MATH. SCAND. 14 (1964), 21–34

ZERO-FREE INTERVALS OF SEMI-STABLE MARKOV PROCESSES

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1. Introduction.

Let $\{x(t)\}$ be a right-continuous stationary strong Markov process starting at the origin. The process is called *semi-stable of order* α (Lamperti [10]) if for any $c>0$ the processes $\{x(ct)\}$ and $\{c^\alpha x(t)\}$ have the same joint distributions at any times $t_1,t_2,\ldots,t_n$. Clearly the familiar stable processes are semi-stable. Let Z be the set of zeros of $x(t)$. Apart from the trivial cases $Z=\{0\}$ a.s. (almost surely), or $Z=[0,\infty)$ a.s., Z is a.s. an unbounded Cantor set of Lebesgue measure zero. These facts as well as a complete determination of the stochastic structure of Z may be found in [10]. Examination of the argument there shows that it applies equally well when Z is taken to be the closure of the *range* of an increasing semi-stable Markov process. The same extension, then, is valid for the results presented here, but for simplicity we hold to the terminology "set of zeros".

In the non-trivial case a.s. the open set $[0,1)\setminus Z$ is the union of countably many open intervals e_n, which we take to be arranged in decreasing order of length L_n. The purpose of this paper is to determine (in terms of suitable transforms) the distribution functions $F_n(x)$ of the lengths L_n. We also determine the "tied-down" distribution functions $F_n^*(x)$; loosely speaking, these are the conditional distributions of the L_n given that $1\in Z$. (In the sequel it will be convenient systematically to affix a star in order to distinguish the tied-down situation from the free one.)

The results obtained include work of Rosén [unpublished] on $F_1^*(x)$ for the tied-down Wiener process, and Lamperti's [9, § 6] formula giving $F_1(x)$ in the general case. It is easy to use the results to obtain the limit theorem of Getoor [5] (cf. also Stone [12]) on the number of $L_n \geqq x$, $x\to 0$, and to obtain the corresponding limit theorem for the tied-down case. The results may have some relation to those of Kesten [7].

The plan of the paper is as follows. Rosén's results are outlined in § 2.

Received March 18, 1964.

[1] Research supported by the U. S. National Science Foundation. A preliminary announcement appeared in [13].

In § 3 we assemble some known facts about Z, while § 4 is devoted to the definition and basic properties of the tied-down set Z^*. $F_n(x)$ and $F_n^*(x)$ are evaluated in § 5. Some auxiliary quantities $M_n^*(x)$ are discussed in § 6; these lead to the limit theorems appearing in § 7.

2. Rosén's results.

Part of the motivation of the present work was the desire to collate two seemingly different expressions for $F_1^*(x)$ obtained by Rosén; for that reason we devote this section to a brief sketch of his methods and results, included with his kind permission.

The basic idea is to approximate the tied-down Wiener process by means of a tied-down random walk, and then pass to the limit. Let S_n denote the sum of independent random variables taking values ± 1 with probabilities $\frac{1}{2}$ each. Let

$$u_N = \Pr\{S_{2N}=0\} = (-1)^N \binom{-\frac{1}{2}}{N},$$

and

$$f_n = \Pr\{\text{first return to zero occurs at time } 2n\} = (-1)^{n-1}\binom{\frac{1}{2}}{n}.$$

Let I_N be the longest interval between zeros of S_n, during time $0 \leqq n \leqq 2N$; we seek $\Pr\{I_N \leqq 2k \mid S_{2N}=0\}$. Write

$$v_{0,k} = 1 \qquad \text{and} \qquad v_{N,k} = \Pr\{S_{2N}=0 \text{ and } I_N \leqq 2k\}.$$

By classifying paths according to times of first return to zero we obtain

$$v_{N,k} = \sum_{n=1}^{k} f_n v_{N-n,k};$$

hence the generating functions

$$v_k(t) = \sum_{N=0}^{\infty} v_{N,k} t^N \qquad \text{and} \qquad f_k(t) = \sum_{n=1}^{k} f_n t^n$$

are connected by the relation $v_k(t) = \big(1 - f_k(t)\big)^{-1}$. By a careful analysis of the latter expression's partial fraction decomposition Rosén obtained

$$\begin{aligned}(2.1) \qquad F_1^*(x) &= \lim_{N\to\infty} \Pr\{I_N \leqq 2[Nx] \mid S_{2N}=0\} \\ &= \lim v_{N,[Nx]}/u_N \\ &= 2\pi x^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} (-s_k) \exp s_k(1+x^{-1}), \qquad 0<x<1 .\end{aligned}$$

The numbers s_k range over the zeros of the confluent hypergeometric function $\Phi(1,\frac{1}{2};s)$ (notation of [4, § 6.1])[1].

