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DISTRIBUTIONS INVARIANT UNDER THE GROUP OF COMPLEX ORTHOGONAL TRANSFORMATIONS

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Introduction.

Let G be a group of linear transformations on R^n and G' the space of all distributions on R^n that are invariant under G. When G is the Lorentz group, a description of G' was made by Gårding and Roos [5] in the following manner. Consider the invariant mean value

$$(Mf)(t) = \int \delta(x_0^2 - x_1^2 - x_2^2 - x_3^2 - t)\, f(x)\, dx\,,$$

where f belongs to the space $\mathscr{D}(R^4)$ of infinitely differentiable functions with compact support. (We use Schwartz's notations, see [6].) Mf belongs to C^∞ for $t \neq 0$, and has an expansion around $t=0$ involving powers of t and a singular function $\gamma(t) = \log|t|$ in the following way: For every positive integer m there exists a polynomial $P_m f$ of degree m such that

$$Mf(t) - \gamma(t)(P_m f)(t) \in C^m$$

at the origin. With a suitable topology on the space $H = M\mathscr{D}$ the dual H' of H will be linearly homeomorphic to G'. We may say that H' gives us a parametrization of the invariant distributions.

When G is an orthogonal group of arbitrary signature, G' can be described in a similar way. (See [7].)

In this paper we shall study K', K being the group of complex orthogonal transformations on $R^n \times R^n$ If

$$z = (x,y) = (x_1,\ldots,x_n;\, y_1,\ldots,y_n)$$

belongs to $R^n \times R^n$. we write

$$xy = x_1 y_1 + \ldots + x_n y_n\,.$$

Thus $xx - yy$ and xy are invariant under K. We exclude the trivial case $n=1$.

We are going to consider the mean value

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$$(Mf)(s,t) = \int \delta(xx-yy-s)\, \delta(2xy-t) f(x,y)\, dx dy$$

for $f \in \mathscr{D}(R^n \times R^n)$. It will be shown that the space $H = M\mathscr{D}(R^n \times R^n)$ is of the same nature as in the preceding case. In fact, if $\varphi \in H$ there exists for every m a polynomial $P_m\varphi$ such that

$$\varphi(s,t) - \gamma_n(s,t) P_m \varphi(s,t) \in C^m$$

at the origin. Here

$$\gamma_n(s,t) = \begin{cases} (s^2+t^2)^{\frac{1}{2}} & \text{if } n \text{ is odd}, \\ \log(s^2+t^2)^{\frac{1}{2}} & \text{if } n \text{ is even}. \end{cases}$$

Outside the origin, $\varphi \in C^\infty$.

Our paper runs as follows. In section 1 we introduce the infinitesimal transformations and prove some lemmas which will be needed later on. In section 2 we state a result by Gårding on the distributions invariant under the real orthogonal group. In section 3 we investigate the mean value M, especially its singularity at the origin. Considering the results in section 3, we define, in section 4, the function space H. We topologize H so that the mapping M of $\mathscr{D}$ onto H be continuous. Finally, in section 5, we carry out the parametrization of the invariant distributions.

The subject of this paper was suggested to me by professor Lars Gårding. I wish to thank him for his advice and generous interest.

1. Infinitesimal transformations, some general lemmas.

Let K_+ be all elements of K with the determinant $+1$, and let $K_+' \supset K'$ be the corresponding distributions. K_+ is a connected analytic group with a Liealgebra k_+ of infinitesimal transformations. We are now going to use some facts to be found in [3], see p. 8 (prop. 5), p. 16 (prop. 3), Ch. IV, §§ II, III, VIII. Each $X \in k_+$ corresponds to a complex skew symmetric $n \times n$ matrix S, and conversely. If $S = A + iB$ where A and B are real, we identify S with the real $2n \times 2n$ matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Then to each $X \in k_+$ there is associated a differential operator acting on functions or distributions in the following way:

$$XT(x,y) = S \begin{pmatrix} x \\ y \end{pmatrix} \cdot \operatorname{grad} T(x,y),$$

where $\binom{x}{y}$ is a column matrix. These infinitesimal operators are spanned by

$$L_{lk} = x_k \frac{\partial}{\partial x_l} - x_l \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_l} - y_l \frac{\partial}{\partial y_k}\,, \qquad k \neq l\,.$$
$$K_{lk} = y_k \frac{\partial}{\partial x_l} - y_l \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial y_l} + x_l \frac{\partial}{\partial y_k}\,,$$

LEMMA 1.1. *$T \in K_+$ if and only if $XT = 0$ for every $X \in k_+$.*

PROOF. Put $V = \{e^S;\ S \text{ skew symmetric}\}$. We know that V is a neighbourhood of the unit element in K_+. If t is a real parameter we have

$$\frac{d}{dt} T(e^{St}z) = e^{St} S \begin{pmatrix} x \\ y \end{pmatrix} \cdot \operatorname{grad} T(x,y)\,.$$

