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APPROXIMATION THEOREMS OF BOREL AND FUJIWARA

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1. Introduction.

The purpose of this note is to give a unified proof of one theorem of Borel [1] and two theorems of Fujiwara [2] [3], stated as theorem I, II, III, respectively, below. However, I believe that the part of theorem III dealing with the approximation of a rational number ξ is new. The method of proof is based on an idea of Khintchine [4].

For any real number ξ let $[a_0, a_1, \ldots]$ be the regular continued fraction expansion (if ξ is rational, we consider any of the two continued fraction expansions), and let $p_0/q_0, p_1/q_1, \ldots$ be the corresponding convergents. The properties of continued fractions to be used are

(1) a_0 is an integer; a_n is a positive integer, $n \geqq 1$.
(2) $q_{n+1} = a_{n+1}q_n + q_{n-1}$, $n \geqq 1$, with $q_0 = 1$, $q_1 = a_1$.
(3) $p_n q_{n-1} - p_{n-1} q_n = \pm 1$, $n \geqq 1$.
(4) ξ lies between two consecutive convergents.

With this notation theorems I, II, III may be formulated as follows.

THEOREM I. *At least one of* p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1}, $n \geqq 1$, *satisfies the inequality*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{5^{\frac{1}{2}} q^2}. \tag{5}$$

THEOREM II. *If* $a_{n+1} \geqq 2$, $n \geqq 1$, *then at least one of* p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1} *satisfies the inequality*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{8^{\frac{1}{2}} q^2}. \tag{6}$$

THEOREM III. *If* $a_{n+1} \geqq 2$, $n \geqq 1$, *then either* p_n/q_n *or both of* p_{n-1}/q_{n-1}, p_{n+1}/q_{n+1} *satisfy the inequality*

$$\left|\xi - \frac{p}{q}\right| \leqq \frac{1}{\frac{5}{2} q^2}, \tag{7}$$

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where the equality sign can only occur if $\xi = a_0 + \frac{2}{5}$, $n = 1$, *or* $\xi = a_0 + \frac{3}{5}$, $n = 2$, *and for such* ξ *only if the shorter form of the two continued fraction expansions is considered.*

2. Two lemmas.

LEMMA 1. *Let* q, q' *be positive integers. Then*

$$\frac{1}{qq'} < \frac{1}{K}\left(\frac{1}{q^2}+\frac{1}{q'^2}\right) \tag{8}$$

whenever $q'/q > f(K)$ *or* $q/q' > f(K)$, *where* $f(K) = \frac{1}{2}(K + (K^2-4)^{\frac{1}{2}})$. *In particular* $f(5^{\frac{1}{2}}) = \frac{1}{2}(5^{\frac{1}{2}}+1)$, $f(8^{\frac{1}{2}}) = 2^{\frac{1}{2}}+1$, $f(\frac{5}{2}) = 2$.

PROOF. (8) is equivalent to $(q'/q)^2 - Kq'/q + 1 > 0$, which immediately yields the lemma.

LEMMA 2. *Let* $p/q \leqq \xi \leqq p'/q'$, *where* p, p', q, q' *are integers with* q, $q' > 0$ *and* $p'q - pq' = 1$. *If either* $q'/q > f(K)$ *or* $q/q' > f(K)$, *then either*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{Kq^2} \quad \textit{or} \quad \left|\xi - \frac{p'}{q'}\right| < \frac{1}{Kq'^2}.$$

PROOF. By lemma 1, q, q' satisfy (8), i.e.

$$\frac{p'}{q'} - \frac{p}{q} = \frac{1}{qq'} < \frac{1}{K}\left(\frac{1}{q^2}+\frac{1}{q'^2}\right),$$

whence

$$\frac{p}{q} + \frac{1}{Kq^2} > \frac{p'}{q'} - \frac{1}{Kq'^2}.$$

This proves lemma 2.

3. Proof of theorems I, II, III.

1) If $q_n/q_{n-1} > \frac{1}{2}(5^{\frac{1}{2}}+1)$, either p_{n-1}/q_{n-1} or p_n/q_n satisfies (5) by (3), (4) and lemma 2 ($K = 5^{\frac{1}{2}}$). If on the contrary $q_n/q_{n-1} < \frac{1}{2}(5^{\frac{1}{2}}+1)$, then $q_{n-1}/q_n > \frac{1}{2}(5^{\frac{1}{2}}-1)$. Hence $q_{n+1}/q_n = a_{n+1} + q_{n-1}/q_n > 1 + \frac{1}{2}(5^{\frac{1}{2}}-1) = \frac{1}{2}(5^{\frac{1}{2}}+1)$ by (1), (2), consequently in this case either p_n/q_n or p_{n+1}/q_{n+1} satisfies (5). This proves theorem I.