A different analysis led Rosén to a surprisingly simple expression for $F_1^*(x)$ on the subinterval $\frac{1}{2}<x<1$. Let

$$w_{N,k} = \Pr\{S_{2N}=0 \text{ and } I_N=2k\}\,.$$

When $I_N=2k>N$ then the longest interval is unique. By classifying paths according to its left-hand endpoint there results

$$w_{N,k} = \sum_{n=0}^{N-k} u_n f_k u_{N-n-k}, \qquad 2k>N\,,$$

and therefore simply

$$w_{N,k} = f_k, \qquad k = [N/2]+1,\ldots,N\,. \tag{2.2}$$

Since $f_{[Nx]}/u_N \sim \frac{1}{2}N^{-1}x^{-3/2}$ it follows from (2.2) that for $x>\frac{1}{2}$

$$\begin{aligned} F_1^*(x) &= 1-\lim_{N\to\infty}\sum_{k=[Nx]}^{N} w_{N,k}/u_N \\ &= 1-\int_x^1 \tfrac{1}{2}y^{-3/2}\,dy = 2-x^{-\frac{1}{2}}, \qquad \tfrac{1}{2}<x<1\,. \end{aligned} \tag{2.3}$$

3. Properties of Z.

Lamperti's results [9] [10] on the structure of Z were obtained by an elegant study of an auxiliary Markov process $y(t)$, whose transition function

$$P(t,x,B) = \Pr\{y(t)\in B \mid y(0)=x\}$$

is that of

$$y_0(t) = t - \sup\{z:\ t\geqq z\in Z\}\,.$$

The process $y(t)$ is stationary, strongly Markov, and has right-continuous sample functions, a.s. A sample path $y(t)$ starting at x consists of a stretch of the 45°-line $y=x+t$ until a jump to 0 takes place, at a random time $t=U$. Thereafter it behaves like a (shifted) copy of $y_0(t)$. The process $y_0(t)$ is semi-stable of order 1, and its set of zeros is precisely Z. Its sample paths consist of slope 1 line segments resting on the t-axis at the points of Z.

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[1] All s_k have negative real parts, and $s_0=-0{,}854\ldots$ is the only real one. Writing $s_{\pm k}=-t_k\pm iu_k$ one has [2, § 17] $u_k\sim 2\pi k$, $t_k\sim 3\log k$. Thanks are due to the Danish computing centre Regnecentralen for the approximate values $s_{\pm 1}=-4{,}25\pm 6{,}38i$, $s_{\pm 2}=-5{,}18\pm 12{,}89i$. Using only s_0 in (2.1) one obtains the approximations 0,97, 0,55 to the exact values $F_1^*(1)=1$, $F_1^*(\frac{1}{2})=2-2^{\frac{1}{2}}\approx 0{,}59$.

The transition function P is uniquely determined up to a parameter $\beta \in (0,1)$; in particular

$$(3.1) \qquad P(t,x,\{x+t\}) = \Pr\{U>t \mid y(0)=x\} = x^{\beta}(x+t)^{-\beta}, \qquad x>0\,,$$

and

$$(3.2) \qquad P(t,0,dy) = \begin{cases} c_\beta(t-y)^{\beta-1}y^{-\beta}\,dy, & 0<y<t \\ 0, & y \geqq t \end{cases}$$

$$(3.3) \qquad c_\beta = (\sin \pi\beta)/\pi\,.$$

Thus $y_0(t)$ has a density, and the Lebesgue measure of Z vanishes, a.s.

