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# COMMUTATIVE PARTIALLY ORDERED RECURSIVE ARITHMETICS

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## 1.

In [3] V. Vučković considers a partially ordered recursive arithmetic with the following properties.

(1.1) Every number $x$ has $n$ successors, denoted by $S_0x, S_1x, \ldots, S_{n-1}x$.

(1.2) Numeral variables are denoted by $x, y, \ldots$, and definite numerals by $a, b, \ldots$. The numerals are $0, S_0 0, S_1 0, \ldots, S_{n-1}0, S_0 S_0 0, S_1 S_0 0, \ldots$, and so on.

(1.3) The initial functions are the zero function, $Z(x)$; the identity function, $I(x)$; and $n$ successor functions, $S_v x$ with $v=0,\ldots,n-1$.

(1.4) Functions are defined recursively using a schema which has $n+1$ equations as follows.

$$F(x,0) = a(x) ,$$
$$F(x,S_v y) = b_v(x,y,F(x,y)), \qquad v = 0,\ldots,n-1 ,$$

where $a(x)$ and $b_v(x,y,z)$ are functions previously defined. Functions can also be defined explicitly by substitution.

(1.5) The arithmetic is made commutative by introducing the axiom

$$S_u S_v x = S_v S_u x ,$$

and by stipulating that the functions used in a defining schema of the type given above satisfy the condition

$$b_v\big(x,S_u y,b_u(x,y,F(x,y))\big) = b_u\big(x,S_v y,b_v(x,y,F(x,y))\big) .$$

## 2.

In recursive number theory with only a single successor function, $Sx$, the numerals are $0, S0, SS0, \ldots$. Such numerals will be referred to as

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members of $M_1$. We now introduce an ordered set of members of $M_1$, which will be written

$$(x_1, x_2, x_3, \ldots, x_n) .$$

An ordered set of $n$ members of this type will be called a member of $M_n$, and $x_1, \ldots, x_n$ will be referred to as components. We construct a recursive arithmetic which has as its numerals the members of $M_n$.

Functions in this arithmetic are defined by means of their components. The recursive arithmetic produced is isomorphic to that described by Vučković in [3] and several of his results will be examined in the light of this component treatment.

(2.1) A function in $M_n$, $\big(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big)$, will be said to be primitive recursive when $f_1,\ldots,f_n$ are primitive recursive functions defined in $M_1$. In the general case it will be necessary to consider a defining schema for simultaneous recursion, as given by Rózsa Péter in [2]. This schema is as follows

$$\text{(2.2)} \qquad \begin{aligned} f_i(0,x_2,\ldots,x_n) &= a_i(x_2,\ldots,x_n) , \\ f_i(Sx_1,x_2,\ldots,x_n) &= b_i(x_1,\ldots,x_n, f_1,\ldots,f_n), \qquad i = 1,\ldots,n . \end{aligned}$$

However the functions required in most cases can usually be defined much more simply, for example by using the schema

$$\text{(2.3)} \qquad \begin{aligned} f_i(0) &= a_i \\ f_i(Sx) &= b_i\big(x,f_i(x)\big), \qquad i = 1,\ldots,n . \end{aligned}$$

The functions $f_1,\ldots,f_n$ will be called component functions. Since the component functions are functions in $M_1$, many of them are already familiar in single successor recursive number theory. When this is the case their defining schemas will often be omitted.

We shall also allow primitive recursive function to be defined in $M_n$ by substitution.

(2.4) The notion of equality in $M_n$ is introduced in the following way:

$$(x_1,\ldots,x_n) = (y_1,\ldots,y_n) \quad \text{if, and only if,} \quad x_i = y_i \text{ for } i = 1,\ldots,n .$$

**3.**

We shall establish a correspondence between the numerals of Vučković's system and members of $M_n$. Such a correspondence will be written

$$\text{(3.1)} \qquad \boldsymbol{x} \leftrightarrow (x_1,x_2,\ldots,x_n) .$$

This relationship will hold if, and only if, $\boldsymbol{x}$ is such that it contains $x_1$

successors of type $\boldsymbol{S}_1$, $x_2$ of type $\boldsymbol{S}_2, \ldots$, and $x_n$ of type $\boldsymbol{S}_n$. (It is assumed for simplicity that the successors of Vučković's system have suffixes ranging from 1 to $n$ instead of 0 to $n-1$ as in the original paper.)

Hence, for example, working with two successors,

$$\boldsymbol{S}_1\boldsymbol{S}_1\boldsymbol{S}_2\boldsymbol{S}_2\boldsymbol{S}_2\boldsymbol{0} \leftrightarrow (SS0, SSS0) .$$

The above interpretation enables the following correspondences to be stated

$$\boldsymbol{0} \leftrightarrow (0,0,\ldots,0) , \tag{3.11}$$

$$\boldsymbol{S}_v\boldsymbol{x} \leftrightarrow (x_1,\ldots,Sx_v,\ldots,x_n) . \tag{3.12}$$

In order to establish correspondences between functions of Vučković's system ($V$-functions) and functions of the component system ($C$-functions), a rule of inference is required.

Suppose the $V$-function, $\boldsymbol{F}(\boldsymbol{x})$, is defined as follows:

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{0}) &= \boldsymbol{a} , \\ \boldsymbol{F}(\boldsymbol{S}_v\boldsymbol{x}) &= \boldsymbol{b}_v(\boldsymbol{x},\boldsymbol{F}(\boldsymbol{x})), \qquad v = 1,\ldots,n , \end{aligned}$$

with the commutativity condition satisfied.

Suppose the $C$-function consists of an ordered set of $n$ primitive recursive functions in $M_1$, namely

$$\big(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big) ,$$

and that from the defining schema of these functions the following equations can be derived:

$$\begin{aligned} f_i(0,\ldots,0) &= a_i , \\ f_i(x_1,\ldots,Sx_v,\ldots,x_n) &= b_{iv}(x_1,\ldots,x_n, f_1,\ldots,f_n), \qquad i,v=1,\ldots,n . \end{aligned}$$

Then the rule of inference can be stated as follows:

$$\boldsymbol{F}(\boldsymbol{x}) \leftrightarrow \big(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big)$$

(3.2) if, and only if ,

and

$$\boldsymbol{a} \leftrightarrow (a_1,\ldots,a_n),$$

$$\boldsymbol{b}_v(\boldsymbol{x},\boldsymbol{y}) \leftrightarrow \big(b_{1v}(x_1,\ldots,x_n,y_1,\ldots,y_n),\ldots,b_{nv}(x_1,\ldots,x_n,y_1,\ldots,y_n)\big)$$

for $v = 1,\ldots,n$ .

