nice" semi-group of commuting and continuous functions containing T_2 and T_3.

Regarding Isbell's conjecture, Theorem 1$'$ gives only negative information concerning one possible method of attack. Moreover, if σ_f generates an admissible sequence, Theorem 1$'$ gives no information about the conjecture even though there may exist a semi-group containing f and g. However, we can prove Isbell's conjecture directly in the case that σ_f generates an admissible sequence of permutations. We now state a special case of the second of our main theorems.

THEOREM 2′. *Let f and g be a pair of commuting and continuous functions on $[a,b]$ such that the set H of fixed points of the composite function $h=fg$ is finite, and let σ_f generate an admissible sequence of permutations of H. Then, σ_f has a fixed point, that is, f and h have a common fixed point.*

Finally, on the basis of the Ritt theorem, our discussion of the Tchebycheff polynomials, and Theorem 2′ above, we would like to make a conjecture. Let us say that the two commuting and continuous functions f and g on $[a,b]$ are irreducible if there is no proper sub-interval I of $[a,b]$ such that $f(I)\subset I$ and $g(I)\subset I$. Then, we conjecture that if the composite function $h=fg$ has only a finite number of fixed points and if f and g are irreducible functions for which σ_f does *not* generate an admissible sequence of permutations of H, f and g have a common fixed point at either a or b.

The proofs of our main theorems are contained in sections 3 and 4. In section 5 we show that if σ is any permutation satisfying properties (α), (β) and (γ), then its inverse will also, giving a nice symmetry to the statements and conditions of the theorems. Appropriate notation and terminology is given in section 2.

2. Notation and terminology.

For the sake of simplifying proofs, and at the same time, proving more general results, we wish to abstract the essential properties of the set H and the permutation σ_f. We do this in three stages. It shall be assumed throughout that all functions involved are defined on the interval $[a,b]$, take values in $[a,b]$, and are continuous.

DEFINITION 1. Let H be a subset of $[a,b]$ (finite or infinite). Then, a permutation σ of H will be called *φ-admissible* if there exists a pair of commuting functions f and g so that H is the set of fixed points of the composite function $h=fg$ and the permutation σ_f of H which is induced by f is exactly σ.

It is clear that the inverse of any φ-admissible permutation will be φ-admissible. We turn next to the finite case.

DEFINITION 2. Let $H=\{x_1,x_2,\ldots,x_n\}$, $x_i<x_{i+1}$, be a finite set with each interval $[x_i,x_{i+1}]$ specified (arbitrarily) as an up-interval or down-interval. Let the points in H be classified according to the rules (a)–(e) given in the introduction. Then, H with the specifications given to the intervals and points will be called an *s-set*. A permutation σ of an s-set H will be called *s-admissible* if it satisfies properties (α), (β) and (γ) given in the introduction.

Using this terminology, we remark that [1] was concerned with showing that any φ-admissible permutation of an s-set H is also s-admissible if the specification of the intervals in H coincides with that which is induced by the composite function h. It is also interesting to note that in the proof given in [1] that σ_f has properties (β) and (γ), the fact that σ_f and $\sigma_f^{-1}=\sigma_g$ came from functions f and g was utilized in an essential manner. However, the proof that σ_f satisfies (α) was based only on the fact that σ_f and σ_f^{-1} satisfied properties (β) and (γ). We shall show, in section 5, that the inverse of any s-admissible permutation is also s-admissible.

In the second of our main results, we shall be concerned only with the up and down-crossings in H and shall ignore the touchings. With this in mind we make the following definition.

DEFINITION 3. Let $H=\{x_1,x_2,\ldots,x_n\}$, $x_i<x_{i+1}$, be a finite set with n odd. Let the intervals be specified as alternately down and up with $[x_1,x_2]$ a down-interval, and let the points be classified as before (they will be alternately down-crossings and up-crossings with x_1 and x_n down-crossings). Then, H with the specifications given the intervals and points will be called a *w-set*. A permutation σ of a w-set H will be called *w-admissible* if it statisfies properties (α), (β), and (γ).

