MATH. SCAND. 14 (1964), 65–74

ON THE CONSISTENCY OF THE AXIOM OF COMPREHENSION IN THE ŁUKASIEWICZ INFINITE VALUED LOGIC

JENS ERIK FENSTAD

In "naive", i.e. ordinary mathematics the "natural" principle of set existence is the axiom (or rather axiom scheme) of comprehension: Every "well-defined property" determines a set.

In axiomatic set theory one would like to identify "well-defined property" with what can be expressed by a formula build up from the identity and the membership relation by use of the customary logical connectives. However, on the basis of the classical two-valued logic the axiom of comprehension cannot consistently be maintained,—the Russel paradox, involving the set of all sets x such that $x \notin x$, being the well known counterexample.

At least two ways out of the dilemma present itself, either restrict the axiom scheme, or keep the axiom but change the logic. Mathematicians take the first course. The logician is free to take the second, seing if a logic incorporating the axiom of comprehension consistently can be constructed.

Some work has from time to time been done on this topic. Recently Th. Skolem [2] and C. C. Chang [1] have investigated the situation within the infinite valued logic of Łukasiewicz. To give these results and to state the theorems proved in this paper we briefly describe the logic of Łukasiewicz.

The logic L has the following primitive symbols. Propositional connectives are $\vee$, $\neg$ and $\rightarrow$. The quantifier is $\exists$. The only predicate is the membership relation $\in$. The class of formulas is inductively defined as usual. We are also going to use a logic L_i obtained from L by adding the identity relation $\equiv$.

The intended interpretation will be described through the notion of model. A *model* is a pair $M = \langle S, e \rangle$ where S is some set and $e : S^2 \rightarrow I$, where I is the closed interval $[0,1]$ of the real line. We here treat the logic L_i, leaving the obvious modification concerning the logic L to the

Received February 24, 1964.

reader. An *interpretation* of $Ł_i$ into a model M is a map $w : V \to S$, where V is the set $\{v_1, v_2, \ldots, v_n, \ldots\}$ of all variables of $Ł_i$. The truth-value of each formula U under the interpretation w will be defined inductively. If $a, b \in S$ we set $d_{aa} = 1$ and $d_{ab} = 0$, if $a \neq b$. The definition of truth-value is then as follows:

i. $w(x \in y) = e(w(x), w(y))$;
ii. $w(x \equiv y) = d_{w(x),\, w(y)}$;
iii. $w(\neg U) = 1 - w(U)$;
iv. $w(U_1 \vee U_2) = \max\{w(U_1), w(U_2)\}$;
v. $w(U_1 \to U_2) = \min\{1, 1 - w(U_1) + w(U_2)\}$;
vi. $w(\exists x U) = \max_{w'}\{w'(U)\}$;

where $w' : V \to S$ satisfies $w'(y) = w(y)$ for all $y \neq x$. In this way w associates with each formula U of $Ł_i$ a definite numerical value in the range I, called the truth-value of U under the interpretation w of $Ł_i$ in the model $M = \langle S, e \rangle$.

A set Δ of formulas of $Ł_i$ (or $Ł$) is called satisfiable if there exists a model M and an interpretation w into M such that $w(U) = 1$ for all $U \in \Delta$. Δ is called *consistent* if it is satisfiable.

THEOREM A. *Let Δ_0 be the set of all quantifier free (open) formulas $U(t, y, x_1, \ldots, x_n)$ of $Ł_i$ having at most the variables $t, y, x_1, \ldots, x_n$ free. Then the set of all formulas*

$$\forall x_1 \ldots \forall x_n \exists y \forall t\, (t \in y \leftrightarrow U(t, y, x_1, \ldots, x_n)),$$

where $U \in \Delta_0$, is consistent within $Ł_i$.

This result, restricted to the logic $Ł$, was proved in Skolem [2] by a combinatorial argument. In this paper we are going to present a short proof of theorem A using an extension of the Brouwer fixed point theorem.

Our methods are inspired by the paper Chang [1] in which the main theorem is the following result:

THEOREM B. *Let Δ_1 be the set of all formulas $U(t, y, x_1, \ldots, x_n)$ of $Ł_i$ with at most the variables $t, y, x_1, \ldots, x_n$ free and such that in every atomic formula $u \in v$ of U, if u is a bound variable of U then $u = v$. Then the set of all formulas*

$$\forall x_1 \ldots \forall x_n \exists y \forall t\, (t \in y \leftrightarrow U(t, y, x_1, \ldots, x_n)),$$

where $U \in \Delta_1$, is consistent within $Ł_i$.