Let $0<a<b=t$ and $B=(0,b-a)$. Then $P(t,0,B)$ becomes

$$(3.4) \qquad P\big(b,0,(0,b-a)\big) = p(a,b) = \Pr\{Z \text{ meets } (a,b)\} = c_\beta \int_1^{b/a} y^{-1}(y-1)^{-\beta}\,dy\,.$$

For the Wiener process, $\beta=\frac{1}{2}$, and (3.4) goes back to Lévy [11, § 44]. Blumenthal and Getoor [1] obtained (3.4) for Z = the closure of the range of the drift-free increasing stable process of index β, and for the set of zeros of the symmetric stable process of index $\alpha \in (1,2)$ with the identification $\beta=1-\alpha^{-1}$. (The symmetry restriction was removed in [5, note added in proof].)

It is convenient to note that as $\varepsilon \to 0$

$$(3.5) \qquad p(a,a+\varepsilon) \sim c_\beta(\varepsilon/a)^{1-\beta}$$

uniformly in $a \geqq \delta > 0$.

There are two important first-passage random variables connected with the process $y(t)$: U, the time of first passage from $y=1$ to $y=0$, and T, defined by interchanging "from" and "to". As (3.1) shows, the distribution of U is given by

$$(3.6) \qquad \Pr\{U \in dt\} = \beta(t+1)^{-\beta-1}\,dt, \qquad t>0\,.$$

It follows by an easy calculation that

$$(3.7) \qquad \begin{aligned} E(\exp -sU) &= \Phi - \Psi, \qquad s>0\,, \\ \Phi &= \Phi(s) = \Phi(1,1-\beta;s), \qquad [4;\text{ op. cit.}] \\ \Psi &= \Psi(s) = \Gamma(1-\beta)e^{s}s^{\beta}\,. \end{aligned}$$

Lamperti [8] found the corresponding expression for T as a byproduct of other considerations. We present a direct proof of

THEOREM 1. $E(\exp -sT) = 1/\Phi(s)$.

PROOF. For any random variable Y we abbreviate $E(Y \mid y(0)=x)$ to E_xY. Let g be a real continuous function on $[0,\infty)$ which vanishes at ∞. Define

$$f(x) = \int_0^\infty e^{-st}E_xg(y(t))\,dt, \qquad s>0\,. \tag{3.8}$$

In particular, from (3.2), (3.3) we have

$$f(0) = \Gamma(1-\beta)^{-1}s^{-\beta}\int_0^\infty e^{-sy}y^{-\beta}g(y)\,dy\,. \tag{3.9}$$

By the strong Markov property we can write

$$E_1g(y(t)) = \int_0^t E_0g(y(t-u))\,d\Pr\{U \leqq u\} + g(t+1)\Pr\{U>t\}\,.$$

Taking Laplace transforms yields

$$f(1) = f(0)\,E(\exp -sU) + \int_0^\infty e^{-st}g(t+1)(t+1)^{-\beta}\,dt\,. \tag{3.10}$$

We now make the further assumption

$$g(t) \geqq 0, \qquad g(t) \not\equiv 0, \qquad g(t) \text{ vanishes on } [0,1]\,. \tag{3.11}$$

Then comparison of (3.9) and (3.10) yields

$$\begin{aligned} 0 \neq f(1) &= f(0)\,E(\exp -sU)+f(0)\Psi(s) \\ &= f(0)\Phi(s)\,. \end{aligned} \tag{3.12}$$

On the other hand, the assumption (3.11) implies that if $y(0)=0$ and $T>t$, then $g(y(t))=0$. It follows that

$$E_0g(y(t)) = \int_0^t E_1g(y(t-\tau))\,d\Pr\{T \leqq \tau\}\,.$$

A Laplace transform then gives

$$f(0) = f(1)\,E(\exp -sT)$$

and the result follows from (3.12).

4. The tied-down set of zeros, Z^*.

It will be recalled that one of the equivalent ways of constructing the tied-down Wiener process, x^*, from the free one, x, is that displayed in the equations

$$(4.1)\qquad \begin{aligned} x^*(t^*)/t^* &= x(t)/t \\ t^* &= t/(1+t), \qquad 0 \leqq t < \infty, \qquad 0 \leqq t^* < 1\ , \\ x^*(1) &= 0\ . \end{aligned}$$

Thus, if we agree to call ∞ a zero of $x(t)$ the correspondence $t \leftrightarrow t^*$ carries the zero-sets for the free and tied-down processes onto each other. Although (4.1) seems to have no analogue in the case of other stable processes it motivates the following

DEFINITION. $Z^* = \{z/(1+z) : z \in Z\} \cup \{1\}$.