Since $T \in K_+$ implies that the left side is zero, $XT = S\binom{x}{y} \cdot \operatorname{grad} T$ must be zero. Conversely, if $XT = 0$ for all $X \in k_+$, then $T(\Lambda z) = T(z)$ for all $\Lambda \in V$, and for a given $\Lambda_0 \in K_+$ we also have

$$T(\Lambda\Lambda_0 z) = T(\Lambda_0 z) \quad \text{for every } \Lambda \in V\,.$$

Hence the set of $\Lambda \in K_+$, for which $T(\Lambda z) = T(z)$, is open and closed in K_+. Since K_+ is connected the proof is complete.

Under left multiplication by C^∞-functions and addition, the infinitesimal operators generate a left C^∞-module M_K of first order differential operators. Let us consider a module M of such operators

$$\sum_1^p a_j(x) \frac{\partial}{\partial x_j}$$

defined in an open set $\Omega \subset R^p$. The module M is said to be *free* if:

1) The dimension of M is constant in Ω.
2) If A and B belongs to M, so does $AB - BA$.

The module M_K is free in $\Omega = R^n \times R^n - \{0\}$, and its dimension is $2n-2$. In fact, 2) is a classical property of infinitesimal operators, and 1) can be proved as follows: If $z = (x,y)$ belongs to Ω there exists an l such that $(x_l, y_l) \neq (0,0)$. Then it is easily seen that the $2n-2$ operators L_{lk}, K_{lk}, $k \neq l$, are linearly independent in (x,y). Hence the dimension of M_K is larger than $2n-2$. On the other hand M_K annihilates the invariants $xx - yy$ and xy, whose gradients $(2x, -2y)$ and (y,x) are linearly independent in Ω. Hence the dimension is $2n-2$.

LEMMA 1.2. *Let M be a free module of dimension $m < p$ in $\Omega \subset R^p$ and let N be the set of all distributions which are annihilated by M. Then there* *exist locally in Ω $p-m$ independent C^∞-functions $g_{m+1},\ldots,g_p \in N$. Any such set of functions generates N locally, that is, $T \in N$ if and only if*

$$T(x) = F\big(g_{m+1}(x),\ldots,g_p(x)\big) \quad locally\ ,$$

where F is a unique distribution in R^{p-m}.

Proof. By a theorem of Frobenius (see e.g. [2, Ch. X.1]) there exists locally in Ω a coordinate transformation $x=x(y) \in C^\infty$ such that M is spanned by

$$\frac{\partial}{\partial y_1};\ldots;\frac{\partial}{\partial y_m}\ .$$

Then $y_{m+1}(x),\ldots,y_p(x)$ are independent and belong to N. Any g in $N \cap C^\infty$ is of the form

$$g(x) = h\big(y_{m+1}(x),\ldots,y_p(x)\big), \qquad h \in C^\infty\ .$$

If

$$g_k = h_k(y_{m+1},\ldots,y_p), \qquad k = m+1,\ldots,p\ ,$$

are independent, we can choose coordinates z_k as follows:

$$z_l = y_l, \quad l \leqq m; \qquad z_k = g_k, \quad k > m\ .$$

Then $T \in N$ if and only if $\partial T/\partial z_l = 0$, $l \leqq m$, so the lemma follows from a well-known result by Schwartz.

This lemma will be used in section 5. We also prove the following simple lemma:

Lemma 1.3. *Let $s_j(u,v) \in C^\infty(U \times V)$, $j=1,\ldots,m$, where $U \subset R^n$, $V \subset R^m$ are open and $m<n$. Let the surfaces*

$$S_j(v) = \{u;\, s_j(u,v)=0\}$$

be in general position for $v \in V$; that is, let $\operatorname{grad}_u s_j(u,v)$ be linearly independent if $u \in S_j(v)$ for all $v \in V$. Then for $f \in \mathscr{D}(U)$, we have

$$(Sf)(v) = \int \delta\big(s_1(u,v)\big)\ldots\delta\big(s_m(u,v)\big) f(u)\, du \in C^\infty(V)\ .$$

Proof. Suppose $u_0 \in S_j(v_0)$, $j=1,\ldots,m$. There exist open neighbourhoods U_0 of u_0 and V_0 of v_0 so that the functions

$$\xi_1 = s_1(u,v),\ldots,\ \xi_m = s_m(u,v), \quad \xi_{m+1} = u_{j_{m+1}},\ldots,\ \xi_n = u_{j_n}\ ,$$

for suitable j_k, and for each $v \in V_0$, form a coordinate system in U_0. If $f \in \mathscr{D}(U_0)$ we have

$$(Sf)(v) = \int [f(u(\xi,v))\,|d(u(\xi,v))/d\xi|]_{\xi_1=\ldots=\xi_m=0}\, d\xi_{m+1}\cdots d\xi_n\,,$$

which is infinitely differentiable. By a partition of unity, we get the same result for an arbitrary f in $\mathscr{D}(U)$.