It is easily seen that neither the axiom of infinite union, nor the axiom of powerset are covered by this theorem. On the other hand theorem B includes theorem A. We are going to prove a theorem C which covers the axiom of infinite union, but which does not include theorem A. It does include instances which are contradictory within classical logic. For technical reasons we state our theorem for the logic $\mathcal{L}$. An analogous result could be proved for the logic $\mathcal{L}_i$. However, both versions are of a rather special nature, so we feel justified in keeping to the simpler situation.

THEOREM C. *Let Δ_2 be the set of all formulas $U(t,y,x_1,\ldots,x_n)$ of $\mathcal{L}$ with at most the variables $t,y,x_1,\ldots,x_n$ free and such that the variable t of U can only occur as a variable u of an atomic formula $u \in v$ of U. Then the set of all formulas*

$$\forall x_1 \ldots \forall x_n \exists y \forall t \left(t \in y \leftrightarrow U(t,y,x_1,\ldots,x_n)\right),$$

where $U \in \Delta_2$ is consistent within $\mathcal{L}$.

The restriction on the variable t is easy to state, but quite serious as regard applications of the theorem. The axiom of infinite union is included as the U in this case is the formula $\exists z\ (t \in z \wedge z \in x)$. The axiom of powerset is not covered as the U this time should be the formula $\forall z\ (z \in t \rightarrow z \in x)$, and $z \in t$ violates the restriction of the theorem. A non-classical case is included simply by choosing $\neg(t \in y)$ for the formula U.

The model we construct for the consistency proof is not in any sense "natural", as indeed the truth-value of an atomic formula $u \in v$ will be independent of the interpretation of the variable u.

PROOF OF THEOREM A. As preparation for the proof proper we present a simple extension of the Brouwer fixed point theorem. This extension is almost identical to a lemma given in Dunford and Schwartz: *Linear Operators*, vol. I, p. 453, but for convenience we repeat the short argument.

Let E be a countable product of intervals $I = [0,1]$ given the product topology. Then E is a compact metric space in the metric defined by

$$d(x,y) = \sum_{m=1}^{\infty} \frac{|x_m - y_m|}{2^m}.$$

It will be shown that each continuous map $f : E \rightarrow E$ has the fixed point property.

To do this define the "projections" $\pi_n(x) = x'$ where $x_i' = x_i$ if $i \leqq n$ and $x_i' = 0$ if $i > n$. The subset $E_n = \pi_n(E)$ of E has (in the induced topology) the fixed point property by Brouwer's theorem. Let $f_n = \pi_n \circ f \circ \text{inj}$, this map is obviously continuous form E_n into E_n, hence has a fixed point $y_n \in E_n \subseteq E$. As E is compact the sequence y_n contains a convergent subsequence y_{n_i}. Let $y_0 = \lim y_{n_i}$, we propose to show that $f(y_0) = y_0$. To this end consider the inequality

$$d(f(y_0), y_0) \leqq d(f(y_0), f(y_{n_i})) + d(f(y_{n_i}), f_{n_i}(y_{n_i})) + d(y_{n_i}, y_0) .$$

Here the first and last term of the right hand sum can be made arbitrary small as f is continuous and $y_{n_i} \to y_0$. Further

$$\text{pr}_j \circ f(y_{n_i}) = \text{pr}_j \circ f_{n_i}(y_{n_i}), \qquad j \leqq n_i ,$$

thus

$$d(f(y_{n_i}), f_{n_i}(y_{n_i})) = \sum_{m=n_i+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^{n_i}},$$

hence the middle term of the sum can also be made arbitrary small by choosing i large enough. Therefore, $f(y_0) = y_0$.

The set Δ_0 of theorem A can be enumerated in a sequence $U_1, U_2, \ldots, U_m, \ldots$. With each U_m we may associate a number n_m such that U_m can be written $U_m(t, y, x_1, \ldots, x_{n_m})$. It is no restriction to assume that $n_m \geqq 1$. Further for any $n \geqq 1$, let λ_n denote a bijection of N^n onto N, where N is the set of natural numbers.