To show that this definition is "correct" we want to connect it with something that is intuitively closer to the concept of "tying down" the set Z. Let $C(\varepsilon)$ be the event that Z meets the interval $(1, 1+\varepsilon)$; let J denote a finite system of disjoint open intervals in $(0, \infty)$, and let J^* be the system obtained from the correspondence $t^* = t/(1+t)$. Write $p(J)$ for the probability that Z meets each interval of J; this, by the definition of Z^*, equals $p^*(J^*)$, the probability that Z^* meets each interval of J^*. Finally, let $p^\varepsilon(J^*)$ be the conditional probability, given $C(\varepsilon)$, that Z meets each interval of J^*.

THEOREM 2. $\lim_{\varepsilon\to 0} p^\varepsilon(J^*) = p(J)\ \big(= p^*(J^*)\big)$.

PROOF. We give only the first two steps of the required induction on the number of intervals in J.

If $J = (a,b)$ then, by the Markov property,

$$p\big((a^*,b^*),\,(1,1+\varepsilon)\big) = \int_{a^*}^{b^*} p(1-t, 1-t+\varepsilon)\, d_t p(a^*,t)\ .$$

Dividing by $p(1, 1+\varepsilon)$ and letting $\varepsilon \to 0$ it follows from (3.5) that

$$\lim_{\varepsilon\to 0} p^\varepsilon(J^*) = \int_{a^*}^{b^*} (1-t)^{\beta-1}\, d_t p(a^*,t)\ .$$

The change of variable $t = y/(1+y)$ throws the last integral into the integral (3.4) giving $p(a,b)$, that is, $p(J)$.

If $J = \{(a,b),\,(c,d)\}$ then

$$\begin{aligned} &p\big((a^*,b^*),\,(c^*,d^*),\,(1,1+\varepsilon)\big) \\ &\quad = \int_{a^*}^{b^*} p\big((c^*-t, d^*-t),\,(1-t, 1-t+\varepsilon)\big)\, d_t p(a^*,t) \\ &\quad = \int_{a^*}^{b^*} p^*\left(\left(\frac{c^*-t}{1-t}, \frac{d^*-t}{1-t}\right),\, \left(1, 1+\frac{\varepsilon}{1-t}\right)\right) d_t p(a^*,t)\,; \end{aligned}$$

for semi-stability of $y_0(t)$ implies that $p(kJ)=p(J)$, $k>0$. Again we divide by $p(C(\varepsilon))$, let $\varepsilon \to 0$, and use (3.5). By the first part of the proof there results

$$\int_{a^*}^{b^*} p^*\left(\frac{c^*-t}{1-t}, \frac{d^*-t}{1-t}\right) (1-t)^{\beta-1}\, d_t p(a^*,t);$$

the change of variable used above converts this into

$$\int_a^b p(c-y,d-y)\, d_y\, p(a,y) \;=\; p(J)\;,$$

as required.

Analogous calculations show that the random sets Z and Z^* share certain invariance properties with the zero sets of the free and tied-down Wiener processes. For example, if J^{-1} denotes the result of applying the mapping $t \to t^{-1}$ to J it can be shown that $p(J^{-1})=p(J)$, i.e. that Z and Z^{-1} are stochastically the same. Then applying the transformation * we find that $p^*(J^*)=p^*(1-J^*)$, so that Z^* is unchanged by the symmetry $t \to 1-t$. Combining this result with the fact that $t \to kt$ leaves Z stochastically fixed we find that Z^* is stochastically unchanged by any linear fractional transformation f leaving the set $\{0,1\}$ invariant. Proofs are omitted, as we have no further use for the results at present.

