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QUASI-SPECTRAL THEORY

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Dunford [3] has shown that a necessary and sufficient condition for an operator to be spectral is that it have a canonical representation in the form $T=S+N$, where N is a generalized nilpotent operator commuting with S, and S is an operator of scalar type; that is, S is spectral with resolution of the identity $\{E(\delta)\}$, and

$$S = \int_{\sigma(S)} \lambda E(d\lambda) ,$$

with the integration in the strong operator topology. The question arises, whether for some non-spectral operator T, there exists a representation in the form

$$T = \int_{\sigma(T)} \lambda E(d\lambda) + N ,$$

where N, though a generalized nilpotent operator is not required to commute with T, and where $\{E(\delta)\}$ is required only to be a projection valued measure. That is, for $\emptyset$ the empty set, p the complex plane, and σ and δ subsets of the complex plane in some Boolean algebra of subsets, we have

1. $E(\emptyset)=0,\ E(p)=I.$
2. $E(\delta')=E(\delta)'$, where δ' is the complement of δ, and $E(\delta)'=I-E(\delta)$.
3. $E(\delta\cap\sigma)=E(\delta)\cap E(\sigma)$, where $E(\delta)\cap E(\sigma)=E(\delta)E(\sigma)$.
4. $E(\delta\cup\sigma)=E(\delta)\cup E(\sigma)$, where $E(\delta)\cup E(\sigma)=E(\delta)+E(\sigma)-E(\delta)E(\sigma)$.

If there exists such a projection valued measure, and such a generalized nilpotent operator, then T is said to be quasi-spectral, and $\{E(\delta)\}$ its quasi-resolution of the identity. This paper is concerned with the existence of such quasi-spectral operators.

1. An example.

In $L^2[0,1]$ the operator $Tf(x)=xf(x)$ is a spectral operator with resolution of the identity $\{E(\delta)\}$; $E(\delta)f(x)$ being the function equal to $f(x)$

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on δ and equal to zero on δ'. But in $C[0,1]$ T is not spectral (in general $E(\delta)f(x)$ is not in $C[0,1]$). However, T does have a quasi-resolution of the identity, given by the following definitions:

$$E(\emptyset) = 0 .$$

If δ intersects $\sigma(T)=[0,1]$ in a single interval $[a,b]$ or (a,b) or $[a,b)$ or $(a,b]$, with $a \neq 0$, then

$$\begin{aligned} E(\delta)f(x) &= 0 & 0 \leqq x \leqq a , \\ &= f(x)-f(a) & a \leqq x \leqq b , \\ &= f(b)-f(a) & b \leqq x \leqq 1 . \end{aligned}$$

If $a=0$, then

$$\begin{aligned} E(\delta)f(x) &= f(x) & 0 \leqq x \leqq b , \\ &= f(b) & b \leqq x \leqq 1 . \end{aligned}$$

If δ intersects $[0,1]$ in a finite number of intervals, then $E(\delta)$ is the sum of the projections associated with the individual intervals. The algebra of sets consists of those sets whose intersection with $[0,1]$ consists of a finite number of intervals. It follows directly that

$$\int_0^1 E(d\lambda) = I ,$$

and that

$$\int_0^1 \lambda E(d\lambda) = T-N ,$$

where

$$Nf(x) = \int_0^x f(t)\, dt ,$$

which is a generalized nilpotent operator. Thus T is a quasi-spectral operator. The same sort of operator on the space of continuous functions on a monotone sequence contained in $[0,1]$ or on the space of functions on a finite subset of $[0,1]$, yields an analogous result.

Clearly, quasi-resolutions of the identity are in general not unique, since in $L^2[0,1]$ the operator $Tf(x)=xf(x)$ possesses both a resolution of the identity (which is *a fortiori* a quasi-resolution of the identity) and the above quasi-resolution of the identity.

2. Conditions for the existence of a quasi-resolution of the identity.

Keeping the example of the previous paragraph in mind, we set up the following conditions, which are satisfied by the example, and which are sufficient to assure that the operator T on the Banach space X has a quasi-resolution of the identity.