With every $e \in E$ we may associate a model $M = \langle N, e \rangle$ of L_i by defining $e(i,j) = \text{pr}_{\lambda_2(i,j)}(e)$. By use of this model a map $f : E \to E$ will be introduced by the following coordinate equations:

$$\text{pr}_{\lambda_2(i,j)} \circ f(e) = w(U_m(t, y, x_1, \ldots, x_{n_m})) ,$$

where w is any interpretation of L_i into M such that $w(t) = i$, $w(y) = j$, and $w(x_1) = k_1, \ldots, w(x_{n_m}) = k_{n_m}$, where $m, k_1, \ldots, k_{n_m}$ are the unique numbers such that

$$j = \lambda_2(m, \lambda_{n_m}(k_1, \ldots, k_{n_m})) .$$

To show that the map f defined above is continuous it is sufficient to prove that each coordinate map $\text{pr}_n \circ f : E \to I$ is continuous. But the value of $\text{pr}_n \circ f(e)$ is equal to $w(U_m(t, y, x_1, \ldots, x_{n_m}))$ for some m and w, and this truth-value is determined by a finite number of coordinates of e, a fact which taken in conjunction with the definition of an interpretation, immediately yields the continuity of the map $\text{pr}_n \circ f(e)$. This argument also shows why we cannot allow bound quantifiers in a kernel formula U_m, because then $w(U_m)$ would in general depend on an infinite number of coordinates of e, hence the map need not be continuous.

By the above fixed point lemma the map f has a fixed point, say e_0. Define a model $M_0 = \langle N, e_0 \rangle$, where N is the set of natural numbers and $e_0(i,j)$ is given by

$$e_0(i,j) = \mathrm{pr}_{\lambda_2(i,j)}(e_0) .$$

Let w be any interpretation of L_i into M_0, it will be shown that

$$w\big(\forall x_1 \ldots \forall x_{n_m} \exists y \forall t\, (t \in y \leftrightarrow U_m(t,y,x_1,\ldots,x_{n_m}))\big) = 1 ,$$

for all m. This will complete the proof of theorem A. But the truth of this equality is almost immediate by the definition of an interpretation. Assume that $w(x_1) = k_1, \ldots, w(x_{n_m}) = k_{n_m}$. Define w' equal to w for all variables different from y and set

$$w'(y) = \lambda_2\big(m, \lambda_{n_m}(k_1, \ldots, k_{n_m})\big) .$$

Then let w'' be any interpretation agreeing with w' except possibly for the variable t. Let $w''(t) = i$, for some $i \in N$. We must show that

$$w''\big(t \in y \leftrightarrow U_m(t,y,x_1,\ldots,x_{n_m})\big) = 1 .$$

But this is the case if and only if $w''(t \in y)$ and $w''\big(U_m(t,y,x_1,\ldots,x_{n_m})\big)$ are equal. But

$$w''(t \in y) = e_0(i,j) = \mathrm{pr}_{\lambda_2(i,j)}(e_0) ,$$

and

$$w''\big(U_m(t,y,x_1,\ldots,x_{n_m})\big) = \mathrm{pr}_{\lambda_2(i,j)}\big(f(e_0)\big) .$$

This concludes the proof, because e_0 is a fixed point of the map f.

Proof of Theorem C. 1° A first reduction consists in translating the consistency problem within L into a similar problem within a logic L^* obtained from L by replacing the membership relation $u \in v$ by a monadic predicate $\varepsilon(v)$. It should be clear how to translate each formula V of L into a formula V^* of L^*: replace each occurrence of an atomic formula $u \in v$ in V by the atomic formula $\varepsilon(v)$ to obtain V^*. The semantic notions as regard the logic L^* are defined in the obvious way. In the translation one also omits superfluous quantifiers according to the semantic rules: $\forall z V$, $\exists z V$ and V all have the same truth-value in every model under every interpretation if the variable z does not occur free in the formula V.

Under the translation V to V^* of L into L^* it is seen that the formula U^* where $U \in \Delta_2$ (defined in theorem C) does not contain the variable t free. Further one observes that with every model $M^* = \langle N, e^* \rangle$ of L^* and interpretation w^* of L^* into M^* such that

$$w^*\big(\forall x_1 \ldots \forall x_n \exists y\, (\varepsilon(y) \leftrightarrow U^*(y,x_1,\ldots,x_n))\big) = 1\,,$$

one may associate a model $M = \langle N, e\rangle$ of $Ł$ and an interpretation w of $Ł$ into M such that

$$w\big(\forall x_1 \ldots \forall x_n \exists y \forall t (t \in y \leftrightarrow U(t,y,x_1,\ldots,x_n))\big) = 1\,,$$

where it is assumed that $U \in \Delta_2$. In fact one defines $e(i,j) = e^*(j)$ and $w = w^*$ for all variables z.