Let t be a given point of $(0,1)$, and define

$$a \;=\; \max\{z^*: t \geqq z^* \in Z^*\}\;.$$

It should be clear from the preceeding that the random variable a has a probability density, given by

$$(4.2)\quad f_t^*(a) \;=\; -\frac{d}{da}\,p^*(a,t) \;=\; c_\beta(1-t)^\beta(t-a)^{-\beta}a^{\beta-1}(1-a)^{-1}, \quad 0<a<t<1\;.$$

Theorem 2 shows that for Z^* the probabilities of *finitary* events, those depending on the answers to a finite number of questions of the form "does Z^* meet (a,b)?", can be evaluated as the limits of the corresponding conditional probabilities for Z, using the *same* intervals (a,b). We shall now extend this algorithm to the nonfinitary events $L_n^* \leqq x$, where $L_1^* \geqq L_2^* \geqq \ldots$ run through the lengths of the open intervals constituting $[0,1]\setminus Z^*$. Let

$$F_n^*(x) \;=\; \Pr\{L_n^* \leqq x\} \qquad \text{and} \qquad F_n^\varepsilon(x) \;=\; \Pr\{L_n \leqq x \mid C(\varepsilon)\}\;.$$

An easy argument shows that these are continuous functions.

THEOREM 3. $\lim_{\varepsilon\to 0} F_n^\varepsilon(x) = F_n^*(x)$, $0 \leqq x \leqq 1$.

PROOF. Let J be a system of n disjoint open intervals in $(0,1)$ which have rational endpoints and lengths exceeding x. Let $\{J_k\}$ be an enumeration of all such J, and write W_k or $W_k{}^*$ for the event that at least one interval belonging to J_k meets Z or Z^*, respectively. Let Q_k or $Q_k{}^*$ be the intersection of the first k W's or W^*'s. These are finitary events, and it follows that

$$\Pr\{Q_k{}^*\} = \lim_{\varepsilon\to 0} \Pr\{Q_k \mid C(\varepsilon)\} .$$

Clearly

$$\{L_n \leqq x\} = \bigcap_{k=1}^{\infty} W_k = \bigcap_{k=1}^{\infty} Q_k ,$$

and the same with stars throughout. Therefore

$$\begin{aligned} F_n{}^*(x) = \lim_k \Pr\{Q_k{}^*\} &= \lim_k \lim_\varepsilon \Pr\{Q_k \mid C(\varepsilon)\} \\ &\geqq \overline{\lim}_\varepsilon \lim_k \Pr\{Q_k \mid C(\varepsilon)\} \\ &= \overline{\lim}_\varepsilon F_n{}^\varepsilon(x) , \end{aligned} \tag{4.3}$$

since the Q_k decrease as k increases.

To finish the proof suppose that x_0 is a point and $\{\varepsilon_m\}$ is a sequence tending to zero, such that for some n_0 $F_{n_0}^{\varepsilon_m}(x_0) \nrightarrow F_{n_0}^*(x_0)$, while for each n the function $F_n{}^{\varepsilon_m}(\cdot)$ tends to a (right-continuous) distribution function $G_n(\cdot)$ on its continuity set C_n. Then, by (4.3)

$$\begin{aligned} F_n{}^*(x) &\geqq G_n(x), \qquad x \in C_n \\ F_{n_0}^*(x_0) &> G_{n_0}(x_0 - 0) . \end{aligned} \tag{4.4}$$

Since Z is a Lebesgue null-set with probability one, so are Z^* and Z conditioned by $C(\varepsilon_m)$. Hence

$$\sum_{n=1}^{\infty} E(L_n{}^*) = \sum_{n=1}^{\infty} E(L_n \mid C(\varepsilon)) = 1 .$$

Then

$$\begin{aligned} 1 &= \int_0^1 \sum \left(1 - F_n{}^{\varepsilon_m}(x)\right) dx = \underline{\lim}_{m\to\infty} \int_0^1 \sum \left(1 - F_n{}^{\varepsilon_m}(x)\right) dx \\ &\geqq \int_0^1 \sum \underline{\lim} \left(1 - F_n{}^{\varepsilon_m}(x)\right) dx = \int_0^1 \sum \left(1 - G_n(x)\right) dx , \end{aligned} \tag{4.5}$$

by Fatou's lemma and the fact that $m(\bigcap_{n=1}^{\infty} C_n) = 1$. In view of (4.4) and the continuity of $F_{n_0}^*$ the last member of (4.5) *exceeds*

$$\int_0^1 \sum \left(1 - F_n{}^*(x)\right) dx = 1 .$$

The resulting contradiction completes the argument.