2. The real orthogonal group.

Let $O_n = O$ be the group of real orthogonal transformations, and let O' be the invariant distributions in $R^n \times R^n$, that is,

$$T \in O' \quad \text{if and only if} \quad T(\Lambda x, \Lambda y) = T(x,y) \quad \forall \Lambda \in O\,.$$

For $f \in \mathscr{D}(R^n \times R^n)$ we define the mean value

$$Nf(r,s,t) = (rs-t^2)^{-\frac{1}{2}(n-3)} \int \delta(xx-r)\,\delta(yy-s)\,\delta(xy-t)\,f(x,y)\,dx dy\,.$$

The surfaces $xx=r$, $yy=s$, $xy=t$ are in general position in the interior of the cone

$$C = \{(r,s,t);\ r \geqq 0,\ s \geqq 0,\ rs \geqq t^2\}\,,$$

where Nf hence belongs to C^∞, by lemma 1.3. We also define

$$(Pf)(\xi,\eta) = \int_O f(\Lambda\xi,\Lambda\eta)\,d\Lambda\,,$$

where $d\Lambda$ is an invariant measure over $O=O_n$ with $\int d\Lambda = 1$. In the sequel we always suppose

$$\xi\xi = r,\quad \eta\eta = s,\quad \xi\eta = t\,.$$

LEMMA 2.1. *$Pf(\xi,\eta) = c_n Nf(r,s,t)$ where c_n only depends on the dimension n.*

PROOF. Let $Z = Z(r,s,t)$ be the manifold $xx=r$, $yy=s$, $xy=t$. Since O acts transitively on Z we have

$$\int_O f(\Lambda\xi,\Lambda\eta)\,d\Lambda = \int_Z f(x,y)\,\omega_1(x,y)\,,$$

where ω_1 is an invariant form on Z. But

$$\int \delta(xx-r)\,\delta(yy-s)\,\delta(xy-t)\,f(x,y)\,dx dy = \int_Z f(x,y)\,\omega_2(x,y)\,,$$

where ω_2 is another invariant form on Z. Hence

$$\omega_2(x,y) = \varphi(r,s,t)\,\omega_2(x,y)$$

and

$$(Pf)(\xi,\eta) = \varphi(r,s,t)^{-1}\int \delta(xx-r)\,\delta(yy-s)\,\delta(xy-t)\,f(x,y)\,dx\,dy\,.$$

We determine φ by putting $f=1$

$$\varphi(r,s,t) = \int \delta(xx-r)\,\delta(yy-s)\,\delta(xy-t)\,dx\,dy\,.$$

In the integral

$$I(x) = \int \delta(yy-s)\,\delta(xy-t)\,dy$$

we may suppose

$$x = ((xx)^{\frac{1}{2}},0,\ldots,0) = (|x|,0,\ldots,0)\,.$$

After introducing polar coordinates,

$$y_2{}^2+\ldots+y_n{}^2 = \sigma\,,$$

we get

$$\begin{aligned} I(x) &= c_n'\int \delta\big(\sigma-(s-y_1{}^2)\big)\,\delta(|x|y_1-t)\,\sigma^{\frac{1}{2}(n-3)}\,d\sigma\,dy_1 \\ &= c_n'\int \delta(|x|y_1-t)\,(s-y_1{}^2)^{\frac{1}{2}(n-3)}\,dy_1 = (c_n'/|x|)(s-t^2/|x|^2)^{\frac{1}{2}(n-3)}\,. \end{aligned}$$

Hence

$$\begin{aligned} \varphi(r,s,t) &= \int \delta(xx-r)\,I(x)\,dx \\ &= c_n''\int \delta(\varrho-r)\,(s-t^2/\varrho^2)^{\frac{1}{2}(n-3)}\,\varrho^{\frac{1}{2}(n-3)}\,d\varrho = c_n''(rs-t^2)^{\frac{1}{2}(n-3)}\,, \end{aligned}$$

and the lemma is proved.