A. X is separable.

B. $\sigma(T) \subset [0,1]$.

C. $\forall \lambda \in \sigma(T)$, $\overline{(\lambda I - T)X}$ is a hyperplane.

From C, we know that it is possible to choose for each $\lambda \in \sigma(T)$ a linear functional $\varphi(\lambda)$ of norm 1, whose nullspace is $\overline{(\lambda I - T)X}$. We further assume that:

D. $\exists\, M > 0$, such that $\forall x \in X$, $\|x\| \leqq M \sup_{\lambda \in \sigma(T)} |\langle x, \varphi(\lambda)\rangle|$.

E. $\forall x \in X$, $\forall \lambda \in \sigma(T)$, $\exists x' \in X$ such that

$$\begin{aligned} \langle x', \varphi(\nu)\rangle &= 0 && \nu \leqq \lambda \\ &= \langle x, \varphi(\nu)\rangle - \langle x, \varphi(\lambda)\rangle && \lambda \leqq \nu . \end{aligned}$$

F. $\forall x \in X$, $\langle x, \varphi(\lambda)\rangle$ is a continuous function of λ.

Assuming these conditions, we construct a quasi-resolution of the identity for T in a series of lemmas and theorems.

LEMMA 1. *$\langle Tx, \varphi(\lambda)\rangle = \lambda\langle x, \varphi(\lambda)\rangle$ for all $\lambda \in \sigma(T)$.*

PROOF.

$$\begin{aligned} \langle(\lambda I - T)x, \varphi(\lambda)\rangle &= 0 . \\ \langle \lambda x, \varphi(\lambda)\rangle &= \langle Tx, \varphi(\lambda)\rangle . \\ \lambda\langle x, \varphi(\lambda)\rangle &= \langle Tx, \varphi(\lambda)\rangle . \end{aligned}$$

THEOREM 1. *Condition* D *implies that*

$$1/d(\lambda) \leqq \|(\lambda I - T)^{-1}\| \leqq M/d(\lambda) ,$$

for all λ in the resolvent set of T, for some M independent of λ (in fact, the M in condition D).

By $d(\lambda)$ we denote the distance from λ to the spectrum of T.

PROOF.

$$\begin{aligned} \|(\lambda I - T)^{-1}\| &= \sup_x \frac{\|x\|}{\|(\lambda I - T)x\|} \\ &\leqq \sup_x \frac{M \sup_{\nu\in\sigma(T)}|\langle x, \varphi(\nu)\rangle|}{\sup_{\nu\in\sigma(T)}|(\lambda - \nu)\langle x, \varphi(\nu)\rangle|} \\ &\leqq \sup_x \frac{M \sup_{\nu\in\sigma(T)}|\langle x, \varphi(\nu)\rangle|}{\inf_{\nu\in\sigma(T)}|\lambda - \nu| \sup_{\nu\in\sigma(T)}|\langle x, \varphi(\nu)\rangle|} \\ &= \frac{M}{\inf_{\nu\in\sigma(T)}|\lambda - \nu|} = \frac{M}{d(\lambda)} . \end{aligned}$$

Since $1/d(\lambda) \leqq \|(\lambda I - T)^{-1}\|$ [6, Theorem VII 3.31], the result follows.

THEOREM 2. *Condition* D *implies that there exists a* K *such that* $\|x\| \leqq K\|x+y\|$, *if* $\langle x,\varphi(\lambda)\rangle \langle y,\varphi(\lambda)\rangle = 0$ *for all* $\lambda \in \sigma(T)$.

PROOF.

$$\begin{aligned} \|x\| \leqq M \sup\nolimits_{\lambda\in\sigma(T)} |\langle x,\varphi(\lambda)\rangle| &\leqq M \sup\nolimits_{\lambda\in\sigma(T)} \max\nolimits_{z=x,y} |\langle z,\varphi(\lambda)\rangle| \\ &= M \sup\nolimits_{\lambda\in\sigma(T)} |\langle x,\varphi(\lambda)\rangle + \langle y,\varphi(\lambda)\rangle| \\ &= M \sup\nolimits_{\lambda\in\sigma(T)} |\langle x+y,\varphi(\lambda)\rangle| \\ &\leqq M\|x+y\| . \end{aligned}$$

LEMMA 2. *Condition* D *implies* $\bigcap_\lambda \overline{(\lambda I - T)X} = 0$.