As an example of the use of this rule consider the $V$-function, $\boldsymbol{C}_i\boldsymbol{x}$, defined by

$$\boldsymbol{C}_i\boldsymbol{0} = \boldsymbol{0}\,,$$

$$\boldsymbol{C}_i\boldsymbol{S}_v\boldsymbol{x} = \begin{cases} \boldsymbol{C}_i\boldsymbol{x} & \text{if } v \neq i\,, \\ \boldsymbol{S}_i\boldsymbol{C}_i\boldsymbol{x} & \text{if } v = i\,, \end{cases}$$

and the $C$-function

$$(0,\ldots,0,x_i,0,\ldots,0)\,.$$

Then,

$$f_j(0,\ldots,0) = 0\,,$$

$$f_j(x_1,\ldots,Sx_v,\ldots,x_n) = \begin{cases} f_j(x_1,\ldots,x_n) & \text{if } j \neq i \text{ or } v \neq i\,, \\ Sf_j(x_1,\ldots,x_n) & \text{if } j = v = i\,. \end{cases}$$

From (3.11) $\quad \boldsymbol{0} \leftrightarrow (0,\ldots,0)\,.$

From (3.1) $\quad \boldsymbol{y} \leftrightarrow (y_1,\ldots,y_n), \quad$ if $v \neq i\,.$

From (3.12) $\quad \boldsymbol{S}_v\boldsymbol{y} \leftrightarrow (y_1,\ldots,Sy_v,\ldots,y_n), \quad$ if $v = i\,.$

Hence the rule of inference given in (3.2) can be applied to give

(3.3) $$\boldsymbol{C}_i\boldsymbol{x} \leftrightarrow (0,\ldots,0,x_i,0,\ldots,0)\,.$$

LEMMA. *The V-function for addition, $\boldsymbol{x}+\boldsymbol{y}$, is defined as follows:*

$$\boldsymbol{x}+\boldsymbol{0} = \boldsymbol{x}\,,$$

$$\boldsymbol{x}+\boldsymbol{S}_v\boldsymbol{y} = \boldsymbol{S}_v(\boldsymbol{x}+\boldsymbol{y}), \quad v = 1,\ldots,n\,.$$

*The corresponding C-function is*

$$(x_1+y_1,\ldots,x_n+y_n)\,.$$

*By an analysis similar to that given above rule* (3.2) *can be applied to give*

(3.4) $$\boldsymbol{x}+\boldsymbol{y} \leftrightarrow (x_1+y_1,\ldots,x_n+y_n)\,.$$

Various other correspondences between $V$-functions and $C$-functions can be established using (3.2). Some of these relationships are important in establishing a complete isomorphism between the two systems. These are listed briefly below and will be examined in more detail later.

(3.51) $$\boldsymbol{P}_v\boldsymbol{x} \leftrightarrow (x_1,\ldots,Px_v,\ldots,x_n)\,,$$

where $\boldsymbol{P}_v\boldsymbol{x}$ is defined by

$$\boldsymbol{P}_v\boldsymbol{0} = \boldsymbol{0}$$

$$\boldsymbol{P}_v\boldsymbol{S}_u\boldsymbol{x} = \begin{cases} \boldsymbol{S}_u\boldsymbol{P}_v\boldsymbol{x} & \text{if } v \neq u \\ \boldsymbol{x} & \text{if } v = u \end{cases}$$

(3.52) $$\boldsymbol{x} \mathbin{\dot{-}} \boldsymbol{y} \leftrightarrow (x_1 \mathbin{\dot{-}} y_1,\ldots,x_n \mathbin{\dot{-}} y_n)$$

where $\boldsymbol{x} \mathbin{\dot{-}} \boldsymbol{y}$ is defined by

$$\boldsymbol{x} \dot{-} 0 = \boldsymbol{x}$$
$$\boldsymbol{x} \dot{-} \boldsymbol{S}_v \boldsymbol{y} = \boldsymbol{P}_v(\boldsymbol{x} \dot{-} \boldsymbol{y}), \quad v = 1,\ldots,n .$$

(3.53) $$\boldsymbol{\emptyset}_v \boldsymbol{x} \leftrightarrow (x_{n+1-v}, x_{n+2-v}, \ldots, x_{n+n-v}) ,$$

where $\boldsymbol{\emptyset}_v \boldsymbol{x}$ is defined by

$$\boldsymbol{\emptyset}_v 0 = 0 ,$$
$$\boldsymbol{\emptyset}_v \boldsymbol{S}_u \boldsymbol{x} = \boldsymbol{S}_{u+v} \boldsymbol{\emptyset}_v \boldsymbol{x}, \quad v = 1,\ldots,n .$$

(If a suffix exceeds $n$, its value is taken within the range 1 to $n$. This is done by taking the excess over $n$. For example, $\boldsymbol{S}_{n+3}$ would be read as $\boldsymbol{S}_3$.)

(3.54) $$\boldsymbol{x} \cdot \boldsymbol{y} \leftrightarrow (x_1 \cdot y_1, \ldots, x_n \cdot y_n) ,$$

where $\boldsymbol{x} \cdot \boldsymbol{y}$ is defined by

$$\boldsymbol{x} \cdot 0 = 0$$
$$\boldsymbol{x} \cdot \boldsymbol{S}_v \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{C}_v \boldsymbol{x}, \quad v = 1,\ldots,n .$$

(3.6) LEMMA. *If*

$$\boldsymbol{F}(\boldsymbol{x}) \leftrightarrow \big(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big)$$

*and*

$$\boldsymbol{G}(\boldsymbol{x}) \leftrightarrow \big(g_1(x_1,\ldots,x_n),\ldots,g_n(x_1,\ldots,x_n)\big) ,$$

*then there exists a V-function,* $\boldsymbol{H}(\boldsymbol{x})$, *such that*

$$\boldsymbol{H}(\boldsymbol{x}) \leftrightarrow \big(f_1(g_1(x_1,\ldots,x_n),x_2,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big) .$$

PROOF. From (3.3),

$$\boldsymbol{C}_1 \boldsymbol{F}(\boldsymbol{x}) \leftrightarrow \big(f_1(x_1,\ldots,x_n),0,\ldots,0\big) ,$$
$$\boldsymbol{C}_1 \boldsymbol{G}(\boldsymbol{x}) \leftrightarrow \big(g_1(x_1,\ldots,x_n),0,\ldots,0\big) .$$

From (3.3) and (3.4),

$$\boldsymbol{C}_1 \boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{C}_2 \boldsymbol{x} + \cdots + \boldsymbol{C}_n \boldsymbol{x} \leftrightarrow \big(g_1(x_1,\ldots,x_n),x_2,\ldots,x_n\big) .$$

