MATH. SCAND. 14 (1964), 19–20

INVARIANT UNIFORMITIES FOR COSET SPACES

R. W. BAGLEY

S. N. Hudson [2] and T. S. Wu [4] have shown that, if H is a compact subgroup of a topological group G, then G/H has a right invariant uniformity (invariant under action of G on G/H) and that the uniformity is pseudometrizable if G/H satisfies the first axiom of countability. It is known that, if G is Hausdorff and satisfies the first axiom of countability and H is closed then G/H is metrizable (Montgomery and Zippin [3, p. 36]). In this paper we obtain a result which includes those of Hudson and Wu and covers some other cases not included in the theorem of Montgomery and Zippin.

For each symmetric neighborhood U of the identity e of G let

$$U^* = \{(Hx,Hy) \mid Hx \subset UHy \text{ and } Hy \subset UHx\},$$
$$U^{\sim} = \{(Hx,Hy) \mid hx \in Uky \text{ for some } h,k \in H\}.$$

The sets U^* form a base for the partition uniformity $\mathcal{U}^*$ on the set G/H and, if condition A below is satisfied, then the sets $U^{\sim}$ form a base for a uniformity $\mathcal{U}^{\sim}$.

(A). For each neighborhood U of e there is a neighborhood V such that $HV \subset UH$.

It is easy to see that $\mathcal{U}^*$ and $\mathcal{U}^{\sim}$ are right invariant in the sense that they have bases, each element of which is right invariant under action of G. It is known that, if H is compact, then the topology t^* induced by $\mathcal{U}^*$ is equal to the quotient topology [1]. We denote by $t^{\sim}$ the topology induced by the sets $U^{\sim}$. A subset of G/H is a neighborhood of Hx in $t^{\sim}$ if it contains a set $U^{\sim}(Hx) = \{Hy \mid (Hx,Hy) \in U^{\sim}\}$. If condition A is satisfied, then $t^{\sim}$ is the topology induced by the uniformity $\mathcal{U}^{\sim}$.

Denote the quotient topology on G/H by q.

LEMMA. *Each of the following three statements implies the other two:* (1) *condition A is satisfied.* (2) $t^{\sim} = q$. (3) $t^* = q$.

PROOF. We note that $t^{\sim} \subset q \subset t^*$. Suppose condition A is satisfied and U^* is any element of the base for $\mathcal{U}^*$. Let V be a symmetric neighbor-

Received March 20, 1964.

hood of e such that $HV \subset UH$. If (Hx, Hy) is in $V^{\sim}$, then $(Hx, Hy) \in U^*$. Hence, $\mathscr{U}^{\sim} \supset \mathscr{U}^*$. Obviously $\mathscr{U}^{\sim} \subset \mathscr{U}^*$; so (1) implies both (2) and (3). Now, suppose $t^{\sim} = q$ and let U be any neighborhood of e. There is a neighborhood V such that $V^{\sim}(H) \subset UH$. But $V^{\sim}(H) = HVH$. Thus, $HV \subset UH$. It follows that (2) implies (1) and (3). The proof that (3) implies (1) and (2) is as routine as the above.

THEOREM. *If G is a topological group and H is a subgroup for which condition* A *is satisfied, then $\mathscr{U}^* = \mathscr{U}^{\sim}$ is a right invariant uniformity for the quotient space G/H. If G/H satisfies the first axiom of countability, this uniformity is pseudo-metrizable (metrizable if G is Hausdorff and H is closed). Since the uniformity is right invariant, the pseudo-metric (metric) can be chosen so that it is right invariant. This metric is unique in the sense that each right invariant metric on G/H has $\mathscr{U}^*$ as its uniformity.*

PROOF. By virtue of the above lemma and remarks it is sufficient to point out that a right invariant uniformity for G/H has a countable base if the topology it induces satisfies the first axiom of countability.

It is obvious that condition A is satisfied when H is compact. Thus, the theorem above includes Wu's and Hudson's theorems. Actually, compactness implies a stronger condition.

REMARK. If C is a compact subset of a topological group G, then, for each neighborhood U of e, there is a neighborhood V of e such that $xV \subset Ux$ for all $x \in C$.

This follows from a straightforward argument using nets.

To see other situations in which the theorem holds we note that, if the left uniformity of G is equal to the right uniformity or the cosets Hx, $x \in G$, form a star-closed partition of G, then condition A is satisfied. If the partition is star-closed and U is an open neighborhood of e, then there is a neighborhood V of e such that HV, the saturation of V, is contained in UH; so condition A is satisfied.

REFERENCES

1. R. W. Bagley, *The uniformizing of partitions*, Amer. Math. Monthly 69 (1962), 767–768.
2. S. N. Hudson, *A note on invariant uniformities in coset spaces*, Math. Scand. 12 (1963), 36–38.
3. D. Montgomery and L. Zippin, *Topological transformation groups*, New York, 1955.
4. T. S. Wu, *Invariant uniformities for coset spaces*, Amer. Math. Monthly (to appear).

UNIVERSITY OF MIAMI, FLA., U.S.A.