PROOF. Suppose $\bigcap_\lambda (\lambda I - T)X \neq 0$. Then there would exist an $x \neq 0$, such that $x \in \overline{(\lambda I - T)X}$ for all $\lambda \in \sigma(T)$. Therefore $\|x\| \neq 0$ but $\langle x,\varphi(\lambda)\rangle = 0$ for all $\lambda \in \sigma(T)$. This contradicts condition D.

LEMMA 3. $\{\varphi(\lambda)\}$ *is total over* X.

PROOF. $\langle x,\varphi(\lambda)\rangle = 0$ for all λ implies $x \in \bigcap_\lambda \overline{(\lambda I - T)X}$ which in turn implies $x = 0$.

LEMMA 4. *Condition* A *implies that the weak closure of* $L\{\varphi(\lambda)\}$ *is* X.

PROOF. This is Theorem 7, p. 126 of [1].

Let a and b be two points in $\sigma(T)$, $a \leqq b$, and let $\delta = [a,b]$. If $a = 0$, define $E(\delta)$ by the relationships

$$\begin{aligned} \langle E(\delta)x,\varphi(\lambda)\rangle &= \langle x,\varphi(\lambda)\rangle \qquad \lambda \leqq b , \\ &= \langle x,\varphi(b)\rangle \qquad b \leqq \lambda . \end{aligned}$$

If $a = 0$ is not in $\sigma(T)$, this still makes sense, since $\varphi(0)$ is not used. We shall continue to use $E([0,b])$ for $b \in \sigma(T)$ whether $0 \in \sigma(T)$ or not. If $a \neq 0$, define $E(\delta)$ by the relationships

$$\begin{aligned} \langle E(\delta)x,\varphi(\lambda)\rangle &= 0 \qquad\qquad\qquad\qquad\quad \lambda \leqq a , \\ &= \langle x,\varphi(\lambda)\rangle - \langle x,\varphi(a)\rangle \qquad a \leqq \lambda \leqq b , \\ &= \langle x,\varphi(b)\rangle - \langle x,\varphi(a)\rangle \qquad b \leqq \lambda . \end{aligned}$$

Let $\delta_\lambda = [\lambda, 1]$. Define $E(\lambda) = E(\delta_\lambda)$. By condition D there exists an $x'_a = E(a)x$, there exists an $x'_b = E(b)x$, and therefore there exists an

$$E(\delta)x = E(a)x - E(b)x = \big(E(a) - E(b)\big)x .$$

By Lemma 2, $E(\delta)x$ is unique. Since the right sides of our defining equations are linear, $E(\delta)$ is linear.

THEOREM 3. *Condition E implies that there exists an* $\bar{x} \in X$ *such that for all* $\lambda \in \sigma(T)$, $\langle \bar{x},\varphi(\lambda)\rangle = 1$.

PROOF. Let λ_0 be the smallest number in the spectrum of T, and x be a point such that $\langle x, \varphi(\lambda_0)\rangle = 1$. Then let $\bar{x} = x - E(\lambda_0)x$.

$$\begin{aligned}\langle \bar{x}, \varphi(\lambda)\rangle &= \langle x - E(\lambda_0)x, \varphi(\lambda)\rangle \\ &= \langle x, \varphi(\lambda)\rangle - \langle E(\lambda_0)x, \varphi(\lambda)\rangle \\ &= \langle x, \varphi(\lambda)\rangle - \langle x, \varphi(\lambda)\rangle + \langle x, \varphi(\lambda_0)\rangle = 1\end{aligned}$$

for all $\lambda \in \sigma(T)$.

THEOREM 4. *Conditions* D, E, *and* F *imply that for* $\bar{x}$ *as in Theorem* 3, $L\{T^n\bar{x}\}$ *is dense in* X.

PROOF. By condition F, $\langle x, \varphi(\lambda)\rangle$ is a continuous function of λ, and therefore it may be approximated uniformly by polynomials. That is, given $\varepsilon > 0$, there exists an n, and numbers a_i, $i = 0, 1, \ldots, n$ such that

$$|\langle x, \varphi(\lambda)\rangle - \sum_{i=0}^{n} a_i \lambda^i| < \varepsilon/M \qquad \text{for all} \quad \lambda \in \sigma(T) .$$

By Lemma 1, and by Theorem 3, this may be written

$$|\langle x - \sum_{i=0}^{n} a_i T^i \bar{x}, \varphi(\lambda)\rangle| < \varepsilon/M \qquad \text{for all} \quad \lambda \in \sigma(T) .$$

By condition D we have

$$\|x - \sum_{i=0}^{n} a_i T^i \bar{x}\| < \varepsilon ,$$

and the proof is complete.