Hence, if $\boldsymbol{K}(\boldsymbol{x})$ is defined by

$$\boldsymbol{K}(\boldsymbol{x}) = \boldsymbol{C}_1 \boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{C}_2 \boldsymbol{x} + \cdots + \boldsymbol{C}_n \boldsymbol{x} ,$$

then

$$\boldsymbol{C}_1 \boldsymbol{F}\big(\boldsymbol{K}(\boldsymbol{x})\big) \leftrightarrow \big(f_1(g_1(x_1,\ldots,x_n),x_2,\ldots,x_n),0,\ldots,0\big) .$$

Next the function $\boldsymbol{H}(\boldsymbol{x})$ is defined by

$$\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{C}_1 \boldsymbol{F}\big(\boldsymbol{K}(\boldsymbol{x})\big) + \boldsymbol{C}_2 \boldsymbol{F}(\boldsymbol{x}) + \cdots + \boldsymbol{C}_n \boldsymbol{F}(\boldsymbol{x}) .$$

From (3.3) and (3.4), it follows that

$$\boldsymbol{H}(\boldsymbol{x}) \leftrightarrow \big(f_1(g_1(x_1,\ldots,x_n),x_2,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big) .$$

This establishes the lemma. This lemma, of course, deals with a particular instance of the use of substitution. However, the above method can be generalised to deal with all possible substitutions in the $C$-functions.

(3.7) THEOREM. *Any primitive recursive $C$-function has a corresponding primitive recursive $V$-function.*

PROOF. (I am indebted to R. L. Goodstein for this method of proof.) Let the primitive recursive $C$-function be

$$\bigl(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\bigr).$$

Any primitive recursive function in $M_1$ can be produced in a finite number of steps using just the initial functions $0$, $Sx$, $x+y$, $x \dot{-} y$, $x^2$, iteration and substitution. Hence, for example, starting from $(0,0,\ldots,0)$ we can produce in a finite number of steps the $C$-function

$$\bigl(0,\ldots,0,f_i(x_1,\ldots,x_n),0,\ldots,0\bigr).$$

If it can be shown that to each of the initial functions and operations listed above there is a corresponding $V$-function or operation, then for each step taken in generating $\bigl(0,\ldots,0,f_i(x_1,\ldots,x_n),0,\ldots,0\bigr)$ there will be a corresponding step in the $V$-system.

(3.71) $$\mathbf{0} \leftrightarrow (0,\ldots,0) \quad \text{by } (3.11).$$

(3.72) $$\boldsymbol{C_i S_i x} \leftrightarrow (0,\ldots,Sx_i,\ldots,0) \quad \text{by } (3.3).$$

From (3.3) and (3.4) it follows that

(3.73) $$\boldsymbol{C_i(x+y)} \leftrightarrow (0,\ldots,x_i+y_i,\ldots,0).$$

Similarly it follows from (3.3) and (3.52) that

(3.74) $$\boldsymbol{C_i(x \dot{-} y)} \leftrightarrow (0,\ldots,x_i \dot{-} y_i,\ldots,0).$$

From (3.3) and (3.54),

(3.75) $$\boldsymbol{C_i x \cdot C_i x} \leftrightarrow (0,\ldots,x_i^2,\ldots,0).$$

Results (3.71–5) show that there exist primitive recursive $V$-functions corresponding to each of the functions used to generate $f_i(x_1,\ldots,x_n)$.

In considering iteration it suffices to examine the following schema in $M_1$ for defining a function $f_i(x_j)$:

$$f_i(0) = 0,$$
$$f_i(Sx_j) = b\bigl(f_i(x_j)\bigr).$$

Since $b(y)$ is a previously known function, we shall suppose that a correspondence has been set up of the following kind:

$$\boldsymbol{C_i b(y)} \leftrightarrow (0,\ldots,b(y_i),\ldots,0)\,.$$

Next a $V$-function, $\boldsymbol{F(x)}$, is defined by the following schema:

$$\boldsymbol{F(0)} = \boldsymbol{0}\,,$$
$$\boldsymbol{F(S_v x)} = \begin{cases} \boldsymbol{C_i b(F(x))} & \text{if } v=j\,, \\ \boldsymbol{F(x)} & \text{if } v\neq j\,. \end{cases}$$

Hence rule (3.2) can be applied to give

$$\boldsymbol{F(x)} \leftrightarrow (0,\ldots,f_i(x_j),\ldots,0)\,.$$

The generalisation of lemma (3.6) shows that for any $C$-function defined explicitly by substitution, there exists a corresponding $V$-function. Hence it has been shown that for every step taken in producing the function $(0,\ldots,f_i(x_1,\ldots,x_n),\ldots,0)$, there is a corresponding step in the $V$-system which preserves the fundamental correspondence relationship.

∴ There exists a $V$-function, $\boldsymbol{F_i(x)}$, such that

$$\boldsymbol{F_i(x)} \leftrightarrow (0,\ldots,f_i(x_1,\ldots,x_n),\ldots,0)\,.$$

Clearly the same technique can be applied to each component function, so that there exist $V$-functions, $\boldsymbol{F_1(x)}, \boldsymbol{F_2(x)},\ldots,\boldsymbol{F_n(x)}$, such that

$$\boldsymbol{F_1(x)} \leftrightarrow (f_1(x_1,\ldots,x_n),0,\ldots,0)\,,$$
$$\boldsymbol{F_2(x)} \leftrightarrow (0,f_2(x_1,\ldots,x_n),\ldots,0)\,,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\boldsymbol{F_n(x)} \leftrightarrow (0,0,\ldots,f_n(x_1,\ldots,x_n))\,.$$

Hence, by (3.4), $\boldsymbol{F(x)}=\boldsymbol{F_1(x)}+\boldsymbol{F_2(x)}+\ldots+\boldsymbol{F_n(x)}$, will be a primitive recursive $V$-function such that

$$\boldsymbol{F(x)} \leftrightarrow (f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n))\,.$$

(3.8) THEOREM. *Any primitive recursive $V$-function has a corresponding primitive recursive $C$-function.*

PROOF. Let the $V$-function be defined by

$$\boldsymbol{F(0)} = \boldsymbol{a}\,,$$
$$\boldsymbol{F(S_v x)} = \boldsymbol{b_v(x,F(x))}, \quad v = 1,\ldots,n\,.$$

Suppose that, since $\boldsymbol{a}$ and $\boldsymbol{b_v(x,y)}$ are previously known functions,

$$\boldsymbol{a} \leftrightarrow (a_1,\ldots,a_n)\,,$$
$$\boldsymbol{b}_v(\boldsymbol{x},\boldsymbol{y}) \leftrightarrow \big(b_{1v}(x_1,\ldots,x_n,y_1,\ldots,y_n),\ldots,b_{nv}(x_1,\ldots,x_n,y_1,\ldots,y_n)\big)\,.$$

Hence, by (3.2), component functions can be formulated according to the schema

(3.81)
$$f_i(0,\ldots,0) = a_i\,,$$
$$f_i(x_1,\ldots,Sx_v,\ldots,x_n) = b_{iv}(x_1,\ldots,x_n,f_1,\ldots,f_n), \quad i,v = 1,\ldots,n\,.$$

It remains to be shown that functions defined in this way are primitive recursive.

