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ON THE QUADRATIC INTEGRABILITY OF SOLUTIONS OF $d^2x/dt^2+f(t)x = 0$

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1. Introduction.

In the differential equations considered in this paper, the coefficient functions $f(t)$, $g(t)$ are real and locally integrable in a real, half-open interval (a,b), being either semi-infinite or finite but allowing a non-integrable singularity of f,g at the open end.

The aim of this paper then is to present a few comparison criteria (Theorems 1 through 5, below) relating the numbers of solutions of class $L^2(a,b)$ (shortly, L^2) of two equations of the form announced in the heading. As a solution (and its first derivative) is absolutely continuous on every closed subinterval, our concern is the behaviour of solutions as the open end is approached.

Consider first the equation

$$x'' + [\lambda + f(t)]x = 0 , \tag{1}$$

where λ is a complex constant. If for a given λ this equation has two linearly independent solutions of class L^2, then evidently every solution for the same λ is of class L^2. A classical theorem by H. Weyl states that either (I) for every λ all solutions of (1) are of class L^2, or (II) for every λ at most one solution is of class L^2 (not counting the trivial solution $x=0$). Further, in the alternative (II) for every non-real λ there is exactly one L^2-solution, while for some or all real values of λ there may be none. In the sequel the terms introduced by Weyl will be used: *limit circle case* for the alternative (I), *limit point case* for the alternative (II).

Various criteria have been given which ensure that an equation (1) is of limit point type or of limit circle type, among which are the following:

a) Sufficient that (1) be of limit point type at infinity is (see [8, p. 192]) that (for all sufficiently large t)

$$f(t) \leqq k^2t^2 . \tag{2}$$

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b) Sufficient for (1) to be of limit point type at the origin is [7, Theorem 1] that (for all sufficiently small t)

(3) $$f(t) \leqq -\tfrac{3}{4}t^{-2}.$$

c) A sufficient condition for (1) to be of limit circle type at infinity is each of the following [5, p. 38],

(4) $$|f(t)-k^2t^{2(1+n)}| \leqq (n-\delta)kt^n, \qquad n>0,\ \delta>0.$$

(5) $$|f(t)-k^2e^{2nt}| \leqq (n-\delta)ke^{nt}, \qquad n>0,\ \delta>0.$$

d) Sufficient that (1) be of limit circle type at the origin is [5, p. 37]

(6) $$(-\tfrac{3}{4}+\delta)t^{-2} \leqq f(t) \leqq (\tfrac{13}{4}-\delta)t^{-2},$$

or more generally

(7) $$|f(t)-(\tfrac{1}{4}+k^2)t^{-2}| \leqq (2-\delta)kt^{-2},$$

(8) $$|f(t)-k^2t^{-2(1+n)}| \leqq (2+n-\delta)kt^{-(2+n)}.$$

A few criteria are collected in [4, pp. 24–26 and pp. 11–12] with further references.

2. Generalization of Bellman's comparison theorem.

Weyl's alternative was generalized by Bellman [2, p. 513, Theorem 2] or [3, p. 116, Theorem 6] to the following comparison theorem: *When*

(9) $$x''+f(t)x = 0$$

is of limit circle type, so is

(10) $$y''+g(t)y = 0,$$

provided

(11) $$\sup|g(t)-f(t)| < \infty.$$

By virtue of the symmetry in this condition, also the property of being of limit point type is preserved from (9) to (10) under the condition (11).

A generalisation of Bellman's theorem is given in the following theorem.

THEOREM 1. *When for every solution* $x(t)$ *of* (9)

(12) $$|g(t)-f(t)|^{\frac{1}{2}}\,x(t) \in L^2,$$

(10) *is of limit circle type or of limit point type according as which is the case for the equation* (9). *In addition,* (12) *will be satisfied with any solution* $y(t)$ *of* (10) *in place of* $x(t)$.

REMARK. In this paper, we are concerned principally with real equations, and Theorem 1 is proved below for real f,g only. However, it is not difficult to demonstrate the general validity of Theorem 1, that is for arbitrary complex f,g; this may be done e.g. by modifying Bellman's proof of his theorem [3, pp. 116–117]. In an analogous way, Theorem 2 (below) may be given an appropriate form which is valid for complex equations. Theorems 3, 4 and 5 cannot be generalized in this way, but it is seen that only the comparison coefficient f is required to be real, while g may be complex.

Finally, it is possible to obtain a further extension of Theorem 1, viz. to classes L^p, L^q with $1/p+1/q=1$, (12) being replaced by the two conditions

$$|g-f|^{\alpha}\, x \in L^p \quad \text{and} \quad |g-f|^{1-\alpha}\, x \in L^q ,$$

where the choice of $\alpha = 1/p$, $1-\alpha = 1/q$ yields the simplest generalization.

