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THE STRUCTURE SPACE OF A LEFT IDEAL

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Jacobson [1, p. 206] showed that if A is a two-sided ideal in a ring B, then the structure space of A is homeomorphic to the open subset of the structure space of B consisting of those primitive ideals which do not contain A. For any such P in the structure space of B, its image under the homeomorphism τ is given by $\tau(P)=P\cap A$. We shall show that if we restrict attention to the right structure space, i.e. the space of right primitive ideals, this theorem is actually valid even when A is just a left ideal in B. The homeomorphism τ is, however, no longer of such simple nature; in fact, we cannot state more than $\tau(P)\supseteqq P\cap A$.

We shall write AB for the set of elements ab, $a \in A$, $b \in B$, and $\{A,B\}$ for the set of finite sums of products ab, $a \in A$, $b \in B$.

Let A be a left ideal in the ring B.

A right primitive ideal P in A is the quotient of a maximal modular right ideal I:

$$P = I : A = \{a \mid Aa \subseteqq I\}$$

and P is the largest two-sided ideal contained in I. Even when I is not modular, the ideal $I:A$ is primitive, but we cannot be sure that it is contained in I. We shall make extensive use of the fact that $J_1J_2 \subseteqq P$ implies $J_1 \subseteqq P$ or $J_2 \subseteqq P$ for all right ideals J_1 and J_2.

The sets of primitive ideals in A and B are denoted Π and Π_B, respectively.

For any sets $S \subseteqq A$ and $X \subseteqq \Pi$ define the hull of S in Π by

$$h(S) = \{P \mid S \subseteqq P \in \Pi\}$$

and the kernel of X in A by

$$k(X) = \bigcap \{P \mid P \in X\}.$$

The operation hk defines a closure in Π, and the structure space of A is Π endowed with this hull-kernel topology. Similarly h_B and k_B are defined in Π_B and B.

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THEOREM 1. *To each maximal modular right ideal I in A corresponds a maximal modular right ideal I_B in B such that $I = I_B \cap A$.*

PROOF. It is easily seen that $I_B = \{b \mid bA \subseteq I\}$ is a proper right ideal in B, and if e is a left identity for A modulo I, then e is also a left identity for B modulo I_B. For any $b \notin I_B$ the maximality of I implies $e \in bA + I$. Hence $e \in bB + I_B$ which means that I_B is maximal. Clearly $I = I_B \cap A$.

If $P = I:A$, we shall denote the primitive ideal $I_B:B$ by P_B. It may happen that $P = I:A = I':A$ and accordingly $P_B = I_B:B$ and $P_B' = I_B':B$. But then $BP_BA \subseteq P \subseteq I'$ and thus $P_B \subseteq P_B'$. Conversely, $P_B' \subseteq P_B$ so that we can define a mapping

$$\tau\colon \Pi \to \Pi_B \qquad \text{by} \qquad \tau(P) = P_B .$$

If we denote by $\tilde{\Pi}$ the complete image of Π in Π_B, we have:

THEOREM 2. *The mapping τ is a homeomorphism of Π onto $\tilde{\Pi}$.*

PROOF. First note that

$$A(P_B \cap A) \subseteq P_B \cap A \subseteq I_B \cap A = I$$

and

$$B(AP)A \subseteq AP \subseteq I ,$$

so that $AP \subseteq P_B \cap A \subseteq P$. Then suppose $\tau(P) = \tau(P') = P_B$. Now,

$$PP \subseteq AP \subseteq P_B \cap A \subseteq P' ,$$

and as P' is primitive, $P \subseteq P'$. Conversely, $P' \subseteq P$, and τ is one-to-one. For any $X \subseteq \Pi$ and any $P' \in hk(X)$ the following inclusions are valid:

$$\begin{aligned} Bk_B(\tau(X))A &\subseteq k_B(\tau(X)) \cap A \\ &= \bigcap \{P_B \mid P_B \in \tau(X)\} \cap A \\ &= \bigcap \{P_B \cap A \mid P_B \in \tau(X)\} \\ &\subseteq \bigcap \{P \mid P \in X\} = k(X) \subseteq P' , \end{aligned}$$

so that $k_B(\tau(X)) \subseteq \tau(P')$ which is equivalent to $\tau(P') \in h_Bk_B(\tau(X))$. Hence

$$\tau(hk(X)) \subseteq h_Bk_B(\tau(X)) .$$

Conversely, take any $\tau(P') \in h_Bk_B(\tau(X))$. Then

$$\begin{aligned} k(X)k(X) \subseteq Ak(X) &\subseteq \bigcap \{AP \mid P \in X\} \\ &\subseteq \bigcap \{P_B \cap A \mid P_B \in \tau(X)\} \\ &= A \cap k_B(\tau(X)) \\ &\subseteq A \cap \tau(P') \subseteq P' \end{aligned}$$

which yields $k(X) \subseteqq P'$ or $P' \in hk(X)$. Thus we have proved

$$\tau(hk(X)) = h_B k_B(\tau(X)) \cap \tilde{\Pi} .$$

THEOREM 3. $\tilde{\Pi} = \Pi_B \setminus h_B(A)$.

PROOF. If $P_B = J:B \notin h_B(A)$, then $I = J \cap A$ is a proper right ideal in A. As J is maximal, it may be defined as

$$J = \{b \mid bA \subseteqq I\} ,$$

but then any right ideal I' in A such that I is properly contained in I' satisfies $BA \subseteqq I'$. If e is an identity for B modulo J, then

$$(1-e)A \subseteqq J \cap A = I, \qquad \text{hence} \qquad A \subseteqq eA + I \subseteqq I'$$

and I is maximal. Now, there exists a maximal modular right ideal I'' such that $I:A = I'':A$ and, defining I_B'' as usual, we see that $b \in I_B'':B$ implies $BbA \subseteqq I'':A = I:A$, which means that $ABbA \subset I$, hence $ABb \subset J$ and thus

$$\{A,B\}(I_B:B) \subseteqq J:B = P_B .$$

Now, $\{A,B\}$ is a right ideal in B and as $\{A,B\} \subseteqq P_B$ would imply $AA \subseteqq I$ hence $A \subseteqq J$ since $J = \{b \mid bA \subseteqq I\}$, we conclude that $I_B:B \subseteqq P_B$. To see that the opposite inclusion holds, just note that

$$P_B A \subseteqq P_B \cap A \subseteqq I:A = I'':A ,$$

hence $P_B \subseteqq I_B'':B$. The proof is completed by observing that no primitive ideal in $h_B(A)$ is a member of $\tilde{\Pi}$.

If the left annihilator of A in B is zero, we may use the inclusion

$$k_B(\tilde{\Pi}) \subseteqq \{b \mid bA \subseteqq k(\Pi)\}$$

to obtain

THEOREM 4. *If A is semi-simple, then B is semi-simple too and $\tilde{\Pi}$ is dense in Π_B.*

PROOF. We denote the radical of A and B by $R(A)$ and $R(B)$, respectively, and have

$$R(B) = k_B(\Pi_B) \subseteqq k_B(\tilde{\Pi}) \subseteqq \{b \mid bA \subseteqq R(A)\} = 0$$

and

$$h_B k_B(\tilde{\Pi}) = h_B(0) = \Pi_B .$$

REFERENCE

1. Nathan Jacobson, *Structure of rings* (American Mathematical Society, Colloquium Publication 37), Providence, R. I., 1956.

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