Consider first the functions $f_i(0,\ldots,0,x_n)$. From (3.81) the following equations can be obtained:

$$f_i(0,\ldots,0,0) = a_i$$
$$f_i(0,\ldots,0,Sx_n) = b_{in}\big(0,\ldots,0,x_n,f_1(0,\ldots,0,x_n),\ldots,f_n(0,\ldots,0,x_n)\big)\,,$$
$$\text{for } i = 1,\ldots,n\,.$$

Since these equations constitute a definition by simultaneous recursion, it follows that $f_i(0,\ldots,0,x_n)$ are primitive recursive functions.

Next consider the functions $f_i(0,\ldots,0,x_{n-1},x_n)$.

In the preceding paragraph it has been shown that the functions $f_i(0,\ldots,0,x_n)$ are primitive recursive, whilst form (3.81) the following equations can be obtained:

$$f_i(0,\ldots,0,Sx_{n-1},x_n)$$
$$= b_{in-1}\big(0,\ldots,0,x_{n-1},x_n,f_1(0,\ldots,0,x_{n-1},x_n),\ldots,f_n(0,\ldots,0,x_{n-1},x_n)\big)\,.$$

Hence the functions $f_i(0,\ldots,0,x_{n-1},x_n)$ are also defined by simultaneous recursion and are, therefore, primitive recursive functions.

This inductive process can be continued until it is shown that the functions $f_i(x_1,\ldots,x_n)$ are primitive recursive. Hence it has been shown that

(i) $\boldsymbol{F}(\boldsymbol{x}) \leftrightarrow \big(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big)$,

(ii) The component functions which satisfy (i) are primitive recursive functions in $M_1$.

In order to complete the proof it is necessary to shown that a substitution in the $V$-system can be copied in the $C$-system. This is trivial, for suppose that

$$\boldsymbol{F}(\boldsymbol{x}) \leftrightarrow \big(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big)\,,$$
$$\boldsymbol{G}(\boldsymbol{x}) \leftrightarrow \big(g_1(x_1,\ldots,x_n),\ldots,g_n(x_1,\ldots,x_n)\big)\,,$$

and that

$$\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{F}\big(\boldsymbol{G}(\boldsymbol{x})\big)\,.$$

Then the $C$-function corresponding to $\boldsymbol{H}(\boldsymbol{x})$ is given by

$$(f_1(g_1,\ldots,g_n),\ldots,f_n(g_1,\ldots,g_n)) .$$

Hence the theorem is established.

Theorems (3.7) and (3.8) demonstrate the functional isomorphism between the $V$-system and the $C$-system. That is to say, to any primitive recursive $V$-function there is a corresponding primitive recursive $C$-function, and conversely.

**4.**

In this section a deductive isomorphism will be established between the $V$-system and the $C$-system. That is to say we shall show that for any proof in one system there is a corresponding proof in the other system. We shall assume the following rules of inference for single successor recursive number theory (see R. L. Goodstein, [1, p. 104]).

$$Sb_1 \quad \frac{f(x) = g(x)}{f(a) = g(a)} \qquad\qquad Sb_2 \quad \frac{a = b}{f(a) = f(b)}$$

$$T \quad \frac{\begin{array}{c} a = b \\ a = c \end{array}}{b = c} \qquad\qquad U \quad \frac{\begin{array}{c} f(0) = g(0) \\ f(Sx) = h(x, f(x)) \\ g(Sx) = h(x, g(x)) \end{array}}{f(x) = g(x)}$$

For the $V$-system the following analogous rules of inference will be used:

$$\boldsymbol{Sb_1} \quad \frac{\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{G}(\boldsymbol{x})}{\boldsymbol{F}(\boldsymbol{a}) = \boldsymbol{G}(\boldsymbol{a})} \qquad\qquad \boldsymbol{Sb_2} \quad \frac{\boldsymbol{a} = \boldsymbol{b}}{\boldsymbol{F}(\boldsymbol{a}) = \boldsymbol{F}(\boldsymbol{b})}$$

$$\boldsymbol{T} \quad \frac{\begin{array}{c} \boldsymbol{a} = \boldsymbol{b} \\ \boldsymbol{a} = \boldsymbol{c} \end{array}}{\boldsymbol{b} = \boldsymbol{c}} \qquad\qquad \boldsymbol{U} \quad \frac{\begin{array}{c} \boldsymbol{F}(0) = \boldsymbol{G}(0) \\ \boldsymbol{F}(\boldsymbol{S}_v\boldsymbol{x}) = \boldsymbol{H}_v(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x})) \\ \boldsymbol{G}(\boldsymbol{S}_v\boldsymbol{x}) = \boldsymbol{H}_v(\boldsymbol{x}, \boldsymbol{G}(\boldsymbol{x})) \end{array}}{\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{G}(\boldsymbol{x})}$$

Suppose there exists a proof in the $C$-system showing that

$$(e_1, e_2, \ldots, e_n) = (f_1, f_2, \ldots, f_n) ,$$

then this proof must show that

$$\begin{array}{l} e_1 = f_1 , \\ e_2 = f_2 , \\ \ldots\ldots\ldots \\ e_n = f_n . \end{array}$$

It is necessary to show that to each step in each proof there will be a corresponding step in the $V$-system. This is done by examining each rule of inference in $M_1$ and showing that a corresponding inference can be drawn in the $V$-system.