THEOREM 5. *If there exists an* $\bar{x}$ *as in Theorem* 3, *and if* $L\{T^n\bar{x}\}$ *is dense in* X, *then* φ *is a weak*ly continuous map from* $\sigma(T)$ *into* X^*.

PROOF. We need to show that given an $\varepsilon > 0$, and $x_1, \ldots, x_n \in X$, there exists a $\delta > 0$ such that $|\lambda - \nu| < \delta$ implies that

$$|\langle x_i, \varphi(\lambda)\rangle - \langle x_i, \varphi(\nu)\rangle| < \varepsilon \qquad \text{for} \quad i = 1, \ldots, n .$$

Since $L\{T^k\bar{x}\}$ is dense in X, there exist numbers m_i, $i = 1, \ldots, n$; and numbers a_{ki}, $i = 1, \ldots, n$ and $k = 1, \ldots, m_i$; such that

$$\left\| x_i - \left(\sum_{k=1}^{m_i} a_{ki} T^k \bar{x} \right) \right\| < \varepsilon/3 .$$

Therefore

$$\begin{aligned}\left| \langle x_i, \varphi(\lambda)\rangle - \left(\sum_{k=1}^{m_i} a_{ki} \lambda^k \right) \langle \bar{x}, \varphi(\lambda)\rangle \right| &= \left| \langle x_i - \sum_{k=1}^{m_i} a_{ki} T^k \bar{x}, \varphi(\lambda)\rangle \right| \\ &\leqq \|\varphi(\lambda)\| \left\| x_i - \sum_{k=1}^{m_i} a_{ki} T^k \bar{x} \right\| < \varepsilon/3 .\end{aligned}$$

Similarly,

$$\left| \langle x_i, \varphi(\nu) \rangle - \left(\sum_{k=1}^{m_i} a_{ki} \nu^k \right) \langle \bar{x}, \varphi(\nu) \rangle \right| < \varepsilon/3 .$$

Now pick δ_i such that

$$\left| \left(\sum_{k=1}^{m_i} a_{ki} \lambda^k \right) \langle \bar{x}, \varphi(\lambda) \rangle - \left(\sum_{k=1}^{m_i} a_{ki} \nu^k \right) \langle \bar{x}, \varphi(\nu) \rangle \right| < \varepsilon/3 ,$$

if $|\lambda - \nu| < \delta_i$. Pick $\delta = \min \delta_i$. Then

$$|\langle x_i, \varphi(\lambda) \rangle - \langle x_i, \varphi(\nu) > | < \varepsilon \qquad \text{if} \qquad |\lambda - \nu| < \delta .$$

Corollary. *If there exists an $\bar{x}$ as in Theorem 3, and if $L\{T^n \bar{x}\}$ is dense in X, then condition* F *holds.*

We have now shown that if we assume conditions A, B, C, D, and E, then there exists an $\bar{x}$ as in Theorem 3, and condition F is equivalent to $L\{T^n \bar{x}\}$ being dense in X.

Theorem 6. *$E(\delta)$ is a continuous operator on X.*

Proof.

$$\begin{aligned}
\|E(\delta)\| = \sup\nolimits_{x \in X} \frac{\|E(\delta)x\|}{\|x\|} &\leqq \sup\nolimits_{x \in X} M \sup\nolimits_{\lambda \in \sigma(T)} \frac{|\langle E(\delta)x, \varphi(\lambda) \rangle|}{\|x\|} \\
&= M \sup\nolimits_{x \in X} \sup\nolimits_{a \leqq \lambda \leqq b} \frac{|\langle E(\delta)x, \varphi(\lambda) \rangle|}{\|x\|} \\
&= M \sup\nolimits_{x \in X} \sup\nolimits_{a \leqq \lambda \leqq b} \frac{|\langle x, \varphi(\lambda) \rangle - \langle x, \varphi(a) \rangle|}{\|x\|} \\
&= M \sup\nolimits_{x \in X} \sup\nolimits_{a \leqq \lambda \leqq b} \frac{|\langle x, \varphi(\lambda) - \varphi(a) \rangle|}{\|x\|} \\
&\leqq M \sup\nolimits_{x \in X} \sup\nolimits_{a \leqq \lambda \leqq b} \frac{\|x\| \, \|\varphi(\lambda) - \varphi(a)\|}{\|x\|} \\
&\leqq M \sup\nolimits_{a \leqq \lambda \leqq b} (\|\varphi(\lambda)\| + \|\varphi(a)\|) \\
&= 2M .
\end{aligned}$$

Theorem 7. $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2)$.

