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SKOLEM ARITHMETICS ON CERTAIN CONCRETE WORD SYSTEMS

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Given an abstract or concrete word system $\mathscr{W}$ in a finite or denumerably infinite alphabet, a Skolem arithmetic on $\mathscr{W}$ is an arithmetic constructible by the Skolem method on $\mathscr{W}$, that is, constructible by means of propositional calculus, definition by composition and recursion in $\mathscr{W}$, and proof by induction in $\mathscr{W}$ without the use of unbounded quantifiers.

There are five abstract word systems in general use, namely, the Dedekind word system $\Delta(a)$ in the one-sign alphabet $\{a\}$, the commutative word systems $\nabla(\boldsymbol{a}^{(n)})$ and $\nabla(\mathsf{A})$ in the finite alphabet $\boldsymbol{a}^{(n)}=\{a_1,a_2,\ldots,a_n\}$ and in the denumerably infinite alphabet $\mathsf{A}=\{a_1,a_2,\ldots\}$, respectively, and finally the noncommutative word systems $\Omega(\boldsymbol{a}^{(n)})$ and $\Omega(\mathsf{A})$ in the finite alphabet $\boldsymbol{a}^{(n)}$ and denumerably infinite alphabet A respectively. In our previous notes [8, 9, 10, 6, 11] we have given concrete interpretations of $\Delta(a)$, $\nabla(\mathsf{A})$ and $\Omega(\mathsf{A})$, namely, $\Delta(1)$ in the alphabet $\{1\}$, $\nabla(\mathsf{P})$ in the denumerably infinite alphabet $\mathsf{P}=\{p_1,p_2,p_3,\ldots\}$ (class of prime numbers) and $\Omega(\mathsf{P}^{(1)})$, $\Omega(\mathsf{P}^{(2)}),\ldots$ respectively in the denumerably infinite alphabet $\mathsf{P}^{(1)}=\{p_1{}^{(1)},p_2{}^{(1)},\ldots\}$ (the class of nonpower numbers), $\mathsf{P}^{(2)}=\{p_1{}^{(2)},p_2{}^{(2)},\ldots\}$ (the class of nontetrational numbers) and so on. Moreover, we have indicated the initial fragments of the Skolem arithmetics on $\Delta(1)$, $\nabla(\mathsf{P})$, $\Omega(\mathsf{P}^{(k)})$, $k=1,2,\ldots$, in general up through their corresponding unique resolution theorems.

In this paper, we show that our Skolem arithmetics on $\Delta(1)$, $\nabla(\mathsf{P})$, $\Omega(\mathsf{P}^{(k)})$, $k=1,2,3,\ldots$, denoted respectively as $\Sigma(1)$, $\Pi(\mathsf{P})$, $\Xi^{(k)}(\mathsf{P}^{(k)})$, are in fact interconstructible. We do not give full formal statements of these Skolem arithmetics, for such details follow the paper by Church [2] keeping in mind the difference between Church arithmetics and the Skolem arithmetics. (Actually, Church in his paper [2] deals only with a recursive arithmetic in a one-sign alphabet, but his ideas can be extended to the denumerably infinite alphabetical cases as well, moreover the Church arithmetics can be constructed either as Skolem arithmetics [2, 3] or Goodstein arithmetics [12].) Lately, there has been some super-

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ficial criticism against the Skolem arithmetic $\Sigma(1)$ as a method of treating the foundations of elementary number theory, specifically, that in $\Sigma(1)$ it takes entirely too long to reach the fundamental theorems usually presented at the outset of elementary number theory. We point out that these fundamental theorems of elementary number theory appear in the desired sequence in the Skolem arithmetic $\Pi(\mathsf{P})$, and the higher fundamental theorems such as those given in our notes [8, 9] in the Skolem arithmetics $\Xi^{(k)}(\mathsf{P}^{(k)})$, $k=1,2,3,\ldots$, are again for each k reshuffled in the sequence desired by the critics. The results in chapter 4 of this paper substantiate this point.

1. Abstract recursive word systems.

1.1. Dedekind word system $\Delta(a)$. Given the one-sign alphabet $\{a\}$, the empty word $\#$, the successor function ax satisfying the axiom $ax \neq ay \vee x=y$ and finally the equations $h_0=\{\#\}$, $h_{n+1}=\{ax \mid x \in h_n\}$, the word system $\Delta(a)=\bigcup_{m=0}^{\infty} h_m$ is called a Dedekind word system in $\{a\}$. An immediate consequence of this construction is the mathematical induction theorem.

1.2. Commutative word system $\nabla(\mathsf{A})$. Given the denumerably infinite alphabet $\mathsf{A}=\{a_1,a_2,a_3,\ldots\}$, the empty word $\#$, the denumerably infinite class of successor functions $a_\mu X$, $\mu \in \mathsf{N}$, where $\mathsf{N}=\{1,2,3,\ldots\}$, satisfying the axioms

$$a_\mu \neq a_\nu \vee \mu=\nu\ , \tag{1}$$

$$a_\mu(a_\nu X) \neq a_\nu(a_\mu Y) \vee X=Y\ , \tag{2}$$

$\mu,\nu \in \mathsf{N}$, and finally the equations

$$H_0 = \{\#\}, \qquad H_{n+1} = \{a_\mu X \mid X \in H_n \wedge \mu \in \mathsf{N}\}\ ,$$

the word system $\nabla(\mathsf{A})=\bigcup_{m=0}^{\infty} H_m$ is called a commutative word system in A. A quick consequence of this construction is the stage induction theorem of $\nabla(\mathsf{A})$:

$$\# \in \mathscr{S} \vee X \in \mathscr{S} \wedge a_\mu X \notin \mathscr{S} \vee \mathscr{S} = \nabla(\mathsf{A})\ .$$

For further details, see the papers by Vučković [15] and the author [10].