5. The distributions F_n and F_n^*.

We prove the following result by an extension of the method Lamperti [9] used to find $F_1(x)$.

THEOREM 4. (i) $\int_0^\infty e^{-sx}\, d\big(1-F_n(1/x)\big) = (\Phi-\Psi)^{n-1}/\Phi^n$.

(ii) $\int_0^\infty e^{-sx} x^{\beta-1}\big(1-F_n^*(1/x)\big)\, dx = \Gamma(\beta)s^{-\beta}(\Phi-\Psi)^n/\Phi^n$.

PROOF. Clearly

$$\begin{aligned}\Pr\{L_n > 1/x\} &= \Pr\{y_0(t) \text{ crosses } y=1/x \text{ at least } n \text{ times during } 0<t<1\}\\ &= \Pr\{y_0(t) \text{ crosses } y=1 \text{ at least } n \text{ times during } 0<t<x\},\end{aligned}$$

since $y_0(xt)$ and $xy_0(t)$ have the same finite-dimensional distributions. Let $\{T_n\}$ and $\{U_n\}$ be independent copies of the first passage random variables T and U; let $S_n = T_1+U_1+\ldots+T_n+U_n$. Then

$$1-F_n(1/x) = \Pr\{S_{n-1}+T_n \leqq x\}, \tag{5.1}$$

and part (i) follows at once from (3.7) and Theorem 1.

To prove part (ii) we have, by Theorem 3 and the semi-stability of $y_0(t)$,

$$1-F_n^*(1/x) = \lim_{\varepsilon\to 0} \Pr\{S_{n-1}+T_n \leqq x \mid Z \text{ meets } (x,x+\varepsilon)\}.$$

The event $\{S_{n-1}+T_n \leqq x$ and Z meets $(x,x+\varepsilon)\}$ is the union of the disjoint events $A_n = \{S_n \leqq x$ and Z meets $(x,x+\varepsilon)\}$ and $B_n = \{S_{n-1}+T_n \leqq x < S_n < x+\varepsilon\}$. The event B_n is contained in the event $\bar{B} = \{x<S_n<x+\varepsilon\}$. Since the density of U never exceeds β (cf. (3.6)) the same is true of S_n; therefore $\Pr\{\bar{B}_n\} \leqq \beta\varepsilon$. Taking (3.5) into account it follows that $\Pr\{B_n \mid Z \text{ meets } (x,x+\varepsilon)\}$ vanishes with ε.

By the strong Markov property we have

$$\Pr\{A_n\} = \int_0^x p(x-y, x-y+\varepsilon)\, d\Pr\{S_n \leqq y\}.$$

Therefore

$$\begin{aligned}1-F_n^*(1/x) &= \lim_{\varepsilon\to 0} \Pr\{A_n \mid Z \text{ meets } (x,x+\varepsilon)\}\\ &= x^{1-\beta}\int_0^x (x-y)^{\beta-1}\, d\Pr\{S_n \leqq y\}.\end{aligned}$$

A Laplace transformation then yields (ii), and completes the proof.

Putting $n=1$ in part (ii) we obtain

$$\int_0^\infty e^{-sx}x^{\beta-1}F_1{}^*(1/x)\,dx = e^s/c_\beta\Phi;$$

inversion yields

$$F_1{}^*(x) = \beta^{-1}c_\beta{}^{-1}x^{\beta-1}\sum(-s_k)\exp\{s_k(1+x^{-1})\};$$

$\Phi(1,1-\beta;\,s_k)=0$ defines the numbers s_k. This specializes to Rosén's result (2.1) when $\beta=\frac{1}{2}$.