(4.1) Suppose that

$$\boldsymbol{F}_i(\boldsymbol{x}) \leftrightarrow \big(0,\ldots,f_i(x_1,\ldots,x_n),\ldots,0\big)\,,$$
$$\boldsymbol{G}_i(\boldsymbol{x}) \leftrightarrow \big(0,\ldots,g_i(x_1,\ldots,x_n),\ldots,0\big)\,,$$

and that $\boldsymbol{F}_i(\boldsymbol{x})=\boldsymbol{G}_i(\boldsymbol{x})$. Then we shall write

$$\boldsymbol{F}_i(\boldsymbol{x}) = \boldsymbol{G}_i(\boldsymbol{x}) \leftrightarrow \big(0=0,\ldots,f_i(x_1,\ldots,x_n) = g_i(x_1,\ldots,x_n),\ldots,0=0\big)\,.$$

Suppose now that rule $Sb_1$ is applied in the $C$-system by substituting $a_j$ for $x_j$. This gives

$$\big(0=0,\ldots,f_i(x_1,\ldots,a_j,\ldots,x_n) = g_i(x_1,\ldots,a_j,\ldots,x_n),\ldots,0=0\big)\,.$$

In constructing the corresponding step in the $V$-system we note first that

$$(\boldsymbol{x}\dot{-}\boldsymbol{C}_j\boldsymbol{x})+\boldsymbol{C}_j\boldsymbol{a} \leftrightarrow (x_1,\ldots,a_j,\ldots,x_n)\,.$$

Now apply rule $\boldsymbol{Sb}_1$ to the equation $\boldsymbol{F}_i(\boldsymbol{x})=\boldsymbol{G}_i(\boldsymbol{x})$ in the following way

$$\boldsymbol{F}_i\big((\boldsymbol{x}\dot{-}\boldsymbol{C}_j\boldsymbol{x})+\boldsymbol{C}_j\boldsymbol{a}\big) = \boldsymbol{G}_i\big((\boldsymbol{x}\dot{-}\boldsymbol{C}_j\boldsymbol{x})+\boldsymbol{C}_j\boldsymbol{a}\big)\,.$$

This is the required equation in the $V$-system.

(4.2) Suppose that

$$\boldsymbol{C}_i\boldsymbol{a} = \boldsymbol{C}_i\boldsymbol{b} \leftrightarrow (0=0,\ldots,a_i=b_i,\ldots,0=0)\,,$$
$$\boldsymbol{F}_i(\boldsymbol{x}) \leftrightarrow \big(0,\ldots,f_i(x_1,\ldots,x_n),\ldots,0\big)\,.$$

Let rule $Sb_2$ be applied to effect a substitution in the $j$-th component place to give in the $C$-system

$$\big(0=0,\ldots,f_i(x_1,\ldots,x_{j-1},a_i,x_{j-1},\ldots,x_n)$$
$$= f_i(x_1,\ldots,x_{j-1},b_i,x_{j-1},\ldots,x_n),\ldots,0=0\big)\,.$$

From (3.3), (3.52) and (3.53),

$$(\boldsymbol{x}\dot{-}\boldsymbol{C}_j\boldsymbol{x})+\boldsymbol{\emptyset}_{j-i}\boldsymbol{C}_i\boldsymbol{a} \leftrightarrow (x_1,\ldots,x_{j-1},a_i,x_{j-1},\ldots,x_n)\,,$$
$$(\boldsymbol{x}\dot{-}\boldsymbol{C}_j\boldsymbol{x})+\boldsymbol{\emptyset}_{j-i}\boldsymbol{C}_i\boldsymbol{b} \leftrightarrow (x_1,\ldots,x_{j-1},b_i,x_{j-1},\ldots,x_n)\,.$$

From $\boldsymbol{C}_i\boldsymbol{a}=\boldsymbol{C}_i\boldsymbol{b}$, we deduce by $\boldsymbol{Sb}_2$ that

$$(\boldsymbol{x}\dot{-}\boldsymbol{C}_j\boldsymbol{x})+\boldsymbol{\emptyset}_{j-i}\boldsymbol{C}_i\boldsymbol{a} = (\boldsymbol{x}\dot{-}\boldsymbol{C}_j\boldsymbol{x})+\boldsymbol{\emptyset}_{j-i}\boldsymbol{C}_i\boldsymbol{b}\,.$$

Then, again by $\boldsymbol{Sb}_2$,

$$\boldsymbol{F}_i\big((\boldsymbol{x} \dot{-} \boldsymbol{C}_j\boldsymbol{x}) + \boldsymbol{\emptyset}_{j-i}\boldsymbol{C}_i\boldsymbol{a}\big) = \boldsymbol{F}_i\big((\boldsymbol{x} \dot{-} \boldsymbol{C}_j\boldsymbol{x}) + \boldsymbol{\emptyset}_{j-i}\boldsymbol{C}_i\boldsymbol{b}\big)\,,$$

which is the required equation in the $V$-system.

(4.3) Suppose that

$$\boldsymbol{C}_i\boldsymbol{a} = \boldsymbol{C}_i\boldsymbol{b} \leftrightarrow (0=0,\ldots,a_i=b_i,\ldots,0=0)$$
$$\boldsymbol{C}_i\boldsymbol{a} = \boldsymbol{C}_i\boldsymbol{c} \leftrightarrow (0=0,\ldots,a_i=c_i,\ldots,0=0)\,.$$

Then by rule $T$ it can be infered that

$$(0=0,\ldots,b_i=c_i,\ldots,0=0)\,.$$

The equation corresponding to this in the $V$-system can be derived immediately by rule $\boldsymbol{T}$. This gives

$$\boldsymbol{C}_i\boldsymbol{b} = \boldsymbol{C}_i\boldsymbol{c}\,.$$

(4.4) Suppose that

$$\boldsymbol{F}(\boldsymbol{x}) \leftrightarrow \big(0,\ldots,f_i(x_1,\ldots,x_n),\ldots,0\big)\,,$$
$$\boldsymbol{G}(\boldsymbol{x}) \leftrightarrow \big(0,\ldots,g_i(x_1,\ldots,x_n),\ldots,0\big)\,,$$
$$\boldsymbol{H}_r(\boldsymbol{x},\boldsymbol{y}) \leftrightarrow \big(0,\ldots,h_{ir}(x_1,\ldots,x_n,y_i),\ldots,0\big), \quad r = 1,\ldots,n\,.$$

Suppose, without loss of generality, that rule $U$ is applied in the following way:

$$\frac{\begin{array}{c}\big(0=0,\ldots,f_i(0,x_2,\ldots,x_n) = g_i(0,x_2,\ldots,x_n),\ldots,0=0\big)\\ \big(0=0,\ldots,f_i(Sx_1,x_2,\ldots,x_n) = h_{i1}(x_1,\ldots,x_n,f_i),\ldots,0=0\big)\\ \big(0=0,\ldots,g_i(Sx_1,x_2,\ldots,x_n) = h_{i1}(x_1,\ldots,x_n,g_i),\ldots,0=0\big)\end{array}}{\big(0=0,\ldots,f_i(x_1,x_2,\ldots,x_n) = g_i(x_1,x_2,\ldots,x_n),\ldots,0=0\big)}\,.$$