1.3. Noncommutative word system $\Omega(\mathsf{A})$. Given a denumerably infinite alphabet $\mathsf{A}=\{a_1,a_2,a_3,\ldots\}$, the empty word $\#$, the denumerably infinite class of successor functions $a_\mu X$, $\mu \in \mathsf{N}$, satisfying the axioms

(1) $$a_\mu X \neq a_\nu X \ \vee \ \mu = \nu ,$$

(2) $$a_\mu X \neq a_\mu Y \ \vee \ X = Y ,$$

$\mu, \nu \in \mathsf{N}$, and finally the equations

$$H_0 = \{\#\}, \qquad H_{n+1} = \{a_\mu X \mid X \in H_n \wedge \mu \in \mathsf{N}\} ,$$

the word system $\Omega(\mathsf{A}) = \bigcup_{m=0}^{\infty} H_m$ is called a noncommutative word system in A. Again, an immediate consequence of the above is the stage induction theorem of $\Omega(\mathsf{A})$:

$$\# \in \mathscr{S} \ \vee \ X \in \mathscr{S} \wedge a_\mu X \notin \mathscr{S} \ \vee \ \mathscr{S} = \Omega(\mathsf{A}) .$$

For details confer the papers by Péter [5] and the author [6].

2. Skolem arithmetics in one-sign alphabets.

2.1. Arithmetic $\Sigma(1)$. With the interpretation $\Delta(1)$ of the abstract Dedekind word system $\Delta(a)$, i.e., the word system in the alphabet $\{1\}$, empty word 0, the successor function $x+1$ and so on, we get the Skolem arithmetic $\Sigma(1)$ on $\Delta(1)$, i.e., the arithmetic constructible by the Skolem method on $\Delta(1)$. In $\Sigma(1)$, we shall denote variables in the usual lower-case italic letters. We point out that $\Sigma(1)$ is provided with the familiar primitive recursive scheme and that bounded quantifiers are available in $\Sigma(1)$ as well in all of the following Skolem arithmetics because they can be defined by recursion.

We note several definitions needed in this paper. Recalling our notes [8, 9], the Hilbert–Ackermann class of primitive recursive functions starting with

$$\xi_1(x,y) = [x,y], \qquad \text{where} \qquad [x,y] = x^y ,$$

$$\xi_2(x,y) = \langle x,y \rangle, \qquad \text{where} \qquad \langle x,0 \rangle = 1 , \quad \langle x,y+1 \rangle = [x, \langle x,y \rangle] ,$$

and in which every successive function is defined by the equations

$$\xi_{n+1}(x,0) = 1, \qquad \xi_{n+1}(x,y+1) = \xi_n\big(x, \xi_{n+1}(x,y)\big) ,$$

we define for $k = 1,2,3,\ldots$, the relations

$$R_k(y,x) \Leftrightarrow \bigvee z \leqq x\{x = \xi_k(y,z) \wedge z > 0 \wedge y > 1\} ,$$

$$M_k(x) \Leftrightarrow x \geqq 2 \wedge \bigwedge y \leqq x\{{}^{non}R_k(y,x) \vee x = y\} ,$$

${}^{non}R_k$ means the negation of R_k, and in turn the following classes of natural numbers by the equations

$$p_1^{(k)} = 2, \qquad p_{n+1}^{(k)} = \mu z \leqq [2,[2,n+1]]\{z > p_n^{(k)} \wedge M_k(z)\} .$$

For example, for $k=1$ we have the class of nonpower numbers, for $k=2$ the class of nontetrational numbers, and so on. We denote these classes by $\mathsf{P}^{(k)}$, $k=1,2,3,\ldots$.

Also, we shall have occasion to use the following primitive recursive functions of $\Sigma(1)$. In the following, $k=1,2,3,\ldots$. Firstly, we define

$$l^{(k)}(2,x) = \mu z \leqq x\{x=\xi_k(2,z) \vee x=0\}.$$

In turn, we define the primitive recursive functions γ, $\bar{\gamma}$ by the equations:

$$(1) \qquad \gamma(0) = 1, \qquad \gamma(x+1) = 2\cdot\gamma(x);$$

$$(2) \qquad \begin{aligned} \bar{\gamma}(x) &= 0 && \text{if} \quad {}^{non}R_1(2,x)\,, \\ &= l^{(1)}(2,x) && \text{if} \quad R_1(2,x)\,. \end{aligned}$$

Finally, the primitive recursive functions $\gamma^{(k)}$, $\bar{\gamma}^{(k)}$ by the equations:

$$(3) \qquad \gamma^{(k)}(0) = 1, \ \gamma^{(k)}(x+1) = \xi_k\big(2,\gamma^{(k)}(x)\big);$$

$$(4) \qquad \begin{aligned} \bar{\gamma}^{(k)}(x) &= 0 && \text{if} \quad {}^{non}R_{k+1}(2,x)\,, \\ &= l^{(k+1)}(2,x) && \text{if} \quad R_{k+1}(2,x)\,. \end{aligned}$$

Clearly, $\bar{\gamma}$ is the inverse of γ and $\bar{\gamma}^{(k)}$ the inverses of $\gamma^{(k)}$.

For details, see the papers by Skolem [13, 4], and the author [8, 9].

2.2. Arithmetic $\Pi(p_1)$. Given the interpretation $\Delta(p_1)$ of $\Delta(a)$, i.e., the word system in the alphabet $\{p_1\}$, $p_1=2$, the empty word 1, the successor function $p_1\cdot\boldsymbol{x}$ or briefly $p_1\boldsymbol{x}$ and so on, with $h_0=\{1\}$, $h_{n+1}=\{p_1x \mid x\in h_n\}$ and $\Delta(p_1)=\bigcup_{m=0}^{\infty}h_m$, the following arithmetic constructible by the Skolem method on $\Delta(p_1)$ is denoted by $\Pi(p_1)$. (Note, $\Pi(p_1)$ is the Skolem arithmetic on numbers $[2,\mu](\mu\in\mathsf{N})$.) We shall denote the variables in lowercase boldface letters. The primitive recursive scheme of $\Pi(p_1)$ is of the form:

$$\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,1) = \boldsymbol{g}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\,,$$

$$\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,p_1\boldsymbol{y}) = \boldsymbol{h}\big(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,\boldsymbol{y},\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,\boldsymbol{y})\big)\,.$$