Theorem 8. *$E(\delta_1 \cup \delta_2) = E(\delta_1) + E(\delta_2) - E(\delta_1)E(\delta_2)$, when δ_1 and δ_2 are overlapping intervals.*

The proofs of these two theorems follow directly from the definitions, and are omitted.

So far we have considered $E(\delta)$ only for δ a closed interval. If δ is a finite union of intervals, we now define

$$E(\delta) = E\left(\bigcup_{k=1}^{n} \delta_k\right) = \sum_{k=1}^{n} E(\delta_k) .$$

By this definition we remove the restriction in Theorem 8 that the intervals be overlapping. Theorem 7 is also extended to the case where δ_1 and δ_2 are finite unions of intervals by observing that addition distributes over multiplication just as union distributes over intersection. From our original definition it follows that $E(\emptyset)=0$, and $E\big(\sigma(T)\big)=I$. Therefore we define $E(\delta')=I-E(\delta)$. The mapping $E:\delta \to E(\delta)$ is now a homomorphism of the Boolean algebra of finite unions, complements and intersections of closed intervals onto the Boolean algebra of projections $\{E(\delta)\}$.

THEOREM 9. *Let* $0=x_0<x_1<\ldots<x_{n-1}<x_n=1$ *be a partition of* $\sigma(T)\subset[0,1]$. *Let* δ_i *be the interval* $[x_{i-1},x_i]$. *Then* $\sum_{i=1}^{n} E(\delta_i)=I$.

PROOF.

$$\langle \sum_{i=1}^{n} E(\delta_i)x, \varphi(\lambda)\rangle = \sum_{i=1}^{n} \langle E(\delta_i)x, \varphi(\lambda)\rangle .$$

Let $x_{k-1} \leqq \lambda \leqq x_k$. Then

$$\begin{aligned}\sum_{i=1}^{n} \langle E(\delta_i)x, \varphi(\lambda)\rangle &= \langle x, \varphi(x_1)\rangle + \sum_{i=1}^{k-1} [\langle x, \varphi(x_i)\rangle - \langle x, \varphi(x_{i-1})\rangle] + \\ &\quad + \langle x, \varphi(\lambda)\rangle - \langle x, \varphi(x_{k-1})\rangle + \sum_{i=k+1}^{n} 0 \\ &= \langle x, \varphi(\lambda)\rangle .\end{aligned}$$

Since $\{\varphi(\lambda)\}$ is total, $\big(\sum_{i=1}^{n} E(\delta_i)\big)x=x$ for all $x \in X$, and therefore

$$\sum_{i=1}^{n} E(\delta_i) = I .$$

We now define

$$\int_{\sigma(T)} E(d\lambda) \quad \text{to be} \quad \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} E(\delta_i) ,$$

where the δ_i are as in Theorem 9, and $\|\Delta\| \to 0$ means that for all i, $|x_i - x_{i-1}| \to 0$, as long as there are points in the spectrum of T interior to δ_i. If $\sigma(T)$ only has a finite number of points, we are therefore taking the limit of a finite sequence, which we take to be the last term in the sequence (or we might consider that we have an infinite sequence which becomes constant after a while). Convergence of the limit is in the following sense: $A_n \to A$ means $\langle A_n x, \varphi(\lambda)\rangle \to \langle Ax, \varphi(\lambda)>$ for all $x \in X$ and for all $\lambda \in \sigma(T)$.

COROLLARY. $\int_{\sigma(T)} E(d\lambda) = I$.

