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STABILITY OF LINEAR DIFFERENTIAL EQUATIONS IN BANACH ALGEBRAS

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Introduction.

We consider a Banach algebra A over the real or complex numbers with identity element (denoted e). All norms, $N(\cdot)$, are assumed to be *natural*, that is $N(xy) \leqq N(x)N(y)$ and $N(e)=1$. We study differential equations of the type

$$(1) \qquad u'(t) = a(t)u(t) \qquad \text{where} \qquad u(0)=e .$$

Here u and a are functions from the positive reals R^+ into A. $u'(t)$ is defined by the relation

$$N\left(\frac{u(t+h)-u(t)}{h} - u'(t)\right) \to 0 \quad \text{as} \quad h \to 0 .$$

We always assume that $a(t)$ is Bochner integrable (see [2, p. 79]) over any finite interval of R^+. This implies that there exists a unique solution of (1) (see [6, p. 521]). The differential equation (1) is said to be *stable* if there exists a constant M such that

$$N\big(u(t)\big) \leqq M \qquad \text{for all} \quad t \geqq 0 .$$

It is clear that stability is a topological property, i.e. it is preserved if we introduce an equivalent norm of A. It is therefore natural to try to find *topological* conditions on $a(t)$ making (1) stable.

The first part considers the case when $a(t)$ is constant. Here we can solve (1): $u(t)=\exp(ta)$. The spectrum $\sigma(a)$ of a, is independent of the norm and we find some sufficient conditions for stability, generalizing well known theorems for matrix algebras.

We introduce the *Gâteaux differential* belonging to a norm N

$$\Phi_N(a) = \lim_{\alpha \to +0} \frac{N(e+\alpha a)-1}{\alpha} .$$

A technical condition for stability is obtained:

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THEOREM 3. *The function* $\exp(ta)$ *is bounded for* $t \geqq 0$ *if and only if there exists a norm* N *such that* $\Phi_N(a) \leqq 0$.

In the second part, $a(t)$ is not necessarily constant. We prove

THEOREM 4. $N(u(t)) \leqq \exp \int_0^t \Phi_N(a(s))\, ds$.

From this theorem we get some corollaries concerning stability of (1).

1. $a(t)$ constant.

We study relations between the situation of the spectrum $\sigma(a)$ in the complex plane and stability of (1). If $\sigma(a)$ contains a point λ_0 with $\operatorname{Re}\lambda_0 > 0$ then

$$N(\exp(ta)) \geqq \nu(\exp(ta)) \geqq \exp(t \operatorname{Re}\lambda_0) ,$$

where $\nu(x) = \sup\{|\lambda| \mid \lambda \in \sigma(x)\}$ is the spectral radius of x. Hence

$$\sigma(a) \subset \{\lambda \mid \operatorname{Re}\lambda \leqq 0\}$$

is necessary for stability. We also have the following sufficient condition.

THEOREM 1. *If* $\sigma(a) \subset \{\lambda \mid \operatorname{Re}\lambda < 0\}$ *then* $\exp(ta) \to 0$ *when* $t \to \infty$.

PROOF. Put $b = \exp(a)$. There exists a $\delta > 0$ such that

$$\sigma(b) \subset \{\lambda \mid |\lambda| \leqq \exp(-\delta)\} .$$

We get

$$\nu(b) = \lim_{n\to\infty} (N(b^n))^{1/n} = \sup_{\lambda\in\sigma(b)} |\lambda| \leqq e^{-\delta} < 1 .$$

It follows that $N(b^n) \to 0$ as $n \to \infty$ and this implies $\exp(ta) \to 0$ as $t \to \infty$.

If $\sigma(a)$ contains points on the imaginary axis the problem of finding a sufficient condition of stability is more difficult. Theorem 2 is a generalization of a well known theorem for matrix algebras.

THEOREM 2. *Let* $A = B(X)$ *be the algebra of bounded linear operators from a Banach space* X *into itself.*

Let further $a = b + c$ *be a spectral operator in* A *with resolution* $E(\Delta)$ *of the identity,* b *the scalar and* c *the quasinilpotent part of* a. *If*

$$\sigma(a) \subset \{\lambda \mid \operatorname{Re}\lambda \leqq 0\} ,$$

and

$$\Delta_1 = \sigma(A) \cap \{\lambda \mid \operatorname{Re}\lambda = 0\}$$

is both open and closed as a subset of $\sigma(A)$ *and* $cE(\Delta_1) = 0$, *then* $\exp(ta)$ *is bounded for* $t \geqq 0$.