Let $\mathscr{D}(C)$ consist of the restrictions to C of all functions in $\mathscr{D}(R^3)$. If $g\in\mathscr{D}(C)$, it is defined and belongs to C^∞ in the interior of C and every derivative of g has a continuous extension to the boundary of C. Conversely, since C is closed and convex, and hence regular in the sense defined by H. Whitney (see [9, p. 482]), every g with the above property, vanishing outside a compact set, belongs to $\mathscr{D}(C)$. If K_n is an increasing sequence of compact sets with $\bigcup_1^\infty K_n \supset C$, we define the topology on $\mathscr{D}(C)$ as the inductive limit of the spaces $\mathscr{D}(C\cap K_n)$ of all functions in $\mathscr{D}(C)$ with supports in K_n. It is easy to verify that $\mathscr{D}(C\cap K_n)$ are Fréchet spaces. The dual $\mathscr{D}'(C)$ of $\mathscr{D}(C)$ is isomorphic to the space of all distributions in R^3 with supports contained in C. (See [6, p. 99]).

The following theorem is due to Gårding [4].

THEOREM 2.1. *The mapping* N:

$$\mathscr{D}(R^n \times R^n) \ni f \to Nf \in \mathscr{D}(C) ,$$

is linear, continuous and surjective. The adjoint mapping

$$N' \colon \mathscr{D}'(C) \to O'$$

is a linear homeomorphism.

PROOF. We only prove the first part of the theorem. As before, we suppose $\xi\xi = r$ etc. By lemma 2.1

$$c_n Nf(r,s,t) = Pf(\xi,\eta) .$$

Suppose $r > 0$. Then we can choose

$$\xi = (\lambda,0,\ldots,0), \qquad \eta = (\mu,\nu,0,\ldots,0) ,$$

where

$$\lambda = r^{\frac{1}{2}}, \ \mu = t/r^{\frac{1}{2}}, \ \nu = (rs-t^2)^{\frac{1}{2}}/r^{\frac{1}{2}} .$$

This leads us to define

$$Qf(\lambda,\mu,\nu) = Pf(\xi,\eta) ,$$

where $\xi = (\lambda,0\ldots0)$, $\eta = (\mu,\nu,0,\ldots,0)$. It is clear that Q is even in ν and $\in C^\infty$. Since

$$c_n Nf(r,s,t) = Qf(r^{\frac{1}{2}},t/r^{\frac{1}{2}},(rs-t^2)^{\frac{1}{2}}/r^{\frac{1}{2}}) , \tag{2.1}$$

$Nf \in C^\infty$ for $r > 0$, $rs \geqq t^2$ and also, by symmetry, for $s > 0$, $rs \geqq t^2$. We develop f around the origin:

$$f(\xi,\eta) = \sum_{|\alpha|+|\beta|<2m} f_{\alpha,\beta}\, \xi^\alpha \eta^\beta + O(\xi\xi+\eta\eta)^m ,$$

where

$$\alpha = (\alpha_1,\ldots,\alpha_n), \qquad |\alpha| = \alpha_1 + \ldots + \alpha_n ,$$

$$f_{\alpha,\beta} = \frac{(\partial_{\alpha,\beta} f)(0,0)}{\alpha!\,\beta!}, \qquad \partial_{\alpha,\beta} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial y_n}\right)^{\beta_n} ,$$

$$\alpha! = \alpha_1! \ldots \alpha_n! .$$

By integration we obtain

$$(Pf)(\xi,\eta) = \sum_{|\alpha|+|\beta|<2m} f_{\alpha,\beta} \int (\Lambda\xi)^\alpha\, (\Lambda\eta)^\beta \, d\Lambda + O(r^2+s^2+t^2)^{\frac{1}{2}m} , \tag{2.2}$$

where the integral under the summation sign vanishes if $|\alpha| + |\beta|$ is odd, and is a homogenous polynomial of degree $\frac{1}{2}(|\alpha| + |\beta|)$ in r,s,t if $|\alpha| + |\beta|$ is even. (See [8, p. 31].) Hence $Nf \in C^\infty$ at the origin. Its derivatives have the form

$$(\partial_k Nf)(0) = \sum_{|\alpha|+|\beta|=|2k|} c_{\alpha,\beta,k}\, \partial_{\alpha,\beta} f(0,0) , \tag{2.3}$$

where $c_{\alpha,\beta,k}$ are certain constants not depending on f.

Evidently Nf has compact support, so Nf belongs to $\mathscr{D}(C)$. N is surjective, for if $g \in \mathscr{D}(C)$ then

$$g_1(x,y) = g(xx,yy,xy) \in \mathscr{D}(R^n \times R^n)$$

and $N(cg_1)=g$ for some c. It is clear that N is linear, and the continuity follows easily from the closed graph theorem. Suppose $f_\nu \to 0$ in $\mathscr{D}(R^n \times R^n)$ and $Nf_\nu \to g$ in $\mathscr{D}(C)$. Then $f_\nu \to 0$ uniformly, and lemma 2.1 shows that $Nf_\nu \to 0$ at least pointwise so that $g=0$. The use of the closed graph theorem, and sequences (instead of filters) is legitimate, because both $\mathscr{D}(R^n \times R^n)$ and $\mathscr{D}(C)$ are inductive limits of Fréchet spaces. (See [1, p. 35–38, 61–65].)