Now let $S = \int_{\sigma(T)} \lambda E(d\lambda)$ where the integral is defined similarly to $\int_{\sigma(T)} E(d\lambda)$, that is

$$\langle Sx, \varphi(\lambda)\rangle = \lim_{\|\Delta\|\to 0} \langle \sum_{i=1}^{n} x_i E(\delta_i)x, \varphi(\lambda)\rangle .$$

Notice that while one might expect $E(\delta_i)$ to be multiplied by any $\lambda_i \in \delta_i$, we insist that the right hand end point be used. This is because in the case of a finite spectrum, choice of the left hand endpoint would lead to difficulty.

THEOREM 10. *S exists as a continuous linear operator.*

PROOF.

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} x_i E(\delta_i)x, \varphi(\lambda) = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} E(\delta_i)x, \varphi(\lambda) .$$

Let $\lambda \in \delta_{k(\lambda, n)}$, $1 \leqq k \leqq n$. Then the second limit above may be written

$$\begin{aligned}
&\lim_{\|\Delta\|\to 0} \left[\sum_{i=1}^{k-1} x_i \langle E(\delta_i)x, \varphi(\lambda)\rangle + x_k \langle E(\delta_k)x, \varphi(\lambda)\rangle + \sum_{i=k+1}^{n} x_i \langle E(\delta_i)x, \varphi(\lambda)\rangle\right] \\
&= \lim_{\|\Delta\|\to 0} \left[\sum_{i=1}^{k-1} x_i \big(\langle x, \varphi(x_i)\rangle - \langle x, \varphi(x_{i-1})\rangle\big) + x_k \big(\langle x, \varphi(\lambda)\rangle - \langle x, \varphi(x_{k-1})\rangle\big)\right] \\
&= \lim_{|\Delta|\to 0} \left[x_k \langle x, \varphi(\lambda)\rangle - \sum_{i=0}^{k-1} \langle x, \varphi(x_i)\rangle (x_{i+1} - x_i)\right].
\end{aligned}$$

Since $\lambda \in \delta_k$ and since x_k is the right hand end point of δ_k, x_k either becomes λ or approaches it. That is, as we subdivide the spectrum, we might pick λ as one of our division points. The interval to the right of λ gives the first zero term in the above sum, and the one to the left the last non-zero term, so $\lambda = x_k$. If λ is never chosen as a division point, then since $\lambda \in \sigma(T)$, δ_k always has interior points in $\sigma(T)$, and so $|x_k - x_{k-1}| \to 0$. That is, $x_k \to \lambda$. Therefore,

$$x_k \langle x, \varphi(\lambda)\rangle \to \lambda \langle x, \varphi(\lambda)\rangle = \langle Tx, \varphi(\lambda)\rangle .$$

The second part of the limit,

$$\lim_{\|\Delta\|\to 0} \sum_{i=0}^{k-1} \langle x, \varphi(x_i)\rangle (x_{i+1} - x_i) ,$$

is the integral

$$\int_0^\lambda \langle x, \varphi(\nu)\rangle \, d\nu$$

if we define $\langle x, \varphi(\nu)\rangle$ for $\nu \in [0,1]$ but not in $\sigma(T)$ by the equation

$$\langle x, \varphi(\nu)\rangle = \langle x, \varphi(\nu^-)\rangle$$

where ν^- is the first point in the spectrum of T to the left of ν. If there are no points in the spectrum of T to the left of ν, set $\langle x, \varphi(\nu)\rangle = 0$. Since $\langle x, \varphi(\nu)\rangle$ is a continuous function on the spectrum of T, if there are at most a countable number of gaps in the spectrum of T, then $\langle x, \varphi(\nu)\rangle$ will be an integrable function, and the integral will exist. We now define the operator N by the equation

$$\langle Nx, \varphi(\lambda)\rangle = \int_0^\lambda \langle x, \varphi(\nu)\rangle \, d\nu \, .$$