Next, in $\Pi(p_1)$, we have word addition defined by

$$\boldsymbol{x}\oplus 1 = \boldsymbol{x}, \qquad \boldsymbol{x}\oplus p_1\boldsymbol{y} = p_1(\boldsymbol{x}\oplus\boldsymbol{y})\,,$$

word multiplication by

$$\boldsymbol{x}\odot 1 = 1, \qquad \boldsymbol{x}\odot p_1\boldsymbol{y} = \boldsymbol{x}\oplus(\boldsymbol{x}\odot\boldsymbol{y})\,,$$

word exponentiation by

$$\boldsymbol{x}\vartriangle 1 = p_1, \qquad \boldsymbol{x}\vartriangle p_1\boldsymbol{y} = \boldsymbol{x}\odot(\boldsymbol{x}\vartriangle\boldsymbol{y})\,,$$

word predecessor function by

$$\bar{p}_1 1 = 1, \qquad \bar{p}(p_1 \boldsymbol{x}) = \boldsymbol{x} ,$$

restricted word subtraction by

$$\boldsymbol{x}[\dot{-}]1 = \boldsymbol{x}, \qquad \boldsymbol{x}[\dot{-}]p_1\boldsymbol{y} = p_1(\boldsymbol{x}[\dot{-}]\boldsymbol{y}) ,$$

and the relation $\boldsymbol{x} \leqq \boldsymbol{y}$ by

$$\boldsymbol{x} \leqq \boldsymbol{y} \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}[\dot{-}](\boldsymbol{y}[\dot{-}]\boldsymbol{x}) ,$$

and for the case $\boldsymbol{x} \neq \boldsymbol{y}$ we use $\boldsymbol{x} \prec \boldsymbol{y}$. In turn, we have the word-divisibility relation

$$\boldsymbol{y} \,\|\, \boldsymbol{x} \Leftrightarrow \bigvee \boldsymbol{z} \leqq \boldsymbol{x}\{\boldsymbol{x} = \boldsymbol{y} \odot \boldsymbol{z} \wedge \boldsymbol{z} \neq 1\} ,$$

primitive-word relation

$$\mathrm{pw}(\boldsymbol{x}) \Leftrightarrow \boldsymbol{x} \neq 1 \wedge \boldsymbol{x} \neq p_1 \wedge \bigwedge \boldsymbol{z} \leqq \boldsymbol{x}\{\boldsymbol{z} \,{}^{non}\|\, x \vee \boldsymbol{z} = \boldsymbol{x} \vee \boldsymbol{z} = p_1\}$$

and finally the class of primitive words of $\Pi(p_1)$ defined by

$$\mathfrak{p}_1 = \mathfrak{n}_2, \qquad \mathfrak{p}_{n+1} = \mu z \leqq (\mathfrak{n}_2 \vartriangle (\mathfrak{n}_2 \vartriangle \mathfrak{n}_{n+1}))\{\mathfrak{p}_n \prec z \wedge \mathrm{pw}(z)\} ,$$

where $\mathfrak{n}_1 = p_1$, $\mathfrak{n}_{n+1} = \mathfrak{n}_n \oplus p_1$. For example, $\mathfrak{p}_n = [2, p_n]$, where p_n is the nth prime number of P. Further, we note that in $\Pi(p_1)$ we also have a primitive-word unique resolution theorem, proved in a parallel way as in $\Sigma(1)$. Moreover, we have the primitive recursive function

$$\exp(\mathfrak{p}_n, \boldsymbol{x}) = \mu \boldsymbol{z} \leqq \boldsymbol{x}\{(\mathfrak{p}_n \vartriangle p_1 \boldsymbol{z}) \,{}^{non}\|\, \boldsymbol{x}\}$$

(the word-exponent of the nth primitive word in the primitive-word resolution of $\boldsymbol{x}$) and

$$\mathrm{gpw}(\boldsymbol{x}) = \mu \mathfrak{p}_n \leqq \boldsymbol{x}\{\mathfrak{p}_n \,\|\, \boldsymbol{x} \wedge \bigwedge \boldsymbol{z} \leqq \boldsymbol{x}\{\mathfrak{p}_n \,{}^{non}\|\, \boldsymbol{z} \vee \boldsymbol{z} \leqq \boldsymbol{x}\}\}$$

(the greatest primitive word which $\boldsymbol{x}$ is word-divisible).

2.3. Arithmetics $\Xi^{(k)}(p_1^{(k)})$. For all cases of the use of k in this section, $k = 1, 2, 3, \ldots$. Given the interpretation $\Delta(p_1^{(k)})$ of $\Delta(a)$, i.e., the word system in the alphabet $\{p_1^{(k)}\}$ (see, 2.1), the empty word 1, the successor function $\xi_k(p_1^{(k)}, \boldsymbol{x})$ or briefly $p_1^{(k)}\boldsymbol{x}$, and so on, the following arithmetic constructible by means of the Skolem method on $\Delta(p_1^{(k)})$ is denoted by $\Xi^{(k)}(p_1^{(k)})$. (Note, $\Xi^{(1)}(p_1^{(1)})$ is the Skolem arithmetic on numbers $\langle 2, \mu \rangle$, $\mu \in \mathsf{N}$, $\Xi^{(2)}(p_1^{(2)})$ is the Skolem arithmetic on numbers $\xi_3(2, \mu)$, $\mu \in \mathsf{N}$, and so on.) As in $\Pi(p_1)$, we shall denote the variables of $\Xi^{(k)}(p_1^{(k)})$ in the lower-case boldface letters. The primitive recursive scheme of $\Xi^{(k)}(p_1^{(k)})$ is of the form:

$$f(x_1,\ldots,x_n,1) = g(x_1,\ldots,x_n)\,,$$

$$f(x_1,\ldots,x_n,p_1^{(k)}y) = h\big(x_1,\ldots,x_n,y,f(x_1,\ldots,x_n,y)\big)\,.$$