The following lemma will be needed in the final section.

Lemma 2.2. *For each bounded set B in $\mathscr{D}(C)$ there exists a bounded set B_1 in $\mathscr{D}(R^n \times R^n)$ such that $NB_1 \supset B$.*

Proof. For each g in B we choose $g_1(x,y)=cg(xx,yy,xy)$ so that $Ng_1=g$. Clearly g_1 belongs to a bounded set $B_1 \subset \mathscr{D}(R^n \times R^n)$ not depending on g.

3. The mean value M.

Let us consider the mean value

$$(Mf)(s,t) = \int \delta(xx-yy-s)\, \delta(2xy-t)\, f(x,y)\, dx\, dy$$

invariant under K. If $f \in \mathscr{D}(R^n \times R^n)$, Mf has compact support and, by lemma 1.3, $Mf \in C^\infty$ for $s^2+t^2>0$. We want to examine Mf near the origin. First we observe that

$$(Mf)(s,t) = \int_{r>(s^2+t^2)^{\frac{1}{2}}} (r^2-s^2-t^2)^{\frac{1}{2}(n-3)}\, Nf(r,s,t)\, dr ,$$

where

$$(Nf)(r,s,t) = (r^2-s^2-t^2)^{-\frac{1}{2}(n-3)} \int \delta(xx+yy-r)\, \delta(xx-yy-s)\, \delta(2xy-t) f(x,y)\, dx\, dy ,$$

is an invariant mean value for the group O, essentially the same as in the preceding section. Putting

$$(M_1 g)(s,t) = \int_{r>(s^2+t^2)^{\frac{1}{2}}} (r^2-s^2-t^2)^{\frac{1}{2}(n-3)}\, g(r,s,t)\, dr$$

we have $M = M_1 \circ N$. By theorem 2.1. the mapping

$$N\colon \mathscr{D}(R^n \times R^n) \to \mathscr{D}(C)\,,$$

where C now means the cone $r \geqq (s^2+t^2)^{\frac{1}{2}}$, is continuous and surjective.

Lemma 3.1. *The mapping $M_1\colon \mathscr{D}(C-\{0\}) \to \mathscr{D}(R^2)$ is linear, continuous and surjective.*

Proof. Clearly M_1 is linear. Suppose that the sequence $g_\nu \to 0$ in $\mathscr{D}(C-\{0\})$. Then all the g_ν have supports contained in a compact set K not containing the origin, and $g_\nu \to 0$ uniformly in K. This implies $M_1 g_\nu \to 0$, at least pointwise, so the continuity follows from the closed graph theorem.

For the proof of the surjectivity, let $\varphi \in \mathscr{D}(R^2)$ with $\operatorname{supp}\varphi = Q$ and let I be an interval such that

$$I \times Q \subset (C-\{0\})\,.$$

There exists $\psi \in \mathscr{D}(I)$ such that $\int_I \psi(r)\,dr = 1$. Then

$$g(r,s,t) = \frac{\psi(r)\,\varphi(s,t)}{(r^2-s^2-t^2)^{\frac{1}{2}(n-3)}} \in \mathscr{D}(C-\{0\})$$

and $M_1 g = \varphi$. Hence M_1 is surjective.

We now examine $M_1 g$ in a neighbourhood of the origin, e.g. $(s^2+t^2)^{\frac{1}{2}} < \frac{1}{2}$. In that neighbourhood

$$M_1 g(s,t) = \int_{(s^2+t^2)^{\frac{1}{2}}}^{1} (r^2-s^2-t^2)^{\frac{1}{2}(n-3)} g(r,s,t)\,dr + \omega_\infty\,,$$

where ω_∞ denotes a C^∞-function. Developing g we get

$$M_1 g(s,t) = \sum_{|\alpha|<m} \frac{(\partial_\alpha g)(0)}{\alpha!} s^{\alpha_2} t^{\alpha_3} \int_{(s^2+t^2)^{\frac{1}{2}}}^{1} r^{\alpha_1} (r^2-s^2-t^2)^{\frac{1}{2}(n-3)}\,dr +$$
$$+ \int_{(s^2+t^2)^{\frac{1}{2}}}^{1} g_m(r,s,t)\,(r^2-s^2-t^2)^{\frac{1}{2}(n-3)}\,dr + \omega_\infty\,,$$

where $g_m(r,s,t)$ is $O(r^2+s^2+t^2)^{\frac{1}{2}m}$ and $\in C^\infty$ outside the origin, so that the last integral $\in C^{m+n-3}$ at the origin. The integrals under the summation sign can be computed by means of $\bar{n}$ partial integrations, where

$$\bar{n} = \begin{cases} \frac{1}{2}(n-3), & n \text{ odd}, \\ \frac{1}{2}(n-2), & n \text{ even}. \end{cases}$$