In this final expression, put $x_1=x_2=\ldots=x_n=0$, and apply $Sb_1$:

$$\big(0=0,\ldots,f_i(0,0,\ldots,0) = g_i(0,0,\ldots,0),\ldots,0=0\big)\,.$$

The $V$-equation corresponding to this is

(4.41) $$\boldsymbol{F}(\boldsymbol{0}) = \boldsymbol{G}(\boldsymbol{0})\,.$$

Put

$$h_{ir}(x_1,\ldots,x_n,y_i) = f_i(x_1,\ldots,Sx_r,\ldots,x_n)\{1 \dot{-} |y_i,f_i|\}, \quad r = 2,\ldots,n\,.$$

Since every function on the right side of this equation is primitive recursive it follows that the functions $h_{ir}$ are primitive recursive. By putting $y_i=f_i(x_1,\ldots,x_n)$, we obtain

$$f_i(x_1,\ldots,Sx_r,\ldots,x_n) = h_{ir}\big(x_1,\ldots,x_n,f_i(x_1,\ldots,x_n)\big), \quad r = 1,\ldots,n\,.$$

The corresponding $V$-equation is

(4.42) $$\boldsymbol{F}(\boldsymbol{S}_r\boldsymbol{x}) = \boldsymbol{H}_r\big(\boldsymbol{x},\boldsymbol{F}(\boldsymbol{x})\big), \quad r = 1,\ldots,n\,.$$

Similarly the following $V$-equation can be obtained

$$\mathbf{G}(\mathbf{S}_r x) = \mathbf{H}_r(x, \mathbf{G}(x)), \quad r = 1,\ldots,n . \tag{4.43}$$

From (4.41–3), rule $\mathbf{U}$ can be applied to infer the $V$-equation

$$\mathbf{F}(x) = \mathbf{G}(x) .$$

Sections (4.1–4) show that any inference in the $C$-system can be matched by a corresponding inference in the $V$-system for the derivation of a result of the form

$$(0 = 0, \ldots, e_i = f_i, \ldots, 0 = 0) .$$

Hence there exists a $V$-equation, $\mathbf{E}_i = \mathbf{F}_i$, such that

$$\mathbf{E}_i = \mathbf{F}_i \leftrightarrow (0=0,\ldots,e_i = f_i,\ldots,0=0) .$$

Clearly equations can be derived to correspond to the final equation in each component place. Hence there exist $V$-functions, $\mathbf{E}_1,\ldots,\mathbf{E}_n$, $\mathbf{F}_1,\ldots,\mathbf{F}_n$ such that

$$\begin{aligned}
\mathbf{E}_1 &= \mathbf{F}_1 \leftrightarrow (e_1=f_1, 0=0,\ldots,0=0) ,\\
\mathbf{E}_2 &= \mathbf{F}_2 \leftrightarrow (0=0, e_2=f_2,\ldots,0=0) ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\mathbf{E}_n &= \mathbf{F}_n \leftrightarrow (0=0, 0=0,\ldots,e_n=f_n) .
\end{aligned} \tag{4.5}$$

In both the $V$-system and the $C$-system a derived rule of inference can be established of the following form:

$$\frac{a = b \qquad c = d}{a+c = b+d} \qquad\qquad \frac{\mathbf{A} = \mathbf{B} \qquad \mathbf{C} = \mathbf{D}}{\mathbf{A}+\mathbf{C} = \mathbf{B}+\mathbf{D}}$$

Applying these rules to equations (4.5),

$$\mathbf{E}_1 + \cdots + \mathbf{E}_n = \mathbf{F}_1 + \cdots + \mathbf{F}_n \leftrightarrow (e_1=f_1,\ldots,e_n=f_n) .$$

This can be written

$$\mathbf{E} = \mathbf{F} \leftrightarrow (e_1 = f_1,\ldots,e_n = f_n) .$$

(4.6) Hence for every proof in the $C$-system there is a corresponding proof in the $V$-system.

The converse of this result, namely that for every proof in the $V$-system there is a corresponding proof in the $C$-system, is more straightforward.

(4.71) Let

$$\mathbf{F}(x) = \mathbf{G}(x) \leftrightarrow \big(f_1(x_1,\ldots,x_n) = g_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n) = g_n(x_1,\ldots,x_n)\big) .$$

Suppose that rule $\boldsymbol{Sb_1}$ is applied in the $V$-system to give

$$\boldsymbol{F(a) = G(a)} \, .$$

The corresponding result in the $C$-system is obtained by applying rule $Sb_1$ $n$ times to each component equation to give

$$\big(f_1(a_1,\ldots,a_n) = g_1(a_1,\ldots,a_n),\ldots,f_n(a_1,\ldots,a_n) = g_n(a_1,\ldots,a_n)\big) \, .$$

(4.72) Suppose that

$$\boldsymbol{a = b} \leftrightarrow (a_1 = b_1,\ldots,a_n = b_n) \, ,$$
$$\boldsymbol{F(x)} \leftrightarrow \big(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big) \, .$$

Let rule $\boldsymbol{Sb_2}$ be applied in the $V$-system to give

$$\boldsymbol{F(a) = F(b)} \, .$$

The corresponding result in the $C$-system can be obtained by repeated application of rule $Sb_2$ to give

$$\big(f_1(a_1,\ldots,a_n) = f_1(b_1,\ldots,b_n),\ldots,f_n(a_1,\ldots,a_n) = f_n(b_1,\ldots,b_n)\big) \, .$$

(4.73) Suppose that

$$\boldsymbol{a = b} \leftrightarrow (a_1 = b_1,\ldots,a_n = b_n) \, ,$$
$$\boldsymbol{a = c} \leftrightarrow (a_1 = c_1,\ldots,a_n = c_n) \, .$$

Let rule $\boldsymbol{T}$ be applied to give

$$\boldsymbol{b = c} \, .$$

The corresponding result in the $C$-system can be obtained by applying rule $T$ in each component place. This gives

$$(b_1 = c_1,\ldots,b_n = c_n) \, .$$

(4.74) Suppose that

$$\boldsymbol{F(0) = G(0)} \leftrightarrow \big(f_1(0,\ldots,0) = g_1(0,\ldots,0),\ldots,f_n(0,\ldots,0) = g_n(0,\ldots,0)\big),$$
$$\boldsymbol{F(S_v x) = H_v(x, F(x))} \leftrightarrow \big(f_1(x_1,\ldots,Sx_v,\ldots,x_n) = h_{1v}(x_1,\ldots,x_n,f_1,\ldots,f_n),\ldots\big) \, ,$$
$$\boldsymbol{G(S_v x) = H_v(x, G(x))} \leftrightarrow \big(g_1(x_1,\ldots,Sx_v,\ldots,x_n) = h_{1v}(x_1,\ldots,x_n,f_1,\ldots,f_n),\ldots\big) \, .$$

Then in the $V$-system rule $\boldsymbol{U}$ can be applied to give

$$\boldsymbol{F(x) = G(x)} \, .$$

It has already been shown in section (3.8) that the equations in the

$C$-system given above constitute a definition by simultaneous recursion. Hence the component functions are primitive recursive and rule $U$ can be applied to give

$$\big(f_1(x_1,\ldots,x_n) = g_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n) = g_n(x_1,\ldots,x_n)\big) .$$

(4.71–4) show that to every step in a proof in the $V$-system, there is a corresponding step in the $C$-system.
(4.8) Hence for every proof in the $V$-system there will be a corresponding proof in the $C$-system.