PROOF. The set $\Delta_2 = \sigma(A) - \Delta_1$ is a closed subset of $\{\lambda \mid \operatorname{Re}\lambda < 0\}$. We have a decomposition $a = a_1 + a_2$ where $a_k = a \mid E(\Delta_k)X$, $k = 1, 2$. Since a_1 is scalar we get from the definition of spectral operator (cf. [2])

$$\sigma(a_1) \subset \bar{\Delta}_1 = \Delta_1 \subset \{\lambda \mid \operatorname{Re}\lambda = 0\},$$
$$\sigma(a_2) \subset \bar{\Delta}_2 = \Delta_2 \subset \{\lambda \mid \operatorname{Re}\lambda < 0\},$$

and

$$N(\exp(ta_1)) \leqq \text{const}\cdot\sup_{\lambda\in\sigma(a_1)}|\exp(t\lambda)| = \text{const}.$$

Theorem 1 implies that $\exp(ta_2)$ is bounded and this finishes the proof.

We now consider the Gateau differential

$$\Phi_N(a) = \lim_{\alpha\to+0} \frac{N(e+\alpha a) - 1}{\alpha}. \tag{2}$$

We write down the following simple consequences of the definition (see [4, p. 25]):

$$|\Phi_N(a)| \leqq N(a), \tag{3}$$

$$\Phi_N(a+b) \leqq \Phi_N(a) + \Phi_N(b), \tag{4}$$

$$\Phi_N(a) = \inf_{\alpha>0} \frac{N(e+\alpha a) - 1}{\alpha} = \lim_{\alpha\to+0} \frac{\log N(\exp \alpha a)}{\alpha}. \tag{5}$$

The next theorem is a precise characterization of those constants a for which (1) is stable.

THEOREM 3. *The function* $\exp(ta)$ *is bounded for* $t \geqq 0$ *if and only if there exists a norm* N *for which* $\Phi_N(a) \leqq 0$.

PROOF. Assume that $N(\exp(ta)) \leqq K$ for all $t \geqq 0$. By a construction due to L. Ingelstam (see [4, p. 26]) it is possible to find an equivalent norm N_2 such that $N_2(\exp(ta)) \leqq 1$ for all $t \geqq 0$. Put

$$N_1(x) = \sup_{t\geqq 0} N(\exp(ta)\,x).$$

Then

$$N(x) \leqq N_1(x) \leqq K\cdot N(x) \qquad \text{for all } x,$$

that is, N and N_1 are equivalent, but $N_1(e) \neq 1$ in general. We therefore introduce

$$N_2(x) = \sup_{y\neq 0} \frac{N_1(xy)}{N_1(y)}.$$

It is easily seen that N_2 is natural, equivalent to N and

$$N_2(\exp(ta)) \leqq 1 \qquad \text{for all} \quad t \geqq 0.$$

From (5) we then get

$$\Phi_{N_2}(a) = \lim_{t\to+0} \frac{\log N_2(\exp ta)}{t} \leqq 0 ,$$

which proves the necessity.

If $\Phi_N(a) \leqq 0$ then Corollary 1 of Theorem 4 (see below) implies that $N(\exp(ta)) \leqq 1$ for all $t \geqq 0$, so the condition is sufficient.

2. $a(t)$ not necessarily constant.

We first observe that $a(t)$ Bochner integrable over any finite interval implies that $\Phi(a(t))$ is integrable over the same type of intervals. This follows from the fact that $\Phi_N(a(t)) \leqq N(a(t))$ where $N(a(t))$ is integrable and that $\Phi_N(a)t))$ is an infimum of a sequence of measurable functions (5).

THEOREM 4. $N(u(t)) \leqq \exp \int_0^t \Phi_N(a(s))\, ds.$

PROOF. For a fixed $t > 0$ we construct the solution by successive approximations. Consider a partition

$$0 = t_0 < t_1 < \ldots < t_n = t .$$

We define the following step functions:

$$b_n(s) = a(\xi_k), \qquad g_n(s) = \Phi(a(\xi_k)) ,$$

$$(6) \qquad v_n(s) = \exp((s-t_k)a(\xi_k)) \cdot \exp((t_k - t_{k-1})a(\xi_{k-1})) \ldots \exp(t_1 a(\xi_0))$$

for s satisfying $t_k \leqq s < t_{k+1}$. Here $t_k \leqq \xi_k < t_{k+1}$. Integrating $v(s)$ we get

$$v_n(t) = e + \int_0^t b_n(s) v_n(s)\, ds .$$

We have a similar equation for $u(t)$ and using a well known lemma (see [1, p. 35]) we find

$$N(u(t)-v_n(t)) \leqq \sup_{0\leqq s\leqq t} N(u(s)) \cdot \int_0^t N(a(s)-b_n(s))ds \cdot \exp \int_0^t N(b_n(s))\, ds .$$