The result is

$$(3.1)\quad \frac{1}{\alpha!} \int_{(s^2+t^2)^{\frac{1}{2}}}^{1} r^{\alpha_1} (r^2-s^2-t^2)^{\frac{1}{2}(n-3)}\, dr = v_\alpha(s,t) + C_{\alpha,n}\gamma_n(s,t)(s^2+t^2)^{\bar{n}+\frac{1}{2}\alpha_1},$$

where $v_\alpha \in C^\infty$ and does not depend on g and

$$C_{\alpha,n} = \begin{cases} 0, & \alpha_1 \text{ odd}, \\ (-1)^{\bar{n}+1} \dfrac{(n-3)!!\,(\alpha_1-1)!!\,\alpha!}{(n+\alpha_1-2)!!}, & \alpha_1 \text{ even}, \quad (-1)!! = 1, \end{cases}$$

$$\gamma_n(s,t) = \begin{cases} (s^2+t^2)^{\frac{1}{2}}, & n \text{ odd}, \\ \log(s^2+t^2)^{\frac{1}{2}}, & n \text{ even}. \end{cases}$$

Hence we get the following expansion of $M_1 g$:

(3.2)

$$M_1 g(s,t) = \omega_{m+n-3} + \gamma_n(s,t)(s^2+t^2)^{\bar{n}} \sum_{|\alpha|<m} C_{\alpha,n} (s^2+t^2)^{\frac{1}{2}\alpha_1} s^{\alpha_2} t^{\alpha_3} (\partial_\alpha g)(0).$$

4. The space *H*.

Guided by the result in the preceding section, we introduce some function spaces. Let $H = H^n$ consist of all functions φ, defined in $R^2 - \{0\}$, with the following properties:

1) $\varphi \in C^\infty$ outside the origin.

2) φ has compact support.

3) For every positive integer m there exists a polynomial $P_m\varphi$ of the form

$$(4.1)\qquad P_m\varphi(s,t) = (s^2+t^2)^{\bar{n}} \sum_{|j|<m} A_j(\varphi)\, s^{j_1} t^{j_2}$$

such that

$$\varphi(s,t) - \gamma_n(s,t)(P_m\varphi)(s,t) \in C^{m+n-3}$$

at the origin.

It is clear that $P_m\varphi$ is unique and that the A_j are linear functionals on H not depending on m. In particular, if $A_j(\varphi) = 0$ for every j, then $\varphi \in \mathscr{D}(R^2)$.

We shall now introduce a topology on H. For $\psi \in \mathscr{D}^m(R^2)$ put

$$|\psi|_m = \max_{|k|\le m} \max_{(s,t)} |\partial_k \psi(s,t)|.$$

Let H_R be all functions in H with supports contained in

$$B_R = \{(s,t);\ s^2+t^2 \leqq R^2\}$$

and take a fixed, positive χ in $\mathscr{D}(R^2)$, equal to 1 in a neighbourhood of the origin. With the seminorms

$$q_m(\varphi) = |\varphi - \chi\gamma_n(P_m\varphi)|_{m+n-3} + \sum_{|j|<m} A_j(\varphi)$$

H_R becomes a Fréchet space. The topology of $H = \bigcup_1^\infty H_N$ shall be the strict inductive limit of the topologies of H_N. (See [1].) It is defined by the following seminorms:

$$q_{h,\mu}(\varphi) = p_h\big(\varphi - \gamma_n\chi(P_m\varphi)\big) + \sum_{|j|<\mu} |A(\varphi)| , \tag{4.2}$$

where p_h are the seminorms on $\mathscr{D}(R^2)$

$$p_h(f) = \sum_\beta \max_{(s,t)} |h_\beta(s,t)(\partial_\beta f)(s,t)| .$$

Here h_β are continuous functions such that for each compact set K there exists a number $\lambda(K,h)$ with the following property:

$$|\beta| > \lambda(K,h) \quad \text{implies} \quad h_\beta(s,t) = 0 \quad \text{for } (s,t) \in K .$$

(See [5] p. 13).

In (4.2) we suppose $m+n-3 > \lambda(\operatorname{supp}\chi, h)$.

THEOREM 4.1. *The mapping M_1: $\mathscr{D}(C) \to H$ is linear, continuous and surjective.*

COROLLARY. *The mapping M: $\mathscr{D}(R^n \times R^n) \to H$ is linear, continuous and surjective.*

Since $M = M_1 \circ N$, the corollary follows immediately from theorem 2.1.