The results established in (4.6) and (4.8) demonstrate the deductive isomorphism of the two systems.

## 5.

In his paper Vučković introduces $n$ additive functions defined by the schema

(5.1)
$$\begin{aligned} \boldsymbol{x\sigma_u 0} &= \boldsymbol{x}\ , & u &= 0,\ldots,n-1\ ,\\ \boldsymbol{x\sigma_u S_v y} &= \boldsymbol{S_{u+v}(x\sigma_u y)}, & v &= 1,\ldots,n\ . \end{aligned}$$

(See note on suffixes in section (3.53)).

The most important of these functions is $\boldsymbol{x\sigma_0 y}$, which Vučković calls addition and represents by $\boldsymbol{x+y}$. From (5.1), it follows that its defining schema will be

$$\begin{aligned} \boldsymbol{x+0} &= \boldsymbol{x}\ ,\\ \boldsymbol{x+S_v y} &= \boldsymbol{S_v(x+y)}, \quad v = 1,\ldots,n\ . \end{aligned}$$

This defining schema in the $V$-system was discussed in (3.4) and it was shown that

$$\boldsymbol{x+y} \leftrightarrow (x_1+y_1,\ldots,x_n+y_n)\ .$$

The other additive functions can also be defined directly in the $C$-system, but it is simpler when the number of successors is finite to use the functions $\boldsymbol{\emptyset_v x}$, which were introduced in (3.53). These will be called rotational functions.

REMARK. In the $V$-system the number of successors is not necessarily finite. That is to say the theory developes along the same lines if an infinite number of successors are introduced. When this is done, a representation by ordered sets becomes less straightforward, and it is preferable to retain the additive functions in their original form rather than to express them it terms of the rotational functions.

(5.2) THEOREM. $$\boldsymbol{x\sigma_u y = x+\emptyset_u y}\ .$$

PROOF. $$x\sigma_u 0 = x; \quad x+\emptyset_u 0 = x+0 = x .$$
$$x\sigma_u S_v y = S_{u+v}(x\sigma_u y) .$$
$$x+\emptyset_u S_v y = x+S_{u+v}(\emptyset_u y) = S_{u+v}(x+\emptyset_u y) .$$

$\therefore$ By rule $U$,
$$x\sigma_u y = x+\emptyset_u y .$$

This theorem shows that the additive functions, other than $x\sigma_0 y$, can be regarded as a combination of addition and the rotational functions. This interpretation helps to explain why the additive functions are not commutative, i.e.

(5.21) $$x\sigma_u y \neq y\sigma_u x \quad (\text{for } u>0) .$$

From (5.2), $x\sigma_u y=x+\emptyset_u y$, whilst $y\sigma_u x=y+\emptyset_u x$. It is clear that in general
$$x+\emptyset_u y \neq y+\emptyset_u x .$$

Hence (5.21) is a result to be expected. The following properties of the rotational functions are noted:

(5.22) $$\emptyset_u\emptyset_v x = \emptyset_v\emptyset_u x = \emptyset_{u+v} x ,$$

(5.23) $$\emptyset_n x = \emptyset_0 x = x ,$$

(5.24) $$\emptyset_u(x+y) = (\emptyset_u x)+(\emptyset_u y) .$$

## 6.

In this section we discuss the key equations
$$x+(y\dot{-}x) = y+(x\dot{-}y) ,$$
and
$$x\dot{-}(x\dot{-}y) = y\dot{-}(y\dot{-}x) .$$

In his paper Vučković has to apply the uniqueness rule to a schema of double recursion in order to prove these equations. However, since the deductive isomorphism of the $V$-system and the $C$-system has been established, it is possible to show that the introduction of double recursion is unnecessary. The existence of a proof in the $V$-system using only primitive recursion follows from the existence of a corresponding theorem in single successor recursive number theory.

(6.1) THEOREM. $$x+(y\dot{-}x) = y+(x\dot{-}y) .$$

PROOF. From (3.4) and (3.52),

(6.11) $$x+(y\dot{-}x) \leftrightarrow \big(x_1+(y_1\dot{-}x_1),\dots,x_n+(y_n\dot{-}x_n)\big) .$$

Similarly,

(6.12) $$\boldsymbol{y}\boldsymbol{+}(\boldsymbol{x}\dot{\boldsymbol{-}}\boldsymbol{y}) \leftrightarrow \big(y_1+(x_1\dot{-}y_1),\ldots,y_n+(x_n\dot{-}y_n)\big)\,.$$

It has been shown by R. L. Goodstein [1, chapter 5], that

$$x+(y\dot{-}x) = y+(x\dot{-}y)$$

can be proved by primitive recursion using only the rules of inference given in section 4. Hence a proof can be constructed in the $C$-system that

$$\big(x_1+(y_1\dot{-}x_1) = y_1+(x_1\dot{-}y_1),\ldots,x_n+(y_n\dot{-}x_n) = y_n+(x_n\dot{-}y_n)\big)\,.$$

It has been shown in (4.6) that to each proof in the $C$-system there is a corresponding proof in the $V$-system.

∴. There exists a proof in the $V$-system, using only primitive recursion, that $\boldsymbol{x}\boldsymbol{+}(\boldsymbol{y}\dot{\boldsymbol{-}}\boldsymbol{x})\boldsymbol{=}\boldsymbol{y}\boldsymbol{+}(\boldsymbol{x}\dot{\boldsymbol{-}}\boldsymbol{y})$.

(6.2) THEOREM. $\boldsymbol{x}\dot{\boldsymbol{-}}(\boldsymbol{x}\dot{\boldsymbol{-}}\boldsymbol{y}) \boldsymbol{=} \boldsymbol{y}\dot{\boldsymbol{-}}(\boldsymbol{y}\dot{\boldsymbol{-}}\boldsymbol{x})$.