We remark that if σ is an s-admissible permutation of an s-set H, and H' is the w-set of up and down crossings in H, then the restriction σ' of σ to H' will be w-admissible.

3. The first main theorem.

THEOREM 1. *Let f and g be elements of a semi-group $\{f_t, t\in T\}$ of commuting and continuous functions on $[a,b]$ whose parameter set T consists of the non-negative elements of an additive subgroup of the real numbers. Let H be the set of fixed points of the composite function $h=fg$, and let σ_f be the permutation of H induced by f. Then, for every $k=1,2,3,\ldots$, σ_f^k is φ-admissible.*

This theorem is an immediate consequence of the following lemma and the fact that the permutation of H induced by the function f^k, the k-fold iterate of f, is exactly σ_f^k.

LEMMA 1. *Let $\{f_t, t\in T\}$ be as in the statement of the theorem above. Let t_0 be a fixed element in T, $t_0\neq 0$, and let H be the set of fixed points of the function f_{t_0}. Then, for every t in T, the restriction of the function f_t to H is a φ-admissible permutation of H.*

Proof. If $t \leqq t_0$, then $s = t_0 - t$ is in T, and we have $f_s f_t = f_{s+t} = f_{t_0}$ as desired. If $t > t_0$, then there exists a positive integer k such that $t = kt_0 + s$, where $0 \leqq s < t_0$. Hence, we have $f_t = f_s(f_{t_0})^k$, so that f_t coincides with f_s on H. But, by the previous case, the permutation of H induced by f_s is φ-admissible since $s < t_0$.

4. The second main theorem.

Theorem 2. *Let H be a w-set, and let σ be a permutation of H such that each of $\sigma, \sigma^2, \ldots, \sigma^{m-1}, \sigma^m = I$ (the identity) is w-admissible. Then, σ has a fixed point.*

Proof. The proof will be by induction on the number of points in H. The theorem is obviously true for $n = 1$. For convenience we assume, without loss of generality, that $x_i = i$. By the period of a point i, we shall mean the length of the cycle to which point i belongs when σ is written as a product of disjoint cycles.

Step 1. If point 2 has period p, then point 1 has period either p or $2p$.

We note first that for any k, $\sigma^k(1)$ and $\sigma^k(2)$ must be successive points in H. This follows since $[1,2]$ is a down-interval and property (γ) of σ^k requires that $j < 1$ whenever $\sigma^k(j)$ is between $\sigma^k(1)$ and $\sigma^k(2)$. In particular, if $\sigma^k(1) = 1$, then it necessarily follows that $\sigma^k(2) = 2$. Thus, the period of point 2 must be a divisor of the period of point 1. On the other hand, if p is the period of point 2 so that $\sigma^p(2) = 2$, then we must have either $\sigma^p(1) = 1$ or $\sigma^p(1) = 3$. If $\sigma^p(1) = 1$, then the periods of point 1 and point 2 each divide the other and hence, must be equal. If $\sigma^p(1) = 3$, then we must have $\sigma^p(3) = 1$. This follows since $[2,3]$ is an up-interval and if $\sigma^p(3) \geqq 5$, then we would have $\sigma^p(1) = 3$ between $\sigma^p(2)$ and $\sigma^p(3)$, and by property (γ) of σ^p, we would have $1 > 3$. But, if $\sigma^p(1) = 3$ and $\sigma^p(3) = 1$, then $\sigma^{2p}(1) = 1$, that is, the period of point 1 is $2p$. We note that if $p = 1$, our proof is complete.

Step 2. Suppose points 1 and 2 each have period $p > 1$. Let

$$H' = H - \{1, \sigma(1), \sigma^2(1), \ldots, \sigma^{p-1}(1), 2, \sigma(2) \sigma^2(2), \ldots, \sigma^{p-1}(2)\}$$

and let σ' be the restriction of σ to H'. Then, for each $k = 1, 2, \ldots$, $(\sigma')^k$ is a w-admissible permutation of the w-set formed from H'. It will follow by our induction hypothesis that σ', and hence σ, has a fixed point in H'. It is clear that σ', and hence $(\sigma')^k$, is a permutation of H'. We shall show that σ' is w-admissible, the same proof being valid for each $(\sigma')^k$ since $(\sigma')^k$ is simply the restriction of σ^k to H'.