PROOF. $M_1\mathscr{D} \subset H$ by formula (3.2). The linearity is trivial. In order to prove the surjectivity, we take an arbitrary φ in H. We want to find $g \in \mathscr{D}(C)$ with

$$A_\beta(\varphi) = A_\beta(M_1 g)$$

for every β. By formulas (3.2) and (4.1)

$$A_\beta(M_1 g) = C_{0,\beta_1,\beta_2}(\partial_{0,\beta_1,\beta_2}g)(0) + \sum_{|\alpha|=|\beta|-2} k_\alpha \left(\frac{\partial^2}{\partial r^2}\partial_\alpha g\right)(0) \tag{4.3}$$

for certain constants k_α. It is possible to find a $g \in \mathscr{D}(C)$ not depending on r in a neighbourhood of the origin, such that

$$(\partial_{0,\beta_1,\beta_2}g)(0) = A_\beta(\varphi)/C_{0,\beta_1,\beta_2} . \tag{4.4}$$

Then $A_\beta(\varphi) = A_\beta(M_1 g)$ as we wished, and hence $\varphi - M_1 g \in \mathscr{D}(R^2)$. According to lemma 3.1 there is a g_0 in $\mathscr{D}(C - \{0\})$ such that

$$M_1 g_0 = \varphi - M_1 g .$$

Hence $M_1(g+g_0) = \varphi$ so that M_1 is surjective.

For the continuity we can use the closed graph theorem. Suppose $g_\nu \to 0$ in $\mathscr{D}(C)$ and $M_1 g_\nu \to \varphi$ in H. We want to show that $\varphi = 0$. Now for $(s,t) \neq (0,0)$

$$(M_1 g_\nu)(s,t) = \int_{(s^2+t^2)^{\frac{1}{2}}}^{\infty} g_\nu(r,s,t)\,(r^2-s^2-t^2)^{\frac{1}{2}(n-3)}\,dr .$$

Since the g_ν vanish outside a fix compact set and $g_\nu \to 0$ uniformly the right side tends to zero. Hence $\varphi(s,t) = 0$ outside the origin. In view of formula (4.3) and the continuity of A_β, it is clear that $A_\beta \varphi = 0$ for every β, so everything is proved.

We also need the following lemma

LEMMA 4.1. *For each bounded set $B \subset H$ there exists a bounded set $B_1 \subset \mathscr{D}(C)$ such that $M_1 B_1 \supset B$.*

PROOF. Take an arbitrary φ in B, and choose g in $\mathscr{D}(C)$ as in the proof of theorem 4.1. so that $A_\beta(M_1 g) = A_\beta \varphi$. Since B is bounded in H there exist constants a_β, b_α depending on B but not on φ, such that (see formula (4.4))

$$|A_\beta(\varphi)| \leqq a_\beta, \qquad |\partial_\alpha g(0)| \leqq b_\alpha .$$

Hence g can be chosen from a bounded set in $\mathscr{D}(C)$. Since M_1 is continuous, and

$$\varphi - M_1 g \in \mathscr{D}(R^2) ,$$

$\varphi - M_1 g$ obviously belongs to a certain bounded set in $\mathscr{D}(R^2)$. Chosing g_0 as in Lemma 3.1 so that

$$M_1 g_0 = \varphi - M_1 g ,$$

g_0 belongs to a bounded set in $\mathscr{D}(C - \{0\})$, and finally $g + g_0$ belongs to a bounded set B_1 in $\mathscr{D}(C)$ with the required property.

5. Parametrization of the invariant distributions.

With the aid of the lemmas in section 1 we can prove

LEMMA 5.1. *For each $T \in K'$ there is a unique $F \in \mathscr{D}'(R^2)$ such that*

$$\langle T, f \rangle = \langle F, Mf \rangle$$

for every $f \in \mathscr{D}(R^n \times R^n - \{0\})$.

PROOF. The module M_K of the infinitesimal operators is free and of dimension $2n-2$ in $\Omega = R^n \times R^n - \{0\}$, where $xx-yy$ and $2xy$ are independent and belong to the nullspace of M_K. Let V be a sufficiently small neighbourhood of an arbitrary point in Ω and put

$$I(V) = \{(xx-yy, 2xy);\ (x,y) \in V\}.$$

Then, by lemma 1.1 and 1.2, there exists a unique $F_V \in \mathscr{D}'(I(V))$ such that for all $f \in \mathscr{D}(V)$

$$\langle T, f\rangle = \int T(x,y) f(x,y)\, dx dy = \int F_V(xx-yy, 2xy) f(x,y)\, dx dy = \langle F, Mf\rangle .$$

Now let V and V' be two such neighbourhoods and let $f \in \mathscr{D}(\Lambda V \cap V')$, $\Lambda \in K$. Since $T \in K'$ we have:

$$\langle F_{V'}, Mf\rangle = \langle T, f\rangle = \langle T(z), f(\Lambda^{-1}z)\rangle = \langle F_V, M(\Lambda f)\rangle = \langle F, Mf\rangle .$$

Observing that K acts transitively on Ω we can thus prove that $F_V = F_{V'}$ on $I(V) \cap I(V')$, and we get a unique distribution $F \in \mathscr{D}'(R^2)$, (see [6, Th. 4, p. 27]).