Since T is linear and continuous, if we can show that N is linear and continuous, then $S = T - N$ will be linear and continuous. Because $\langle x, \varphi(\nu)\rangle$ is linear, and integration is linear, N is linear. To show that N is continuous, we show that $\|N\|$ is finite:

$$\begin{aligned}
\|N\| = \sup_{x\in X} \frac{\|Nx\|}{\|x\|} &\leqq M \sup_{x\in X} \sup_{\lambda\in\sigma(T)} \frac{|\langle Nx, \varphi(\lambda)\rangle|}{\|x\|} \\
&= M \sup_{x\in X} \sup_{\lambda\in\sigma(T)} \frac{|\int_0^\lambda \langle x, \varphi(\nu)\rangle \, d\nu|}{\|x\|} \\
&\leqq M \sup_{x\in X} \sup_{\lambda\in\sigma(T)} \frac{\int_0^\lambda |\langle x, \varphi(\nu)\rangle| \, d\nu}{\|x\|} \\
&\leqq M \sup_{x\in X} \sup_{\lambda\in\sigma(T)} \frac{\lambda \sup_{\nu\leqq\lambda} |\langle x, \varphi(\nu)\rangle|}{\|x\|} \\
&\leqq M \sup_{x\in X} \sup_{\nu\leqq\lambda} \sup_{\lambda\in\sigma(T)} \frac{|\langle x, \varphi(\nu)\rangle|}{\|x\|} \\
&\leqq M \sup_{x\in X} \frac{\|x\|}{\|x\|} = M \, .
\end{aligned}$$

Thus N and S are both linear and continuous.

LEMMA 5. $|\langle N^n x, \varphi(\lambda)\rangle| \leqq \dfrac{\lambda^n}{n!} \|x\|$.

PROOF. The proof is by induction. For $n = 1$,

$$
\begin{aligned}
|\langle Nx, \varphi(\lambda)\rangle| &= \left| \int_0^\lambda \langle x, \varphi(\nu)\rangle \, d\nu \right| \\
&\leqq \int_0^\lambda |\langle x, \varphi(\nu)\rangle| \, d\nu \\
&\leqq \int_0^\lambda \|x\| \, d\nu = \lambda \|x\| .
\end{aligned}
$$

Assume the lemma true for $n=k$. Then

$$
\begin{aligned}
|\langle N^{k+1}x, \varphi(\lambda)\rangle| &= \left| \int_0^\lambda \langle N^k x, \varphi(\nu)\rangle \, d\nu \right| \\
&\leqq \int_0^\lambda |\langle N^k x, \varphi(\nu)\rangle| \, d\nu \\
&\leqq \int_0^\lambda \frac{\nu^k}{k!} \|x\| \, d\nu = \frac{\lambda^{k+1}}{(k+1)!} \|x\| .
\end{aligned}
$$

THEOREM 11. $\|N^n\| \leqq M/n!$

PROOF.

$$
\begin{aligned}
\|N^n\| &= \sup_{x\in X} \frac{\|N^n x\|}{\|x\|} \\
&\leqq M \sup_{x\in X} \sup_{\lambda\in\sigma(T)} \frac{|\langle N^n x, \varphi(\lambda)\rangle|}{\|x\|} \\
&\leqq M \sup_{x\in X} \sup_{\lambda\in\sigma(T)} \frac{\lambda^n \|x\|}{n! \|x\|} \\
&\leqq M \sup_{\lambda\in\sigma(T)} \frac{\lambda^n}{n!} \leqq \frac{M}{n!} .
\end{aligned}
$$

COROLLARY. *N is a generalized nilpotent operator.*

PROOF.

$$
\lim_{n\to\infty} \|N^n\|^{1/n} \leqq \lim_{n\to\infty} \frac{M^{1/n}}{(n!)^{1/n}} = 0 .
$$

Thus, T is a quasi-spectral operator, and $\{E(\delta)\}$ is its quasi-resolution of the identity.

BIBLIOGRAPHY

1. S. Banach, *Théorie des opérations linéares*, New York, 1955.
2. N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. 64 (1958), 217–274.
3. N. Dunford, *Spectral operators*, Pacific J. Math. 4 (1954), 321–354.
4. N. Dunford, *Spectral theory* I, *Convergence to projections*, Trans. Amer. Math. Soc. 54 (1943), 185–217.
5. N. Dunford, *Spectral theory* II, *Resolutions of the identity*, Pacific J. Math. 2 (1952), 559–614.
6. N. Dunford and J. T. Schwartz, *Linear operators*. I, *General theory*, New York, 1958.
7. E. Hille, *Functional analysis and semigroups* (Amer. Math. Soc. Colloquium Publ. 31), Providence, 1948.
8. A. E. Taylor, *Introduction to functional analysis*, New York, 1958.

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