We note first that to show w-admissibility of σ', we need only check properties (α) and (γ), property (β) being an immediate consequence of

(α) on a w-set. Now, under the present assumptions, if we let A_0 be the pair $\{1,2\}$ and A_j be the pair of successive points $\{\sigma^j(1),\sigma^j(2)\}$, then we have determined for us p mutually disjoint pairs $A_0,A_1,\ldots,A_{p-1}$, and the union of the sets A_j is precisely the set of elements in the cycles to which points 1 and 2 belong. Thus, if we write $H'=\{i_1,i_2,\ldots,i_{n-2p}\}$, $(i_s<i_{s+1})$, then i_1 is odd and successive elements are of opposite parity, that is, i_s and s have the same parity. Thus, in classifying the points and intervals in H' so that H' becomes a w-set, the points in H' retain the classification they originally had in H. It follows that σ' has property (α). As for property (γ), we note first that if $i_{r+1}=i_r+1$, then the interval $[i_r,i_{r+1}]$ is an interval in both H and H', and it retains in H' the classification it had in H. It follows that if $\sigma'(u)$ is between $\sigma'(r)$ and $\sigma'(r+1)$, that is, if $\sigma(i_u)$ is between $\sigma(i_r)$ and $\sigma(i_{r+1})$, then u will satisfy the appropriate inequality with respect to r and $r+1$. Suppose now that $i_{r+1}=i_r+2s+1$ with $s>0$. We note first that for each t, $0\leqq t<s$, the pair of points $\{i_r+2t+1,i_r+2t+2\}$ is one of the pairs A_j and that each of the corresponding intervals $[i_r+2t+1,i_r+2t+2]$ in H have the same classification, and for the sake of argument, let us say down. It follows that each of the intervals $[i_r+2t,i_r+2t+1]$, $0\leqq t\leqq s$, are up-intervals in H. We note that in this case $[i_r,i_{r+1}]$ is also an up-interval (in H') since this is determined by the parity of r which is the same as the parity of i_r. Suppose now that $\sigma'(u)$ is between $\sigma'(r)$ and $\sigma'(r+1)$ so that $\sigma(i_u)$ is between $\sigma(i_r)$ and $\sigma(i_{r+1})$. To satisfy property (γ), we need to show that $u>r+1$, or equivalently, that $i_u>i_{r+1}$. Since for $0\leqq t<s$, the pair $\{i_r+2t+1,i_r+2t+2\}$ is one of the pairs A_j, it follows that the pair $\{\sigma(i_r+2t+1),\ \sigma(i_r+2t+2)\}$ is also, that is, $\sigma(i_r+2t+1)$ and $\sigma(i_r+2t+2)$ are successive points in H. Thus, it follows that $\sigma(i_u)$ must be between $\sigma(i_r+2t)$ and $\sigma(i_r+2t+1)$ for some t, $0\leqq t\leqq s$. Since for each such t, $[i_r+2t,i_r+2t+1]$ is an up-interval and σ is w-admissible, it follows that $i_u>i_r+2t+1$. But, i_u is in H' with $i_u\neq i_{r+1}$, and i_r and i_{r+1} are successive points in H'. Thus, we must have $i_u>i_{r+1}$ as desired.

Step 3. Suppose point 2 has period $p>1$ and point 1 has period $2p$. Then, the permutation σ may be modified so as to satisfy the conditions considered in Step 2.