In $\Xi^{(k)}(p_1^{(k)})$, we have word addition defined by

$$x\oplus^{(k)}1 = x, \qquad x\oplus^{(k)}p_1^{(k)}y = p_1^{(k)}(x\oplus^{(k)}y)\,,$$

word multiplication defined by

$$x\odot^{(k)}1 = 1, \qquad x\odot^{(k)}p_1^{(k)}y = x\oplus^{(k)}(x\odot^{(k)}y)\,,$$

word exponentiation by

$$x \mathbin{\triangle^{(k)}} 1 = p_1^{(k)}, \qquad x \mathbin{\triangle^{(k)}} p_1^{(k)}y = x\odot^{(k)}(x \mathbin{\triangle^{(k)}} y)\,,$$

and a word-version in $\Xi^{(k)}(p_1^{(k)})$ of the Hilbert–Ackermann class of primitive recursive functions starting with $\xi_1^{(k)}(x,y)=x \mathbin{\triangle^{(k)}} y$ and in which each successive function is defined by the equations

$$\xi^{(k)}_{n+1}(x,1) = p_1^{(k)}, \qquad \xi^{(k)}_{n+1}(x,p_1^{(k)}y) = \xi^{(k)}_n\big(x,\xi^{(k)}_{n+1}(x,y)\big)\,.$$

Moreover, we have the word predecessor function $\overline{p}_1^{(k)}x$, restricted word subtraction $x(\dot{-})^{(k)}y$ and the relations $x\leqq^{(k)}y$, $x\prec^{(k)}y$ defined along the lines given in 2.2.

In turn, we have the k-divisibility and k-irreducible-word relations defined respectively by

$$y \,\|^{(k)}\, x \Leftrightarrow \bigvee z\leqq^{(k)}x\{x=\xi_k^{(k)}(y,z) \wedge z\neq 1\}$$

and

$$\mathrm{pw}^{(k)}(x) \Leftrightarrow x\neq 1 \wedge x\neq p_1^{(k)} \wedge \bigwedge z\leqq^{(k)}x\{z \;{}^{non}\|^{(k)}\, x \vee z=x\}\,,$$

and finally the equations

$$\mathfrak{p}_1^{(k)} = \mathfrak{n}_2^{(k)}, \qquad \mathfrak{p}_{n+1}^{(k)} = \mu z \leqq^{(k)} \mathfrak{n}_2^{(k)} \mathbin{\triangle} (\mathfrak{n}_2^{(k)} \mathbin{\triangle} \mathfrak{n}_{n+1}^{(k)})\{\mathfrak{p}_n^{(k)} \prec^{(k)} z \wedge \mathrm{pw}^{(k)}(z)\}$$

($\mathfrak{p}_n^{(k)}$ is the nth k-irreducible word), where $\mathfrak{n}_1^{(k)}=p_1^{(k)}$, $\mathfrak{n}_{n+1}^{(k)}=\mathfrak{n}_n^{(k)}\oplus p_1^{(k)}$. For example, $\mathfrak{p}_n^{(1)}=\langle 2,p_n^{(1)}\rangle$, where $p_n^{(1)}$ is the nth nonpower number in $\mathsf{P}^{(1)}$. We point out that in $\Xi^{(k)}(p_1^{(k)})$ we again have a k-irreducible-word unique resolution theorem (see 3.1). Moreover, using the function $x[\![k]\!]y$ defined by the equations

$$1[\![k]\!]x = x, \qquad p_1^{(k)}[\![k]\!]x = \xi_k^{(k)}(p_1^{(k)},x),$$

$$\xi_k^{(k)}(\mathfrak{p}_\mu^{(k)},y)[\![k]\!]x = \xi_k^{(k)}(\mathfrak{p}_\mu^{(k)},y[\![k]\!]x)\,,$$

$\mu \in \mathsf{N}$, which is primitive recursive in $\Xi^{(k)}(p_1^{(k)})$ (see 3.1), and the relation

$$y \subseteq^{(k)} x \Leftrightarrow \bigvee u \leqq^{(k)} x \bigvee v \leqq^{(k)}x\{x=u[\![k]\!](y[\![k]\!]v) \wedge 1 \prec^{(k)} v\}\,,$$

we define the primitive recursive functions

$$\exp^{(k)}(\mathfrak{p}_n^{(k)},x) = \mu z \leqq^{(k)} x\{\mathfrak{S}_{k+1}^{(k)}(\mathfrak{p}_n^{(k)},p_1^{(k)}z) \ {}^{non}\!\subseteqq^{(k)} x\}$$

(the k-word exponent of the nth k-irreducible word in the k-irreducible word resolution of x), and finally

$$\mathrm{gpw}^{(k)}(x) = \mu \mathfrak{p}_n^{(k)} \leqq^{(k)} x\{\mathfrak{p}_n^{(k)} \|^{(k)} x \wedge \bigwedge z \leqq^{(k)} x\{\mathfrak{p}_n^{(k)} \ {}^{non}\|^{(k)} z \vee z \leqq^{(k)} x\}\}$$

(the greatest k-irreducible word by which x is k-word-divisible).

3. Skolem arithmetics in denumerably infinite alphabets.

3.1. Arithmetics $\Xi^{(k)}(\mathfrak{P}^{(k)})$. As pointed out in section 2.3, the class of unique k-irreducible-word resolution theorems ($k=1,2,3,\ldots$) and the function $x[\![k]\!]y$, $k=1,2,3,\ldots$, are in the Skolem arithmetics $\Xi^{(k)}(p_1^{(k)})$. This is established in a parallel way as given in our notes [8, 9] with respect to the interpreation $\Omega(\mathfrak{P}^{(k)})$ of $\Omega(\mathsf{A})$. That is, the word system in the denumerably infinite alphabet $\mathfrak{P}^{(k)}=\{\mathfrak{p}_1^{(k)},\mathfrak{p}_2^{(k)},\ldots\}$, where $\mathfrak{p}_1^{(k)}$, $\mathfrak{p}_2^{(k)},\ldots$ are the consecutive k-irreducible words of $\Xi^{(k)}(p_1^{(k)})$, with the empty word $p_1^{(k)}$, the denumerably infinite class of successor functions $\mathfrak{S}_k(\mathfrak{p}_\mu^{(k)},x)$, $\mu\in\mathsf{N}$, or briefly $\mathfrak{p}_\mu^{(k)}x$ satisfying the axioms given in 1.3, the equations