2° The next step is to eliminate the bound quantifiers of the formulas U^* so as to be able to define a continuous map $f : E \to E$ as was done in the proof of theorem A. The idea behind the elimination procedure is simple: a continuous function on the unit interval obtains its maximum and this maximum will be shown to depend continuously on certain parameters. The technical details, however, are slightly involved, so for the rest of the proof we stick to a typical example.

Let the formula $U^* \in \Delta_2{}^*$ have the form

$$\exists z_1\big(V_1(z_1,x,y) \vee \exists z_2 V_2(z_1,z_2,x,y)\big)\,.$$

Here $n_m = 1$, the formulas V_1 and V_2 does not contain any bound variables and only the displayed variables free.

Let $M^* = \langle N, e^*\rangle$ be a model of $Ł^*$ defined from a point $e^* \in E$ by setting $e^*(j) = \mathrm{pr}_j(e^*)$. Further assume that w^* is any interpretation of $Ł^*$ into M^* such that if $w^*(x) = k$ and U^* is the translation of a formula $U_m \in \Delta_2$, then $w^*(y) = j$ where $j = \lambda_2(m, \lambda_1(k))$. We would like to define $f : E \to E$ by the coordinate equations

$$\mathrm{pr}_j \circ f(e^*) = w^*(U^*)\,,$$

but for the moment this is not possible, because $w^*(U^*)$ may depend on an infinite number of coordinates of e^*, hence f need not be continuous. We are going to define f in a slightly different way which immediately gives the continuity, then conclude the proof by demonstrating the validity of the above coordinate equations.

3° At this point we make a digression to prove a technical lemma needed in the subsequent development. Let $V(z_1,\ldots,z_m,x_1,\ldots,x_n)$ be an open formula of $Ł^*$ and w^* an interpretation into M^*. Define the map g_0 of $w^*(N)^m \subseteq I^m$ into I by setting

$$g_0\big(e^*(w^*(z_1)),\ldots,e^*(w^*(z_m)),b_1,\ldots,b_n\big) = w^*\big(V(z_1,\ldots,z_m,x_1,\ldots,x_n)\big)\,,$$

where g_0 is a map dependent upon the parameters $b_i = w^*(x_i)$, $i = 1,\ldots,n$. It is easy to see that the map g_0 can be extended to a map $g : I^m \to I$ by replacing $e^*(w^*(z_i))$ by a numerical variable a_i in the range I.

Let Γ be some subset of I^m, it will be shown that the family

$$\{g(a_1,\ldots,a_m,b_1,\ldots,b_n)\}_{\langle a_1,\ldots,a_m\rangle\in\Gamma},$$

is uniformly equicontinuous. The proof is an easy induction on the structure of the formula $V(z_1,\ldots,z_m,x_1,\ldots,x_n)$. Either V has the form $\varepsilon(z_j)$ or $\varepsilon(x_j)$, or it has one of the forms $V_1 \vee V_2$, $\urcorner V_1$, or $V_1 \to V_2$. For illustration assume that V is $V_1 \to V_2$ and that the assertion is proved for the families $\{g^1\}$ and $\{g^2\}$ associated with the formulas V_1 and V_2. Then for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$|g^i(a_1,\ldots,a_m,b_1,\ldots,b_n)-g^i(a_1,\ldots,a_m,b_1',\ldots,b_n')| < \varepsilon$$

for all $\langle a_1,\ldots,a_m\rangle\in\Gamma$ if $|b_i-b_i'|<\delta$ for $i=1,\ldots,n$. Now $g=\min(1,1-g^1+g^2)$, and it is easily seen that

$$\begin{aligned}|g(a_1,\ldots,a_m,b_1,\ldots,b_n)-g(a_1,\ldots,a_m,b_1',\ldots,b_n')|\\ \leqq \sum_{i=1}^{2}|g^i(a_1,\ldots,a_m,b_1,\ldots,b_n)-g^i(a_1,\ldots,a_m,b_1',\ldots,b_n')|,\end{aligned}$$

hence the uniform equicontinuity of the family $\{g\}$ follows from the assumed equicontinuity of the families $\{g^i\}$, $i=1,2$.