We recall that for any k, $\sigma^k(1)$ and $\sigma^k(2)$ are successive points in H. We note that since $\sigma^p(2)=2$ and $\sigma^p(1)=3$, it follows that $\sigma^k(2)=\sigma^{k+p}(2)$ and $\sigma^k(3)=\sigma^{k+p}(1)$, that is, $\sigma^k(2)$ and $\sigma^k(3)$ are also successive points in H. This implies that $\sigma^k(1)$, $\sigma^k(2)$ and $\sigma^k(3)$ form a successive triple of points in H with $\sigma^k(2)$ (an up-crossing) between $\sigma^k(1)$ and $\sigma^k(3)$ (down-crossings). Let B_0 be the triple $\{1,2,3\}$ and let B_j be the triple $\{\sigma^j(1),\sigma^j(2),\sigma^j(3)\}$. We note that the *non-ordered* sets $B_0,B_1,\ldots,B_{p-1}$ are mutually disjoint, and that $B_j = B_{j'}$ when $j \equiv j' \pmod{p}$. If we write the permutation σ^k in the form

$$\sigma^k\colon \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ i_1 & i_2 & i_3 & \dots & i_n \end{pmatrix}, \tag{4}$$

where $i_s = \sigma^k(s)$, then each of the triples B_j appears in each row with its elements successive (in one of the two possible orders). We note that we may take any number of these triples in the second row and reverse the order of the elements within these triples, and the resulting permutation will still be w-admissible. In particular, if we let $(\sigma^k)^{(j)}$ be the permutation σ^k with the orders of the elements within the triples $B_0, B_1, \dots, B_{j-1}$ reversed in the second row, when σ^k is written in the form (4), then $(\sigma^k)^{(j)}$ is w-admissible. Note that if $j \geqq p$, then each triple may be reversed several times, possibly restoring the original order. Now let $\sigma' = \sigma^{(1)}$. Then, it can be seen that $(\sigma')^k = (\sigma^k)^{(k)}$. It follows that for each k, $(\sigma')^k$ is a w-admissible permutation of H. But the points 1 and 2 each have period p with respect to σ', and it follows from step 2 that σ' has a fixed point. This fixed point cannot be an element in any of the cycles to which the points 1, 2 and 3 belong since these are each points of period $p > 1$. But σ and σ' agree on the complimentary set (which is necessarily non-empty since any two of the triples B_j are separated in H by at least one up-crossing). Hence, σ has a fixed point.

5. A useful lemma.

We shall show here that the inverse of an s-admissible permutation of an s-set H is also s-admissible. As a special case, it follows that the inverse of a w-admissible permutation of a w-set H is also w-admissible. We assume, as before, that $H = \{1, 2, \dots, n\}$. Before proceeding to the main lemma, we prove two sub-lemmas.

SUB-LEMMA 1. *Let σ be an s-admissible permutation of an s-set H and suppose $\sigma(r)$ and $\sigma(s)$ $(r < s)$ are consecutive points in H. Then, there cannot be a pair of points t and u between r and s such that $\sigma(t) < \sigma(r)$, $\sigma(s)$ and $\sigma(u) > \sigma(r)$, $\sigma(s)$.*

PROOF. If we assume the contrary, then there is a pair of consecutive points t and $t+1$ with $r < t < t+1 < s$, such that $\sigma(t)$ and $\sigma(t+1)$ are on opposite sides of $\sigma(r)$ and $\sigma(s)$, say $\sigma(t) < \sigma(r)$, $\sigma(s) < \sigma(t+1)$. But then, by property (γ), if $[t, t+1]$ is an up-interval, we must have $r > t+1$, or if $[t, t+1]$ is a down-interval, we must have $s < t$, neither of which is possible.

SUB-LEMMA 2. *Let σ be an s-admissible permutation of an s-set H and suppose the point r is an up-crossing (then $r \neq 1, n$). Then, one of $\sigma(r-1) < \sigma(r) < \sigma(r+1)$ or $\sigma(r-1) > \sigma(r) > \sigma(r+1)$ must hold.*

PROOF. The proof of this sub-lemma was given also in [1]. However, we show here that it depends only on the permutation σ satisfying property (γ). Thus, suppose $\sigma(r-1) > \sigma(r)$ and $\sigma(r+1) > \sigma(r)$. Then, either $\sigma(r-1) > \sigma(r+1) > \sigma(r)$, or $\sigma(r+1) > \sigma(r-1) > \sigma(r)$. If the former holds, then property (γ) implies that $r+1 < r-1$ since $[r-1, r]$ is a down-interval. If the latter holds, it again follows that $r+1 < r-1$ since $[r, r+1]$ is an up-interval. A similar argument shows that we cannot have both $\sigma(r-1) < \sigma(r)$ and $\sigma(r+1) < \sigma(r)$.