LEMMA 5.2. *$T \in K'$ and* $\operatorname{supp} T \subset \{0\}$ *if and only if*

$$T = P(\square, \lozenge)\delta ,$$

where P is a polynomial and

$$\square = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} - \frac{\partial^2}{\partial y_k^2}, \qquad \lozenge = 2\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k \partial y_k} .$$

PROOF. Every distribution T with support in the origin is of the form $Q(\partial/\partial x_1, \ldots, \partial/\partial y_n)\,\delta$, where Q is a polynomial. T is invariant if and only if Q is an invariant polynomial, and we need only prove that such a polynomial is of the form $P(xx-yy, 2xy)$. But if Q is invariant under K it is also invariant under O (the real orthogonal group) and hence a polynomial in $xx+yy$, $xx-yy$ and xy (see [8]). It is simple to prove that it can not contain any terms with powers of $xx+yy$.

Let H' be the (strong) dual of H and M' the adjoint of M defined by

$$\langle M'F, f\rangle = \langle F, Mf\rangle .$$

We want to show that $M' : H' \to K'$ is a linear homeomorphism and state some lemmas.

LEMMA 5.3. *$T \in K'$* and $\operatorname{supp} T \subset \{0\}$ *if and only if*

$$T = \sum c_j M' A_j ,$$

where the sum if finite.

PROOF. It is clear that $\sum c_j M'A_j \in K'$ and has support in the origin. Conversely, let G_ν be the space of all such distributions of order $\leqq 2\nu$. The dimension of G_ν is $\frac{1}{2}(\nu+1)(\nu+2)$ according to lemma 5.2. On the other hand the formulas (2.3) and (4.3) show that the $M'A_j$ belong to G_ν for $|j| \leqq \nu$, and are linearly independent. Hence they span G_ν, and the lemma is proved.

LEMMA 5.4. *$F \in H'$ if and only if there exists $F_0 \in \mathscr{D}'(R^2)$ such that*

$$\langle F,\varphi\rangle = \langle F_0, \varphi - \chi\gamma_n P_m\varphi\rangle + \sum c_j A_j(\varphi) \tag{5.1}$$

for all $\varphi \in H$, where $m+n-3$ is not less than the order of F_0 in $\operatorname{supp}\chi$.

PROOF. It is evident that every F of the form (5.1) belongs to H'. Conversely, if F_0 is the restriction of F to $\mathscr{D}(R^2)$, $F_0 \in \mathscr{D}'(R^2)$. For a sufficiently large m we get:

$$0 = \langle F-F_0, \varphi-\gamma\chi(P_m\varphi)\rangle = \langle F,\varphi\rangle - \langle F_0,\varphi-\gamma\chi(P_m\varphi)\rangle - \langle F,\chi\gamma(P_m\varphi)\rangle .$$

But

$$\langle F,\chi\gamma(P_m\varphi)\rangle = \sum_{|j|<m} c_j A_j(\varphi)$$

for certain c_j, and hence F is of the form (5.1).

Now we can prove our main result.

THEOREM *The mapping M': $H' \to K'$ is a linear homeomorphism.*

PROOF. It is clear that $M'H' \subset K'$, and that M' is linear. Furthermore M' is injective because M is surjective. To prove that M' is surjective we take an arbitrary T in K'. There exists, by lemma 5.1, $F_0 \in \mathscr{D}'(R^2)$ such that for every $f \in \mathscr{D}(R^n \times R^n - \{0\}) = \mathscr{D}(\Omega)$

$$\langle T,f\rangle = \langle F_0, Mf\rangle .$$

We define $F_1 \in H'$ by

$$\langle F_1,\varphi\rangle = \langle F_0, \varphi - \chi\gamma(P_m\varphi)\rangle ,$$

m being large enough. Consequently $\langle T,f\rangle = \langle M'F_1, f\rangle$ for every $f \in \mathscr{D}(\Omega)$, so by lemma 5.2

$$T - M'F_1 = \sum c_j M'A_j .$$

Hence

$$M'(F_1 + \sum c_j A_j) = T ,$$

and M' is surjective.

By known topological theorems [1, ch. IV p. 102–103 prop. 5–6], M' is continuous. That the inverse M'^{-1} is continuous follows from the Lemmas 2.2 and 4.1. Hence everything is proved.

According to our theorem, lemma 5.4 gives us a concrete description, "parametrization", of K'.

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