4° For the proof introduce a map g_2 associated with the formula $V_2(z_1,z_2,x,y)$. Let the parameters be $a_1=e^*(w^*(z_1))$, $b_1=e^*(w^*(x))$ and $b_2=e^*(w^*(y))$. As $g_2(a_1,a_2,b_1,b_2)$ is continuous on I to I, there exists an $\hat{a}_2\in I$ such that

$$g_2(a_1,\hat{a}_2,b_1,b_2) = \max_{a_2\in I}g_2(a_1,a_2,b_1,b_2).$$

And this maximum value is a continuous function of the parameters, a fact which follows from the uniform equicontinuity of $\{g_2(a_1,a_2,b_1,b_2)\}_{a_2\in I}$ and the inequality

$$\begin{aligned}|\max_{a_2\in I}g_2(a_1,a_2,b_1,b_2)-\max_{a_2\in I}g_2(a_1',a_2,b_1',b_2')|\\ \leqq \max_{a_2\in I}|g_2(a_1,a_2,b_1,b_2)-g_2(a_1',a_2,b_1',b_2')|.\end{aligned}$$

Next introduce the map g_1 associated with $V_1(z_1,x,y)$ and the map $g: I\to I$ defined by

$$g(a_1,b_1,b_2) = \max\{g_1(a_1,b_1,b_2),\, g_2(a_1,\hat{a}_2,b_1,b_2)\},$$

where $b_1=e^*(w^*(x))$, $b_2=e^*(w^*(y))$ and $\hat{a}_2$ is a number maximizing the function g_2 for the parameter values a_1,b_1,b_2. g is continuous, hence there exists an $\hat{a}_1\in I$ such that

$$g(\hat{a}_1,b_1,b_2) = \max_{a_1\in I}g(a_1,b_1,b_2).$$

It remains to show that $g(\hat{a}_1,b_1,b_2)$ is a continuous function of the parameters b_1,b_2. This will follow from the uniform equicontinuity of the family $\{g(a_1,b_1,b_2)\}_{a_1\in I}$ and the same type of inequality used above. Now

$$|g(a_1,b_1,b_2)-g(a_1,b_1',b_2')| \\ \leqq \max\{|g_1(a_1,b_1,b_2)-g_1(a_1,b_1',b_2')|,\ |g_2(a_1,\hat{a}_2,b_1,b_2)-g_2(a_1,\hat{a}_2',b_1',b_2')|\}\ ,$$

but here

$$|g_2(a_1,\hat{a}_2,b_1,b_2)-g_2(a_1,\hat{a}_2',b_1',b_2')| \\ \leqq \max_{a_2\in I}|g_2(a_1,a_2,b_1,b_2)-g_2(a_1,a_2,b_1',b_2')|\ ,$$

so by use of the above lemma on uniform equicontinuity we may infer the uniform equicontinuity of the family $\{g(a_1,b_1,b_2)\}_{a_1\in I}$, and this shows that $g(\hat{a}_1,b_1,b_2)$ is a continuous function of b_1 and b_2.

5° The definition of the map $f: E \to E$ now reads

$$\mathrm{pr}_j \circ f(e^*) = g(\hat{a}_1,b_1,b_2)\ ,$$

where $b_1=e^*(w^*(x))$, $b_2=e^*(w^*(y))$ and $j=\lambda_2(m,\lambda_1(w^*(x)))$. From what has been shown in 4° it follows that f is continuous in the product topology on $E=I^N$.

Let $e_0{}^*$ be a fixed point of the map f, it remains to prove the equality

$$\mathrm{pr}_j \circ f(e_0{}^*) = w^*(U^*)\ .$$

6° A last lemma is needed. For each n the formula $U_{m(n)}$ equal

$$\big(x \in y \to (x\in y \to \ldots (x\in y \to \neg x\in y)\ldots)\big)\ ,$$

where there are n occurrences of the atomic formula $x \in y$, is included in the set Δ_2. The transform $U^*_{m(n)}$ is here

$$\big(\varepsilon(y) \to (\varepsilon(y) \to \ldots \to (\varepsilon(y) \to \neg\varepsilon(y))\ldots)\big)\ .$$

In the model $M_0{}^*=\langle N,e_0{}^*\rangle$ where $e_0{}^*$ is a fixed point of the map f defined in 5°, we obtain for a certain j and an interpretation w^* of L^* into $M_0{}^*$ such that $w^*(y)=j$, the validity of the following equation

$$e_0{}^*(j) = w^*\Big(\big(\varepsilon(y) \to (\varepsilon(y) \to \ldots \to (\varepsilon(y) \to \neg\varepsilon(y))\ldots)\big)\Big)\ .$$