$$H_0 = \{p_1^{(k)}\}, \qquad H_{n+1} = \{\mathfrak{p}_\mu^{(k)}x \mid x\in H_n \wedge \mu\in\mathsf{N}\}$$

and finally

$$\Omega(\mathfrak{P}^{(k)}) = \bigcup_{m=0}^{\infty} H_m .$$

For example, the words of $\Omega(\mathfrak{P}^{(1)})$ are of the form $\langle 2,[p_{i_r}^{(1)},p_{i_{r-1}}^{(1)},\ldots,p_{i_1}^{(1)}]\rangle$, $i_1,\ldots,i_r\in\mathsf{N}$, where $[x_1]=x_1$, $[x_{n+1},x_n,\ldots,x_1]=\big[x_{n+1},[x_n,\ldots,x_1]\big]$. The Skolem arithmetic on $\Omega(\mathfrak{P}^{(k)})$, with word addition and multiplication defined like in section 3.3, we denote by $\Xi^{(k)}(\mathfrak{P}^{(k)})$. Here we can denote the variables by lower-case boldface letters. The primitive recursive scheme of $\Xi^{(k)}(\mathfrak{P}^{(k)})$ is of the form:

$$F(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,p_1^{(k)}) = G(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\,,$$

$$F(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,\mathfrak{p}_\mu^{(k)}\boldsymbol{y}) = H_\mu\big(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,\boldsymbol{y},F(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n,\boldsymbol{y})\big), \qquad \mu\in\mathsf{N}\,.$$

Lastly, in an exactly parallel way as in [9], we can show that definition by primitive recursion in $\Xi^{(k)}(\mathfrak{P}^{(k)})$ implies definition by primitive recursion in $\Xi^{(k)}(p_1^{(k)})$ and that the k-irreducible-word unique resolution theorems are theorems of $\Xi^{(k)}(p_1^{(k)})$.

3.2. Arithmetic $\Pi(\mathsf{P})$. As in our note [10], given the interpretation $\nabla(\mathsf{P})$ of the commutative word system $\nabla(\mathsf{A})$, i.e., the word system in the prime-number alphabet $\mathsf{P}=\{p_1,p_2,\ldots\}$, $p_1=2$, with the empty word 1, the denumerably infinite class of successor functions $p_\mu\cdot X$, $\mu\in\mathsf{N}$, or briefly $p_\mu X$ and so on, the following arithmetic constructible by the Skolem method on $\nabla(\mathsf{P})$ is the Skolem arithmetic $\Pi(\mathsf{P})$. In $\Pi(\mathsf{P})$, we shall denote the variables in the upper-case italic letters. The primitive recursive scheme in $\Pi(\mathsf{P})$ is of the form:

$$F(X_1,\ldots,X_n,1) = G(X_1,\ldots,X_n)\,,$$

$$F(X_1,\ldots,X_n,p_\mu Y) = H_\mu(X_1,\ldots,X_n,Y,F(X_1,\ldots,X_n,Y)), \qquad \mu\in\mathsf{N}\,.$$

In particular, with μ, $\nu\in\mathsf{N}$, we have word addition defined by

$$X\oplus 1 = X, \qquad X\oplus p_\mu Y = p_\mu(X\oplus Y)\,,$$

the subscript function defined by

$$\sigma_\nu(1) = 1, \qquad \sigma_\nu(p_\mu X) = \sigma_\nu(X)\oplus p_{\nu\cdot\mu}\,,$$

word multiplication defined by

$$X\odot 1 = 1, \qquad X\odot p_\mu Y = \sigma_\mu(X)\oplus(X\odot Y)\,,$$

word exponentiation defined by

$$X\vartriangle 1 = p_1, \qquad X\vartriangle p_\mu Y = \sigma_\mu(X)\odot(X\vartriangle Y)\,,$$

word predecessor function

$$\begin{aligned}\bar{p}_\mu 1 = 1,\ \bar{p}_\nu p_\mu X &= X &&\text{if}\quad \nu=\mu\,,\\ &= p_\mu(\bar{p}_\nu X) &&\text{if}\quad \nu\neq\mu\,,\end{aligned}$$

restricted word subtraction

$$X[\dot{-}]1 = X, \qquad X[\dot{-}]p_\mu Y = \bar{p}_\mu(X[\dot{-}]Y)\,,$$

and finally the relation

$$X\leqq Y \Leftrightarrow X = Y[\dot{-}](Y[\dot{-}]X)\,.$$

Lastly, we have the length function of $\Pi(\mathsf{P})$ defined by

$$\lambda(1) = 0, \qquad \lambda(p_\mu X) = \lambda(X)+1\,,$$

and the index function by

$$\begin{aligned}\operatorname{ind}(X) &= \mu &&\text{if}\quad X=p_\mu\,,\\ &= 0 &&\text{if}\quad X\neq p_\mu\,.\end{aligned}$$

Now we can introduce numeral words in $\Pi(\mathsf{P})$ by $N(1)=1$, $N(p_\mu X)=p_1(N(X))$, and $\operatorname{num}(X) \Leftrightarrow X=N(X)$. Clearly, since numeral words are recursively definable in $\Pi(\mathsf{P})$ and the primitive recursive scheme of $\Pi(p_1)$ is a special case of the primitive recursive scheme of $\Pi(\mathsf{P})$, we can consider the functions exp and gpw defined in 2.2 as primitive recursive functions of $\Pi(\mathsf{P})$. In turn, we define the primitive recursive functions Γ, $\bar{\Gamma}$ by the following equations:

$$(1) \qquad \Gamma(1) = p_1,\ \Gamma(p_\mu X) = \mathfrak{p}_\mu \odot \Gamma(X),\ \mu \in \mathsf{N};$$

$$(2) \quad \bar{\Gamma}(X) = 1 \quad \text{if} \quad X \neq \operatorname{num}(X)\,,$$
$$= \sigma_{\delta(X)}\big(\exp(\mathfrak{p}_{\delta(X)}, X)\big) \oplus \ldots \oplus \sigma_2\big(\exp(\mathfrak{p}_2, X)\big) \oplus \sigma_1\big(\exp(\mathfrak{p}_1, X)\big)$$
$$\text{if} \quad X = \operatorname{num}(X)\,,$$

where $\delta(X) = \operatorname{ind}\big(\lambda(\operatorname{gwp}(X))\big)$. Clearly, $\bar{\Gamma}$ is the inverse of Γ.