We proceed now to the main lemma.

LEMMA 2. *Let σ be an s-admissible permutation of an s-set H. Then, σ^{-1} is also an s-admissible permutation of H.*

PROOF. It is obvious that σ^{-1} satisfies properties (α) and (β). Thus, the burden of the proof is to show that σ^{-1} satisfies property (γ) on each of the intervals $[i, i+1]$. We do this by considering certain possible specifications of the interval $[i, i+1]$ and the allowable specifications of the points i and $i+1$.

Case 1. *$[i, i+1]$ an up-interval and point i an up-crossing.* Suppose σ^{-1} violates property (γ) on the interval $[i, i+1]$. Then, there exists a point $t = \sigma^{-1}(j)$ between $r = \sigma^{-1}(i)$ and $s = \sigma^{-1}(i+1)$ with $j = \sigma(t) < i$. Assume $r < t < s$. Since $\sigma(r) = i$ and $\sigma(s) = i+1$ are consecutive points in H and there exists a point t between r and s with $\sigma(t) < \sigma(r) < \sigma(s)$, it follows by Sub-Lemma 1 that $\sigma(r+1) < \sigma(r)$. Also, since point i is an up-crossing, it follows that $\sigma^{-1}(i) = r$ is an up-crossing. In view of the inequality $\sigma(r+1) < \sigma(r)$ and Sub-Lemma 2, it follows that $\sigma(r+1) < \sigma(r) < \sigma(r-1)$. Also, $\sigma(s) = \sigma(r) + 1$ so that $\sigma(r) < \sigma(s) < \sigma(r-1)$. But point r being an up-crossing, it follows that $[r-1, r]$ is a down-interval. Thus, property(γ) of σ applied to the interval $[r-1, r]$ yields $s < r-1$, a contradiction. Now assume $s < t < r$. In this case, it follows from Sub-Lemma 1 that $\sigma(r-1) < \sigma(r)$ and by Sub-Lemma 2 that $\sigma(r-1) < \sigma(r) < \sigma(r+1)$, and since $\sigma(s) = \sigma(r) + 1$, we have $\sigma(r) < \sigma(s) < \sigma(r+1)$. But now, $[r, r+1]$ is an up-interval and property (γ) implies $s > r+1$, a contradiction.

Case 1′. *$[i, i+1]$ a down-interval and point $i+1$ an up-crossing.* An argument similar to that given in Case 1 will hold here.

Case 2. *$[i, i+1]$ an up-interval and point i a touching.* We shall show below that if σ^{-1} violates property (γ) on the interval $[i, i+1]$, then it will do so also on the interval $[i-1, i]$, where there will *necessarily exist* a point $i-1$ in H. But point i being a touching, it follows that $[i-1,i]$ is again an up-interval. In a finite number of repetitions of our argument, we would conclude that σ^{-1} must have violated property (γ) on an interval of the type considered in Case 1, which we have proven impossible. These are the only cases that need be considered for $[i,i+1]$ an up-interval.

Suppose σ^{-1} violates property (γ) on the interval $[i,i+1]$. Then, there exists a point $t=\sigma^{-1}(j)$ between $r=\sigma^{-1}(i)$ and $s=\sigma^{-1}(i+1)$ with $j<i$ (thus, $i-1$ exists). We assume that $r<t<s$, with a similar argument holding if $r>t>s$. By Sub-Lemma 1, it follows that $\sigma(r+1)<i$. We consider now four subcases according to different values of $\sigma^{-1}(i-1)$:

$$\sigma^{-1}(i-1) > r+1, \quad \sigma^{-1}(i-1) = r+1, \quad \sigma^{-1}(i-1) = r-1$$

and

$$\sigma^{-1}(i-1) < r-1 .$$

For emphasis we repeat that $\sigma(r+1)<i$.