6. The moments M_n^*.

Let $N^*(x)$ be the number of $L_k{}^*$ exceeding x; clearly $N^*(x)\geqq n$ if and only if $L_n{}^*>x$. We define the *modified factorial moments* of $N^*(x)$ by

$$M_n{}^*(x) = E\binom{N^*(x)}{n}, \qquad n = 0,1,2,\ldots;$$

clearly $M_0{}^*(x)=1,\ 0<x<\infty$, while for $n=1,2,\ldots,\ x\geqq 1/n$, $M_n{}^*(x)$ vanishes. We now prove

THEOREM 5. *With $f_t{}^*(a)$ as in (4.2) we have*

$$M^*_{n+1}(x) = \int_0^{1-x} M^*_n(x/t)f^*_{t+x}(t)\,dt, \qquad 0<x<1, \qquad n = 0,1,\ldots \tag{6.1}$$

PROOF. The probability that $N^*(x)$ equals N is evidently $F^*_{N+1}(x)-F_N{}^*(x)$, $N=0,1,\ldots$. Therefore

$$M_n{}^*(x) = \sum_{N=0}^\infty\binom{N}{n}\{F^*_{N+1}(x)-F^*_N(x)\},$$

which is really a finite sum, since the term in braces vanishes as soon as $Nx>1$. Applying Theorem 4(ii) we obtain

$$\int_0^\infty e^{-sx}x^{\beta-1}M_n{}^*(1/x)\,dx = \Gamma(\beta)s^{-\beta}(\Phi\Psi^{-1}-1)^n.$$

Therefore the function $x^{\beta-1}M^*_{n+1}(1/x)$ is obtained from its predecessor by convoluting the latter with the inverse Laplace transform of $\Phi\Psi^{-1}-1$. It is not difficult to verify that the inverse transform is $c_\beta x^{-1}(x-1)^\beta$, $x>1$, zero elsewhere. The desired conclusion now follows from elementary calculations.

There is a similar but more clumsy formula in the free case, which we omit. There is a loss of symmetry due to the presence of the zero-free interval containing $t=1$, which is no longer an interval *between* zeros belonging to $[0,1]$.

Rosén's second formula (2.3) for $F_1{}^*(x)$ in the Wiener case is a consequence of Theorem 5. Putting $\beta=\frac{1}{2}$ and $n=0$ in (6.1) we obtain

$$E(N^*(x)) = M_1{}^*(x) = x^{-\frac{1}{2}}-1, \qquad 0<x<1 .$$

When $x>\frac{1}{2}$ the only possible values of $N^*(x)$ are 0 and 1. Hence also

$$E(N^*(x)) = 1-F_1{}^*(x), \qquad \tfrac{1}{2}<x<1 ,$$

and (2.3) is immediate. In principle the method could be applied to permit computation of the first n of the $F_k{}^*(x)$ on the interval $1/(n+1)<x<1$ from the first n moments $M_k{}^*(x)$.

The steps leading from Theorem 4(ii) to Theorem 5 can evidently be reversed, and it therefore seems worthwhile to sketch in a heuristic way a direct probabilistic proof of Theorem 5.

We first obtain $M_1{}^*(x)$. Let the interval $0<t<1$ be partitioned into small subintervals dt_j. Call dt_j *good* if it contains the left endpoint a_k of some zero-free interval $e_k=(a_k,b_k)$ for which $L_k{}^*=b_k-a_k>x$. Since

$$\Pr\{dt_j \text{ is good}\} \approx f^*_{t_j+x}(t_j)\, dt_j$$

and $M_1{}^*(x)$ is the expected number of good dt_j, it follows on letting $\max_j dt_j \to 0$ that

$$M_1^*(x) = \int_0^{1-x} f^*_{t+x}(t)\, dt ,$$

which is (6.1) for $n=0$.

Knowing $M_n{}^*(x)$ we step up to $M^*_{n+1}(x)$ in the following way. For each good dt_j let N_j be the number of e_k with $L_k{}^*>x$ that are situated in $(0,t_j)$. As we sweep through the good dt_j from left to right the integers N_j run from 0 to $N^*(x)-1$. In view of the elementary identity

$$\binom{N}{n+1} = \sum_{r=0}^{N-1} \binom{r}{n}$$

it follows that

$$\binom{N^*(x)}{n+1} = \sum_j \binom{N_j}{n} Y_j \tag{6.2}$$

where Y_j is the indicator of the event "dt_j good". We use the Markov property to write

$$E\left[\binom{N_j}{n} Y_j\right] \approx E\left[\binom{N_j}{n} \mid t_j \in Z^*\right] \cdot E(Y_j)\,,$$

and evaluate the first factor by stretching the time-scale with the factor $1/t_j$. It follows that the first factor is approximately

$$E\left[\binom{N^*(x/t_j)}{n}\right] = M_n{}^*(x/t_j)\,.$$

Putting these estimates into (6.2) and letting $\max dt_j$ approach zero we obtain (6.1) again.