PROOF. From (3.52),

$$\boldsymbol{x}\dot{\boldsymbol{-}}(\boldsymbol{x}\dot{\boldsymbol{-}}\boldsymbol{y}) \leftrightarrow \big(x_1\dot{-}(x_1\dot{-}y_1),\ldots,x_n\dot{-}(x_n\dot{-}y_n)\big)\,,$$
$$\boldsymbol{y}\dot{\boldsymbol{-}}(\boldsymbol{y}\dot{\boldsymbol{-}}\boldsymbol{x}) \leftrightarrow \big(y_1\dot{-}(y_1\dot{-}x_1),\ldots,y_n\dot{-}(y_n\dot{-}x_n)\big)\,.$$

It can be proved in the $C$-system that

$$x_i\dot{-}(x_i\dot{-}y_i) = y_i\dot{-}(y_i\dot{-}x_i)\,.$$

Hence there exists a primitive recursive proof in the $V$-system that

$$\boldsymbol{x}\dot{\boldsymbol{-}}(\boldsymbol{x}\dot{\boldsymbol{-}}\boldsymbol{y}) \boldsymbol{=} \boldsymbol{y}\dot{\boldsymbol{-}}(\boldsymbol{y}\dot{\boldsymbol{-}}\boldsymbol{x})\,.$$

## 7.

In this section it will be shown that $\boldsymbol{x}\boldsymbol{+}(\boldsymbol{y}\dot{\boldsymbol{-}}\boldsymbol{x})$ is the least upper bound of $\boldsymbol{x}$ and $\boldsymbol{y}$, and that $\boldsymbol{x}\dot{\boldsymbol{-}}(\boldsymbol{x}\dot{\boldsymbol{-}}\boldsymbol{y})$ is the greatest lower bound. From this it follows that the commutative partially ordered recursive arithmetics discussed in this paper are lattices.

The inequality relationship is introduced in the $V$-system as follows:

(7.1) $\boldsymbol{a}\leqq\boldsymbol{b}$ if, and only if, $\boldsymbol{a}\boldsymbol{=}\boldsymbol{b}\dot{\boldsymbol{-}}(\boldsymbol{b}\dot{\boldsymbol{-}}\boldsymbol{a})$

(7.1′) or $\boldsymbol{a}\dot{\boldsymbol{-}}\boldsymbol{b}\boldsymbol{=}\boldsymbol{0}$

(7.1″) or $\boldsymbol{b}\boldsymbol{=}\boldsymbol{a}\boldsymbol{+}(\boldsymbol{b}\dot{\boldsymbol{-}}\boldsymbol{a})$.

The equations given above are not, of course, independent. Given any one of them the other two can be derived.

(7.2) THEOREM. $\boldsymbol{x}\boldsymbol{+}(\boldsymbol{y}\dot{\boldsymbol{-}}\boldsymbol{x})$ *is the least upper bound of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

PROOF.

$$x_i \dot{-} [x_i + (y_i \dot{-} x_i)] = (x_i \dot{-} x_i) \dot{-} (y_i \dot{-} x_i)$$
$$= 0 \dot{-} (y_i \dot{-} x_i) = 0 .$$

Hence, by (4.6), there exists a proof in the V-system that

(7.21) $$\boldsymbol{x} \dot{-} [\boldsymbol{x} + (\boldsymbol{y} \dot{-} \boldsymbol{x})] = \mathbf{0} .$$

Similarly it can be shown that

$$\boldsymbol{y} \dot{-} [\boldsymbol{y} + (\boldsymbol{x} \dot{-} \boldsymbol{y})] = \mathbf{0} .$$

But in (6.1) it was shown that $\boldsymbol{y} + (\boldsymbol{x} \dot{-} \boldsymbol{y}) = \boldsymbol{x} + (\boldsymbol{y} \dot{-} \boldsymbol{x})$.

(7.22) $$\therefore\ \boldsymbol{y} \dot{-} [\boldsymbol{x} + (\boldsymbol{y} \dot{-} \boldsymbol{x})] = \mathbf{0} .$$

Let $\boldsymbol{z}$ be an upper bound of $\boldsymbol{x}$ and $\boldsymbol{y}$. Then $\boldsymbol{x} \leqq \boldsymbol{z}$ and $\boldsymbol{y} \leqq \boldsymbol{z}$, and consequently $x_i \leqq z_i$ and $y_i \leqq z_i$. Hence,

$$[x_i + (y_i \dot{-} x_i)] \dot{-} z_i = (x_i \dot{-} z_i) + [(y_i \dot{-} x_i) \dot{-} (z_i \dot{-} x_i)]$$
$$= (x_i \dot{-} z_i) + [y_i \dot{-} \{x_i + (z_i \dot{-} x_i)\}]$$
$$= (x_i \dot{-} z_i) + (y_i \dot{-} z_i) = 0 + 0 = 0 .$$

Hence by (4.6), there exists a proof in the $V$-system that

$$[\boldsymbol{x} + (\boldsymbol{y} \dot{-} \boldsymbol{x})] \dot{-} \boldsymbol{z} = \mathbf{0} .$$

(7.23) $$\therefore\ \boldsymbol{x} + (\boldsymbol{y} \dot{-} \boldsymbol{x}) \leqq \boldsymbol{z} .$$

Results (7.21) and (7.22) show that $\boldsymbol{x} + (\boldsymbol{y} \dot{-} \boldsymbol{x})$ is an upper bound of $\boldsymbol{x}$ and $\boldsymbol{y}$. (7.23) shows that it is less than or equal to any other upper bound.

$\therefore\ \boldsymbol{x} + (\boldsymbol{y} \dot{-} \boldsymbol{x})$ is the least upper bound of $\boldsymbol{x}$ and $\boldsymbol{y}$.

(7.3) THEOREM. $\boldsymbol{x} \dot{-} (\boldsymbol{x} \dot{-} \boldsymbol{y})$ *is the greatest lower bound of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

The proof of this theorem follows the same pattern as that of the preceding theorem and is, therefore, omitted.

(7.4) THEOREM. *Commutative partially ordered recursive arithmetics are lattices.*

PROOF. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be any two numbers in the $V$-system. Then theorems (7.2) and (7.3) show that they have a least upper bound and a greatest lower bound which are also numbers in the $V$-system. Hence, the numbers of the $V$-system constitute a lattice.

## 8.

The previous sections indicate some of the results which emerge from comparison of the component system with that set up by Vučković. A further paper will show how the limited universal and existential quantifiers can be produced using the component system.

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