2) If $q_n/q_{n-1} > 2^{\frac{1}{2}}+1$, either p_{n-1}/q_{n-1} or p_n/q_n satisfies (6) by (3), (4) and lemma 2 ($K = 8^{\frac{1}{2}}$). If on the contrary $q_n/q_{n-1} < 2^{\frac{1}{2}}+1$, then $q_{n-1}/q_n > 2^{\frac{1}{2}}-1$. Hence

$$q_{n+1}/q_n = a_{n+1} + q_{n-1}/q_n > 2 + 2^{\frac{1}{2}} - 1 = 2^{\frac{1}{2}}+1$$

by (2) and the assumption $a_{n+1} \geqq 2$ of theorem II, consequently in this case either p_n/q_n or p_{n+1}/q_{n+1} satisfies (6). This proves theorem II.

3) If p_n/q_n satisfies (7) with strict inequality, we are finished. On the contrary assume that

$$\left|\xi - \frac{p_n}{q_n}\right| \geqq \frac{1}{\frac{5}{2}q_n^{\,2}},$$

then by (3), (4)

$$\frac{1}{q_n q_{n+1}} = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \left|\xi - \frac{p_n}{q_n}\right| + \left|\xi - \frac{p_{n+1}}{q_{n+1}}\right| \geqq \frac{1}{\frac{5}{2}q_n^{\,2}}$$

or $q_{n+1}/q_n \leqq \frac{5}{2}$, with strict inequality unless $\xi = p_{n+1}/q_{n+1}$ and

$$\left|\xi - \frac{p_n}{q_n}\right| = \frac{1}{\frac{5}{2}q_n^{\,2}}.$$

Now $q_{n+1}/q_n = a_{n+1} + q_{n-1}/q_n$, hence $a_{n+1} = 2$ and $q_{n-1}/q_n \leqq \frac{1}{2}$ with strict inequality unless $\xi = p_{n+1}/q_{n+1}$ and $q_{n+1}/q_n = \frac{5}{2}$.

In any case $q_{n+1}/q_n > 2$, so by lemma 2 ($K = \frac{5}{2}$) either p_n/q_n or p_{n+1}/q_{n+1} satisfies (7) with strict inequality, i.e. p_{n+1}/q_{n+1} does so, since p_n/q_n does not by assumption. If further $q_{n-1}/q_n < \frac{1}{2}$, $q_n/q_{n-1} > 2$, so by lemma 2 ($K = \frac{5}{2}$) either p_{n-1}/q_{n-1} or p_n/q_n satisfies (7) with strict inequality, i.e. p_{n-1}/q_{n-1} does so, since p_n/q_n does not by assumption. This proves the main case of theorem III.

There remains only to show that $\xi = p_{n+1}/q_{n+1}$, $q_{n+1}/q_n = \frac{5}{2}$, $a_{n+1} = 2$, $q_{n-1}/q_n = \frac{1}{2}$ leads to the exceptional case of theorem III.

By (3) q_{n-1}, q_n are relatively prime and by (1), (2) $1 = q_0 \leqq a_1 = q_1 < q_2 < q_3 < \ldots$. This requires $q_{n-1} = 1$, $q_n = 2$, $q_{n+1} = 5$ and either $n = 1$ or $n = 2$ in which case $a_1 = 1$. Hence by (2) either $\xi = [a_0, 2, 2] = a_0 + \frac{2}{5}$ or $\xi = [a_0, 1, 1, 2] = a_0 + \frac{3}{5}$, where a_0 is an integer. In the first case

$$\frac{p_0}{q_0} = \frac{a_0}{1}, \qquad \frac{p_1}{q_1} = \frac{2a_0 + 1}{2}$$

both satisfy (7) with equality. Similarly with

$$\frac{p_1}{q_1} = \frac{a_0 + 1}{1}, \qquad \frac{p_1}{q_2} = \frac{2a_0 + 1}{2}$$

in the second case. This completes the proof of theorem III.

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