7. Limit theorems.

In this section we present the limiting distributions of $N(x)$ and $N^*(x)$ as $x \to 0$. The free case is due to Getoor [5], but the proof is different.

THEOREM 6. *We have*

(i) $$\lim_{x\to 0} \Pr\{\Gamma(1-\beta)x^\beta N(x) < y\} = G_\beta(y)\,,$$

the Mittag–Leffler distribution of index β;

(ii) $$\lim_{x\to 0} \Pr\{\Gamma(1-\beta)x^\beta N^*(x) < y\} = G_\beta{}^*(y)\,,$$

defined by

(7.1) $$dG_\beta{}^*(y) = \Gamma(1+\beta)\, y\, dG_\beta(y), \qquad 0 < y < \infty\,.$$

PROOF. Getoor used the method of moments to prove (i), showing that

$$\lim_{x\to 0} E\big(\{\Gamma(1-\beta)x^\beta N(x)\}^n\big) = n!/\Gamma(1+n\beta)\,,$$

the moment-sequence that uniquely determines $G_\beta(y)$. It is illuminating, however, to obtain the result directly from a limit theorem for partial sums. In (5.1) we set $x = \{\Gamma(1-\beta)n/y\}^{1/\beta}$ and subtract both sides from unity. There results

(7.2) $$\Pr\{\Gamma(1-\beta)x^\beta N(x) < y\} = \Pr\{S_{n-1} + T_n > [\Gamma(1-\beta)n/y]^{1/\beta}\}\,.$$

Let $V_n = U_1 + U_2 + \ldots + U_n$. It follows from (3.6) and [6, § 35] that $V_n/[\Gamma(1-\beta)n]^{1/\beta}$ tends in distribution to the stable random variable X_β defined by $E(\exp - sX_\beta) = \exp - s^\beta$. Since $E(T) = \Phi'(0) = 1/(1-\beta)$ is finite, the law of large numbers implies that $(T_1 + T_2 + \ldots + T_n)/n^{1/\beta}$ tends to zero with probability one. Combining these facts with (7.2) and letting $x \to 0$, $n \to \infty$ in such a way that y remains fixed we get

$$\Pr\{\Gamma(1-\beta)x^\beta N(x) < y\} \to \Pr\{X_\beta > y^{-1/\beta}\} = \Pr\{X_\beta{}^{-\beta} < y\}\,.$$

This is the desired result (i), for an easy calculation shows that the moments of $X_\beta^{-\beta}$ are precisely those defining G_β.

We could follow somewhat the same line of argument to prove part (ii), but it is simpler to work from the moments. Writing out (6.1) in explicit form we obtain (cf. (4.2))

$$M_{n+1}^*(x) = c_\beta x^{-\beta} \int_0^{1-x} M_n{}^*(x/t)(1-t-x)^\beta t^{\beta-1}\, dt/(1-t)$$

from which it follows by induction that

$$\lim_{x\to 0} x^{n\beta} M_n{}^*(x) = \Gamma(\beta)/\{\Gamma((n+1)\beta)\Gamma(1-\beta)^n\}. \tag{7.3}$$

But then another induction shows that $M_n{}^*(x) \sim E(N^*(x)^n)/n!$ as $x \to 0$. It follows from (7.3) that

$$\lim_{x\to 0} E(\{\Gamma(1-\beta)x^\beta N^*(x)\}^n) = n!\,\Gamma(\beta)/\Gamma((n+1)\beta),$$

which uniquely determine the distribution function $G_\beta{}^*$ given by (7.1). This completes the proof of (ii). Analogous pairs of limit theorems for certain free and tied-down random variables may be found in [3]; the same limiting distributions G_β and $G_\beta{}^*$ appear there. It is natural to ask whether there is a natural continuum of limit theorems lying between the free and tied-down cases. For example, one might tie down at a point $a > 1$ rather than at 1 itself, and perhaps obtain limiting distributions $G_\beta{}^{(a)}$ tending to G_β as $a \to \infty$ and to $G_\beta{}^*$ as $a \to 1$.

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