By choosing a convenient partition we can make

$$\left| \int_0^t (\Phi(a(s)) - g_n(s))\, ds \right| \qquad \text{and} \qquad \int_0^t N(a(s)-b_n(s))\, ds ,$$

and hence $N(u(t)-v_n(t))$, as small as we please. From (6) we get

$$\log N(v_n(t)) \leqq \sum_{k=0}^{n-1} \frac{\log N(\exp((t_{k+1}-t_k)\, a(\xi_k)))}{t_{k+1}-t_k} (t_{k+1}-t_k) .$$

If necessary dividing the intervals $[t_k, t_{k+1})$ further (leaving $a(\xi_k)$ unchanged in the whole interval $[t_k, t_{k+1})$) and using (5) we can also obtain

$$\log N(v_n(t)) - \int_0^t g_n(s)\, ds < \varepsilon \quad \text{for an arbitrary} \quad \varepsilon > 0 .$$

From these facts it follows that

$$N(u(t)) \leqq \exp \int_0^t \Phi(a(s))\, ds ,$$

which ends the proof.

Corollary 1. *If there exists a norm N such that*

$$\limsup_{t \to \infty} \int_0^t \Phi_N(a(s))\, ds < \infty ,$$

then $u' = a(t)u$ is stable.

From (3) we get

Corollary 2. *If $\int_0^\infty N(a(t))\, dt < \infty$ for some norm N then $u' = a(t)u$ is stable.*

Corollary 3. *Let A be a C^*-algebra. If $a(t)$ is normal $(aa^* = a^*a)$ for every t and*

$$\sigma(a(t)) \subset \{\lambda \mid \operatorname{Re}\lambda \leqq f(t)\} \qquad \text{where} \qquad \limsup_{t \to \infty} \int_0^t f(s) ds < \infty ,$$

then $u' = a(t)u$ is stable.

Proof. If a is normal then $\exp(\alpha a)$ is also normal and we get

$$N(\exp(\alpha a(t))) = \nu(\exp(\alpha a(t))) \leqq \exp(\alpha f(t))$$

for the C^*-norm N when $\alpha > 0$. Applying (5) we have

$$\Phi(a(t)) = \lim_{\alpha \to +0} \frac{\log N(\exp(\alpha a(t)))}{\alpha} \leqq f(t)$$

and

$$\int_0^t \Phi_N(a(s))\, ds \leqq \int_0^t f(s)\, ds .$$

Finally we use Corollary 1 and the proof is finished.

Remark. If $a(t)$ is not normal the spectrum does not tell us much about the stability. It may for instance happen that $\sigma(a(t)) \subset$

$\{\lambda \mid \operatorname{Re}\lambda = -\frac{1}{4}\}$ for all t and $u' = a(t)u$ has unbounded solutions (see [5, p. 310]).

THEOREM 5. *Assume that $u' = a(t)u$ is stable. If there exists a norm N such that*

$$\limsup_{t\to\infty} \int_0^t \Phi_N(-a(s))\,ds < \infty \quad \text{and} \quad \int_0^\infty N(b(t))\,dt < \infty ,$$

then $u' = (a(t)+b(t))u$ is stable.

PROOF. Let $u_0(t)$ be the bounded solution of $u_0' = a(t)u_0$ with $u_0(0) = e$ as usual. We have the formula

$$u(t) = u_0(t) + \int_0^t u_0(t)u^{-1}(s)b(s)u(s)\,ds , \tag{7}$$

which one easily verifies by calculating the derivatives of both sides (for the existence of $u_0^{-1}(s)$ see [6, p. 521]). Differentiating u_0^{-1} we get

$$(u_0^{-1})' = u_0^{-1}(-a(t)) .$$

The same proof as of Theorem 4 can be used to show that

$$N(u_0^{-1}(t)) \leqq \exp \int_0^t \Phi(-a(s))\,ds \leqq C ,$$

a constant independent of t. From (7) we get (using the lemma again)

$$N(u(t)) \leqq N(u_0(t))\cdot\exp\left\{C\cdot \sup_{0\leqq s\leqq t} N(u_0(s))\cdot \int_0^t N(b(s))\,ds\right\} ,$$

and from the conditions we deduce that $u(t)$ is bounded.

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