We point out that in the Skolem arithmetic $\Pi(\mathsf{P})$ we also have a primitive-word theory up through the primitive-word unique resolution theorem. For details, see the paper by the author [10]. Also, confer the paper by Vučković [15].

3.3. Arithmetics $\Xi^{(k)}(\mathsf{P}^{(k)})$. For all cases of the use of k in this section, $k = 1, 2, 3, \ldots$. As in our notes [8, 9], given the interpretation $\Omega(\mathsf{P}^{(k)})$ of the noncommutative word system $\Omega(\mathsf{A})$, i.e., the interpretation in the denumerably infinite alphabet $\mathsf{P}^{(k)}$ (see 2.1), with the empty word 1, the denumerably infinite class of successor functions $\xi_k(p_\mu{}^{(k)}, X)$, $\mu \in \mathsf{N}$, or briefly $p_\mu{}^{(k)}X$ and so on, the following Skolem arithmetic on $\Omega(\mathsf{P}^{(k)})$ is $\Xi^{(k)}(\mathsf{P}^{(k)})$. We denote the variables of $\Xi^{(k)}(\mathsf{P}^{(k)})$ in the upper-case italic letters. The primitive recursive scheme of $\Xi^{(k)}(\mathsf{P}^{(k)})$ is of the form:

$$F(X_1, \ldots, X_n, 1) = G(X_1, \ldots, X_n)\,,$$

$$F(X_1, \ldots, X_n, p_\mu{}^{(k)}Y) = H_\mu\big(X_1, \ldots, X_n, Y, F(X_1, \ldots, X_n, Y)\big),\ \mu \in \mathsf{N}\,.$$

In $\Xi^{(k)}(\mathsf{P}^{(k)})$, with $\mu, \nu \in \mathsf{N}$, we have word addition defined by

$$X \oplus^{(k)} 1 = X,\ X \oplus^{(k)} p_\mu{}^{(k)} Y = p_\mu{}^{(k)}(X \oplus^{(k)} Y)\,,$$

which is associative but not commutative, the subscript functions

$$(1) \qquad \sigma_\mu^{(k)}(1) = 1, \qquad \sigma_\mu^{(k)}(p_\nu^{(k)}X) = \sigma_\mu^{(k)}(X) \oplus^{(k)} p^{(k)}_{\nu \oplus^{(k)} \mu}\,,$$

$$(2) \qquad \sigma_\mu(1) = 1, \qquad \sigma_\mu(p_\nu^{(k)}X) = \sigma_\mu(X) \oplus^{(k)} p^{(k)}_{\nu \cdot \mu}\,,$$

and in turn word multiplication defined by

$$X \odot^{(k)} 1 = 1,\ X \odot^{(k)} p_\mu{}^{(k)} Y = \sigma_\mu{}^{(k)}(X) \oplus^{(k)} (X \odot^{(k)} Y)\,,$$

word exponentiation

$$X \mathbin{\triangle^{(k)}} 1 = p_1{}^{(k)}, \qquad X \mathbin{\triangle^{(k)}} p_\mu{}^{(k)} Y = \sigma_\mu{}^{(k)}(X) \odot^{(k)} (X \mathbin{\triangle^{(k)}} Y)\,,$$

and a word-version in $\Xi^{(k)}(\mathsf{P}^{(k)})$ of the Hilbert-Ackermann class of primitive recursive functions starting with $\xi_1^{(k)}(X,Y)=X \,\Delta^{(k)}\, Y$ and in which every succeeding function is defined by the equations

$$\xi_{n+1}^{(k)}(X,1) = p_1^{(k)}, \qquad \xi_{n+1}^{(k)}(X,p_\mu^{(k)}Y) = \xi_n^{(k)}\big(\sigma_\mu^{(k)}(X),\xi_{n+1}^{(k)}(X,Y)\big)\,.$$

As given in our note [11], we have a word restricted subtraction $X[\dot{-}]^{(k)}Y$ and the relation $X\leqq^{(k)} Y$. Finally, we have the length function defined by

$$\lambda^{(k)}(1) = 0, \qquad \lambda^{(k)}(p_\mu{}^{(k)}X) = \lambda^{(k)}(X)+1$$

and the index function defined by

$$\begin{aligned}\operatorname{ind}^{(k)}(X) &= \mu \quad \text{if} \quad X = p_\mu{}^{(k)}\,,\\ &= 0 \quad \text{if} \quad X \neq p_\mu{}^{(k)}\,.\end{aligned}$$

Finally, we introduce numeral words in $\Xi^{(k)}(\mathsf{P}^{(k)})$ by

$$N^{(k)}(1) = 1, \qquad N^{(k)}(p_\mu{}^{(k)}X) = p_1{}^{(k)}\big(N^{(k)}(X)\big)\,,$$

and

$$\operatorname{num}^{(k)}(X) \Leftrightarrow X = N^{(k)}(X)\,.$$

As pointed out in 3.2, with the numeral words recursively defined in $\Xi^{(k)}(\mathsf{P}^{(k)})$ and the primitive recursive scheme of $\Xi^{(k)}(p_1{}^{(k)})$ being a special case of the recursive shceme of $\Xi^{(k)}(\mathsf{P}^{(k)})$, we can consider the functions $\exp^{(k)}$ and $\mathrm{gpw}^{(k)}$ defined in 2.3 as primitive recursive functions of $\Xi^{(k)}(\mathsf{P}^{(k)})$. In turn, we define the primitive recursive function $\Gamma^{(k)}$ and its inverse $\bar{\Gamma}^{(k)}$ by the following equations:

$$(1) \qquad \Gamma^{(k)}(1) = p_1{}^{(k)}, \qquad \Gamma^{(k)}(p_\mu{}^{(k)}X) = \xi_k{}^{(k)}\big(\mathfrak{p}_\mu{}^{(k)},\Gamma^{(k)}(X)\big)(\mu\in\mathsf{N})\,;$$

$$\begin{aligned}(2) \qquad \bar{\Gamma}^{(k)}(X) &= 1 \quad \text{if} \quad X \neq \operatorname{num}^{(k)}(X)\,,\\ &= \sigma_{\delta^{(k)}(X)}\big(\exp^{(k)}(\mathfrak{p}^{(k)}_{\delta^{(k)}(X)},X)\big)\oplus\ldots\oplus\sigma_1\big(\exp^{(k)}(\mathfrak{p}_1{}^{(k)},X)\big)\\ &\qquad \text{if} \quad X = \operatorname{num}^{(k)}(X)\,,\end{aligned}$$

where $\delta^{(k)}(X)=\operatorname{ind}^{(k)}\big(\lambda^{(k)}(\mathrm{gwp}^{(k)}(X))\big)$.

For details see the papers by the author [7, 11], where the word arithmetic which runs through certain primitive-word unique resolution theorems carries over to the Skolem arithmetics $\Xi^{(k)}(\mathsf{P}^{(k)})$. Also, see the papers of van Rootselaar [14] and Vučković [16].

4. Interconstructibility.

4.1. Constructibility of $\Sigma(1)$ in $\Pi(\mathsf{P})$. Recalling the primitive recursive functions γ, $\bar{\gamma}$ of $\Sigma(1)$, it is not difficult to see that γ, $\bar{\gamma}$ give an arithmetization between $\Delta(1)$ and $\Delta(p_1)$. Next, following Asser [1], we state some preliminaries. Let f be a function of $\Sigma(1)$. Denote by $\gamma[f]$ the function of $\Pi(p_1)$ such that

$$\gamma[f](\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n) = \gamma\big(f(\bar{\gamma}(\boldsymbol{x}_1),\ldots,\bar{\gamma}(\boldsymbol{x}_n))\big) .$$

On the other hand, if $\boldsymbol{f}$ is a function of $\Pi(p_1)$, denote by $\bar{\gamma}[\boldsymbol{f}]$ the function of $\Sigma(1)$ such that

$$\bar{\gamma}[\boldsymbol{f}](x_1,\ldots,x_n) = \bar{\gamma}\big(\boldsymbol{f}(\gamma(x_1),\ldots,\gamma(x_n))\big) .$$

In turn, recalling the primitive recursive functions Γ, $\bar{\Gamma}$ of $\Pi(\mathsf{P})$, we note that Γ, $\bar{\Gamma}$ also give an arithmetization between $\nabla(\mathsf{P})$ and $\Delta(p_1)$. If $\boldsymbol{f}$ is a function of $\Pi(p_1)$, denote by $\bar{\Gamma}[\boldsymbol{f}]$ the function of $\Pi(\mathsf{P})$ such that

$$\bar{\Gamma}[\boldsymbol{f}](X_1,\ldots,X_n) = \bar{\Gamma}\big(\boldsymbol{f}(\Gamma(X_1),\ldots,\Gamma(X_n))\big) .$$

Finally, the following theorems are evident. (1) If f is a primitive recursive function of $\Sigma(1)$, then $\gamma[f]$ is also a primitive recursive function of $\Pi(p_1)$. (2) If $\boldsymbol{f}$ is a primitive recursive function of $\Pi(p_1)$, then $\bar{\gamma}[\boldsymbol{f}]$ is also a primitive recursive function of $\Sigma(1)$. (3) If $\boldsymbol{f}$ is a primitive recursive function of $\Pi(p_1)$, then $\bar{\Gamma}[\boldsymbol{f}]$ is also a primitive recursive function of $\Pi(\mathsf{P})$.

To prove that $\Sigma(1)$ is constructible in $\Pi(\mathsf{P})$ it is enough to show that (I) the primitive recursive functions of $\Sigma(1)$ are primitive recursively definable in $\Pi(\mathsf{P})$ and (II) induction in $\Sigma(1)$ implies induction in $\Pi(\mathsf{P})$. The rest carries over easily.

(I) Let $f(x_1,\ldots,x_n)$ be any primitive recursive function of $\Sigma(1)$. On the strength of theorem (1) of this section,

$$\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n) = \gamma[f](\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$$

is a primitive recursive function of $\Pi(p_1)$. Next, by theorem (3), $\bar{\Gamma}[\boldsymbol{f}](X_1,\ldots,X_n)$ is a primitive recursive function of $\Pi(\mathsf{P})$. Now note that γ defined in $\Pi(\mathsf{P})$ is nothing more than the primitive recursive function Γ of $\Pi(\mathsf{P})$. Consequently, we have

$$\bar{\Gamma}\big[\Gamma[f]\big](X_1,\ldots,X_n) = \bar{\Gamma}[\boldsymbol{f}](X_1,\ldots,X_n) ,$$

and furthermore, since $\bar{\Gamma}$ is the inverse of Γ, we have

$$f(X_1,\ldots,X_n) = \bar{\Gamma}[\boldsymbol{f}](X_1,\ldots,X_n) ,$$

which means that f is primitive recursively definable in $\Pi(\mathsf{P})$.

(II) The proof that induction in $\Sigma(1)$ implies induction in $\Pi(\mathsf{P})$ parallels the proof given in our note [9]: Induction in $\Sigma(1)$ is equivalent to the principle of uniqueness of primitive recursion in $\Sigma(1)$ and in turn induction in $\Pi(\mathsf{P})$ is equivalent to the principle of uniqueness of primitive recursion in $\Pi(\mathsf{P})$. Using (I) above, we can conclude that induction in $\Sigma(1)$ implies induction in $\Pi(\mathsf{P})$.