Case 2.1. Suppose $\sigma^{-1}(i-1)>r+1$. Then, it follows that we have the point $r+1$ between the points $\sigma^{-1}(i-1)$ and $\sigma^{-1}(i)$ with $\sigma(r+1)<i$, a violation of property (γ) on the up-interval $[i-1,i]$.

Case 2.2. Suppose $\sigma^{-1}(i-1)=r+1$. Then, by property (β), $[r,r+1]$ must be a down-interval, and since point r is a touching, it follows that $[r-1,r]$ is also a down-interval. The point $r-1$ must exist since point 1 is not a touching when $[1,2]$ is a down-interval. We have now $\sigma(r)=i$, $\sigma(r+1)=i-1$ and $\sigma(s)=i+1$, where $s>r+1$. Thus, either $\sigma(r-1)<i-1$ or $\sigma(r-1)>i+1$. If the former holds, then $\sigma(r+1)$ is between $\sigma(r-1)$ and $\sigma(r)$ with $[r-1,r]$ a down-interval and it would follow that $r+1<r-1$. If the latter holds, then $\sigma(s)$ is between $\sigma(r-1)$ and $\sigma(r)$, and it would follow that $s<r-1$. Thus, this case is *impossible.*

Case 2.3. Suppose $\sigma^{-1}(i-1)=r-1$. Then, by property (β) and the fact that point r is a touching, it follows that $[r-1,r]$ and $[r,r+1]$ are both up-intervals. But then $\sigma(r-1)=i-1$ is between $\sigma(r)=i$ and $\sigma(r+1)<i-1$, and hence, $r-1>r+1$, another *impossible* case.

Case 2.4. Suppose $\sigma^{-1}(i-1)<r-1$, so that $\sigma(r+1)<i-1$. Then, the point $i-1$ is between $\sigma(r)$ and $\sigma(r+1)$ with $\sigma^{-1}(i-1)<r$. By property (γ), the interval $[r,r+1]$ must be a down-interval and, as before, $[r-1,r]$ is also. Now, either $\sigma(r-1)>i+1$ or $\sigma(r-1)<i-1$. If $\sigma(r-1)>i+1$, then we have $\sigma(s)=i+1$ between $\sigma(r)$ and $\sigma(r-1)$ with $[r-1,r]$ a down-interval, and we conclude that $s<r-1$, a contradiction. Thus, we must have $\sigma(r-1)<i-1$. But then, point $r-1$ is between points $\sigma^{-1}(i-1)$ and $\sigma^{-1}(i)$ with $\sigma(r-1)<i-1$, while $[i-1,i]$ is an up-interval. Thus, σ^{-1} violates property (γ) on the interval $[i-1,i]$, as desired.

Case 2'. $[i, i+1]$ *a down-interval and point* $i+1$ *a touching.* An argument similar to that given in Case 2 will show that if the permutation σ^{-1} violates property (γ) on the interval $[i, i+1]$, then it will do so also on the down-interval $[i+1, i+2]$. In this case, a finite number of repetitions of the argument will lead to a violation of property (γ) on an interval of the type considered in Case 1'.

Since these are the only cases that need be considered, our proof is complete.

REFERENCES

1. Glen Baxter, *On fixed points of the composite of commuting functions*, USAF Technical Report (Minnesota), submitted for publication to Proc. Amer. Math. Soc.
2. J. R. Isbell, *Commuting mappings of trees* (Research problem), Bull. Amer. Math. Soc. 63 (1957), 419.
3. J. F. Ritt, *Periodic functions with a multiplication theorem*, Trans. Amer. Math. Soc. 23 (1922), 16–25.
4. J. F. Ritt, *Permutable rational functions*, Trans. Amer. Math. Soc. 25 (1923), 399–448.

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