An easy calculation yields that the only value for $e_0{}^*(j)$ satisfying this equality is

$$e_0{}^*(j) = n/(n+1)$$

Also for each t, with $0<t\leqq n+1$, the formula

$$\big(y \in x \to (y \in x \to \ldots \to (y \in x \to \neg y \in x)\ldots)\big)\ ,$$

where there are t occurrences of the atomic formula $y \in x$, is included as a certain formula $U_{m'(t)}$ in Δ_2. The transform $U^*_{m'(t)}$ is here

$$\Big(\varepsilon(x) \to \big(\varepsilon(x) \to \ldots \to \big(\varepsilon(x) \to \neg\varepsilon(x)\big)\ldots\big)\Big)\,.$$

In the model $M_0{}^*$ it is then seen that for $j' = \lambda_2\big(m'(t), \lambda_1(j)\big)$ the following equality obtains

$$e_0{}^*(j') = w_1{}^*\Big(\Big(\varepsilon(x) \to \big(\varepsilon(x) \to \ldots\big(\varepsilon(x) \to \neg\varepsilon(x)\big)\ldots\big)\Big)\Big)\,,$$

where $w_1{}^*(x) = j$, $w_1{}^*(y) = j'$. This equality gives

$$e_0{}^*(j') = t/(n+1), \qquad 0 < t \leqq n+1\,.$$

Hence it is seen that for every rational $q \in \langle 0, 1]$ there exists an $j \in N$ such that $e_0{}^*(j) = q$.

7° The proof of the equality now follows easily. Calculating $w^*(U^*)$, with the U^* of section 2°, we obtain

$$\begin{aligned} w^*(U^*) &= \max_{w_1{}^*} \max\big\{w_1{}^*\big(V_1(z_1,x,y)\big),\, w_1{}^*\big(\exists z_2 V_2(z_1,z_2,x,y)\big)\big\} \\ &= \max_{w_1{}^*} \max\big\{w_1{}^*\big(V_1(z_1,x,y)\big),\, \max_{w_2{}^*} w_2{}^*\big(V_2(z_1,z_2,x,y)\big)\big\}\,, \end{aligned}$$

where $w_1{}^*$ equals w^* except possibly for z_1 and $w_2{}^*$ equals $w_1{}^*$ except possibly for z_2. Now

$$\max_{w_2{}^*} w_2{}^*\big(V_2(z_1,z_2,x,y)\big) = \max_{m \in N} g_2\big(e_0{}^*(n), e_0{}^*(m), e_0{}^*(k), e_0{}^*(j)\big)$$

where g_2 is the map introduced in section 4°. Further there exists an $\hat{a}_2 \in I$ such that

$$\max_{a_2 \in I} g_2\big(e_0{}^*(n), a_2, e_0{}^*(k), e_0{}^*(j)\big) = g_2\big(e_0{}^*(n), \hat{a}_2, e_0{}^*(k), e_0{}^*(j)\big)\,,$$

and from the lemma of 6° we have a sequence m_i such that

$$\lim_{i \to \infty} e_0{}^*(m_i) = \hat{a}_2\,.$$

Combining these results we obtain, remembering that g_2 is continuous,

$$\begin{aligned} \max_{a_2 \in I} g_2(\ldots, a_2, \ldots) &= g_2(\ldots, \hat{a}_2, \ldots) = g_2(\ldots, \lim_{i \to \infty} e_0{}^*(m_i), \ldots) \\ &= \lim_{i \to \infty} g_2(\ldots, e_0{}^*(m_i), \ldots) \\ &\leqq \max_{m \in N} g_2(\ldots, e_0{}^*(m), \ldots)\,. \end{aligned}$$

Hence

$$\max_{w_2^*} w_2^*(V_2) = \max_{a_2 \in I} g_2\bigl(e_0^*(n), a_2, e_0^*(k), e_0^*(j)\bigr) .$$

Repeating the argument we get in the end

$$\mathrm{pr}_j \circ f(e_0^*) = \max_{a_1 \in I} g\bigl(a_1, e_0^*(k), e_0^*(j)\bigr) = w^*(U^*) .$$

This is the desired equality giving the "right" value for the function f, and the proof may be concluded as in the last part of the proof of theorem A.

REFERENCES

1. C. C. Chang, *The axiom of comprehension in infinite valued logic,* Math. Scand. 13 (1963), 9–30.
2. Th. Skolem, *Bemerkungen zum Komprehensionsaxiom,* Z. Math. Logik und Grundlagen Math. 3 (1957), 1–17.

UNIVERSITY OF OSLO, NORWAY