4.2. Constructibility of $\Sigma(1)$ in $\Xi^{(k)}(\mathsf{P}^{(k)})$. In all cases of the use of k in this section, $k=1,2,3,\ldots$. As in 4.1, we use $\gamma^{(k)}$, $\bar{\gamma}^{(k)}$ and $\Gamma^{(k)}$, $\bar{\Gamma}^{(k)}$ respectively, for the arithmetizations of $\Delta(1)$, $\Delta(p_1{}^{(k)})$ and $\Omega(\mathsf{P}^{(k)})$, $\Delta(p_1{}^{(k)})$ respectively. Again if f is a function of $\Sigma(1)$, denote by $\gamma^{(k)}[f]$ the function of $\Xi^{(k)}(p_1{}^{(k)})$ such that

$$\gamma^{(k)}[f](\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n) = \gamma^{(k)}\big(f\big(\bar{\gamma}^{(k)}(\boldsymbol{x}_1),\ldots,\bar{\gamma}^{(k)}(\boldsymbol{x}_n)\big)\big) .$$

On the other hand, if $\boldsymbol{f}$ is a function of $\Xi^{(k)}(p_1{}^{(k)})$, denote by $\bar{\gamma}^{(k)}[\boldsymbol{f}]$ the function of $\Sigma(1)$ such that

$$\bar{\gamma}[\boldsymbol{f}](x_1,\ldots,x_n) = \bar{\gamma}\big(\boldsymbol{f}\big(\gamma^{(k)}(x_1),\ldots,\gamma^{(k)}(x_n)\big)\big) .$$

Lastly, if $\boldsymbol{f}$ is a function of $\Xi^{(k)}(p_1{}^{(k)})$, we denote by $\bar{\Gamma}^{(k)}[\boldsymbol{f}]$ the function of $\Xi^{(k)}(\mathsf{P}^{(k)})$ such that

$$\bar{\Gamma}^{(k)}[\boldsymbol{f}](X_1,\ldots,X_n) = \bar{\Gamma}^{(k)}\big(\boldsymbol{f}\big(\Gamma^{(k)}(X_1),\ldots,\Gamma^{(k)}(X_n)\big)\big) .$$

Finally, the following theorems are not difficult to see.

(1) If f is a primitive recursive function of $\Sigma(1)$, then $\gamma^{(k)}[f]$ is also a primitive redursive function of $\Xi^{(k)}(p_1{}^{(k)})$.

(2) If $\boldsymbol{f}$ is a primitive recursive function of $\Xi^{(k)}(p_1{}^{(k)})$, then $\bar{\gamma}^{(k)}[\boldsymbol{f}]$ is also a primitive recursive function of $\Sigma(1)$.

(3) If $\boldsymbol{f}$ is a primitive recursive function of $\Xi^{(k)}(p_1{}^{(k)})$, then $\bar{\Gamma}^{(k)}[\boldsymbol{f}]$ is also a primitive recursive function of $\Xi^{(k)}(\mathsf{P}^{(k)})$.

To verify that $\Sigma(1)$ is constructible in $\Xi^{(k)}(\mathsf{P}^{(k)})$ we shall outline that (I) the primitive recursive functions of $\Sigma(1)$ are primitive recursively definable in $\Xi^{(k)}(\mathsf{P}^{(k)})$ and (II) induction in $\Sigma(1)$ implies induction in $\Xi^{(k)}(\mathsf{P}^{(k)})$.

(I) Let $f(x_1,\ldots,x_n)$ be any primitive recursive function of $\Sigma(1)$. By theorem (1) of this section, $\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)=\gamma^{(k)}[f](\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$ is a primitive recursive function of $\Xi^{(k)}(p_1{}^{(k)})$. By theorem (3) of this section, $\bar{\Gamma}^{(k)}[\boldsymbol{f}](X_1,\ldots,X_n)$ is a primitive recursive function of $\Xi^{(k)}(\mathsf{P}^{(k)})$. Again, $\gamma^{(k)}$ defined in $\Xi^{(k)}(\mathsf{P}^{(k)})$ coincides with $\Gamma^{(k)}$ of $\Xi^{(k)}(\mathsf{P}^{(k)})$. Consequently, we have

$$\bar{\Gamma}^{(k)}\big[\Gamma^{(k)}[f]\big](X_1,\ldots,X_n) = \bar{\Gamma}^{(k)}[\boldsymbol{f}](X_1,\ldots,X_n)$$

and furthermore, since $\bar{\Gamma}^{(k)}$ is the inverse of $\Gamma^{(k)}$, we have

$$f(X_1,\ldots,X_n) = \bar{\Gamma}^{(k)}[\boldsymbol{f}](X_1,\ldots,X_n) ,$$

which means that f is primitive recursively definable in $\Xi^{(k)}(\mathsf{P}^{(k)})$.

(II) The proof that induction in $\Sigma(1)$ implies induction in $\Xi^{(k)}(\mathsf{P}^{(k)})$ runs parallel to the one indicated in 4.1.

4.3. Interconstructibility of $\Sigma(1)$, $\Pi(\mathsf{P})$, $\Xi^{(k)}(\mathsf{P}^{(k)})$. On the strength of the results in our note [9], it follows that the Skolem arithmetics $\Xi^{(k)}(\mathsf{P}^{(k)})$, $k=1,2,3,\ldots$, are constructible in $\Sigma(1)$. In an exactly parallel way as in [9], we have that $\Pi(\mathsf{P})$ is constructible in $\Sigma(1)$.

Finally, from the preceding results and the results in sections 4.1 and 4.2, we have the following easy consequence. Given any $m, n \in \mathsf{N}$, the Skolem arithmetic $\Xi^{(m)}(\mathsf{P}^{(m)})$ is constructible in the Skolem arithmetic $\Xi^{(n)}(\mathsf{P}^{(n)})$. Therefore, the Skolem arithmetics $\Sigma(1)$, $\Pi(\mathsf{P})$, $\Xi^{(1)}(\mathsf{P}^{(1)})$, $\Xi^{(2)}(\mathsf{P}^{(2)})$, $\Xi^{(3)}(\mathsf{P}^{(3)}),\ldots$ are all constructible within each other. In turn, it also follows that the abstract Skolem arithmetics $\Sigma(a)$, $\Pi(\mathsf{A})$, $\Xi^{(k)}(\mathsf{A})$, $k=1,2,3,\ldots$, are interconstructible.

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