In the proof of Theorem 1 we shall need the following lemma, due to Bellman [1, pp. 644–645], [3, pp. 35–36]:

LEMMA. *Let $u(t) \geqq 0$ be integrable, $v(t) \geqq 0$ continuous for $t_1 \leqq t < t_2$, and denote by k a positive constant. Then, if*

$$v(t) \leqq k + \int_{t_1}^{t} u(\tau)v(\tau)\,d\tau \tag{13}$$

it follows that

$$v(t) \leqq k \exp \int_{t_1}^{t} u(\tau)\,d\tau . \tag{14}$$

PROOF OF THEOREM 1. We denote by x_1, x_2 two solutions of (9) satisfying

$$x_1x_2' - x_1'x_2 = 1 . \tag{15}$$

Then a general solution y of (10) will satisfy the following integral equation:

$$\begin{aligned} y(t) = {} & [y(c)x_2'(c)-y'(c)x_2(c)]x_1(t) - [y(c)x_1'(c)-y'(c)x_1(c)]x_2(t) + \\ & + \int_{c}^{t} [g(\tau)-f(\tau)]\,[x_1(t)\,x_2(\tau)-x_1(\tau)x_2(t)]\,y(\tau)\,d\tau . \end{aligned} \tag{16}$$

This may be verified by substituting $gy = -y''$ and integrating by parts. We now write

$$x_1 = \varrho \cos\omega, \qquad x_2 = \varrho \sin\omega , \tag{17}$$

whence

$$\varrho^2\omega' = x_1x_2' - x_1'x_2 = 1 , \tag{18}$$

$$\omega = \int_c^t \varrho^{-2}dt \quad \text{provided} \quad \omega(c) = 0 . \tag{19}$$

For later use we observe that (9) may be written

$$x'' + (\varrho^{-4} - \varrho^{-1}\varrho'')x = 0 . \tag{20}$$

Substitution of (17) into (16) gves

$$y(t) = x_c(t) + \int_c^t [g(\tau)-f(\tau)]\varrho(\tau)\varrho(t)\sin\left(\int_t^\tau \varrho^{-2}ds\right) y(\tau)\,d\tau . \tag{21}$$

Writing

$$y(t) = \varrho(t)w(t) , \tag{22}$$

one obtains

$$w(t) = \varrho^{-1}x_c(t) + \int_c^t [g(\tau)-f(\tau)]\varrho^2(\tau)\sin\left(\int_t^\tau \varrho^{-2}ds\right) w(\tau)\,d\tau , \tag{23}$$

hence

$$|w(t)| \leqq k + \int_c^t |g(\tau)-f(\tau)|\varrho^2(\tau)|w(\tau)|\,d\tau . \tag{24}$$

Using Bellman's lemma, (24) implies

$$|w| \leqq k \exp\int_c^t |g-f|\varrho^2 d\tau , \tag{25}$$

further

$$|y| \leqq k\varrho \exp\int_c^t |g-f|\varrho^2 d\tau . \tag{26}$$

It is a tacit assumption in the formulae above, that $t \geqq c$. With t and c interchanged, they are valid for $t \leqq c$, as Bellman's lemma may be modified correspondingly.

Under the assumption (12) of the theorem, i.e.

$$|g-f|^{\frac{1}{2}}\varrho \in L^2 , \tag{27}$$

(26) yields

$$|g-f|^{\frac{1}{2}}y \in L^2 .$$

When in addition to (12), $x \in L^2$ holds, i.e. (9) is of limit circle type, $y \in L^2$ ensues, that is, (10) will be of limit circle type as well. This connection being a symmetrical one, Theorem 1 may be implied.

From (26) we deduce the following

THEOREM 2. *When* (9) *is of limit circle type, the same is the case for* (10), *provided*

$$\varrho \exp\left|\int_c^t |g-f|\varrho^2 d\tau\right| \in L^2 , \tag{28}$$

ϱ *being given by* (17) *and* (15).

3. Further limit circle criteria.

Next, a few more limit circle criteria will be deduced, all concerning the case of a finite interval, with the singularity placed at the left end point a. Generalizing a method used by Sears [7, pp. 210–211], and starting with the integral equation (21) one finds

$$|y(t)| \leqq \xi(t) = k\varrho(t) + \int_t^b |g(\tau)-f(\tau)|\varrho(\tau)\varrho(t)\left(\int_t^\tau \varrho^{-2}ds\right)|y(\tau)|\,d\tau . \tag{29}$$

Hence

$$\xi'(t) = k\varrho' + \varrho^{-1}\varrho'(\xi-k\varrho) - \varrho^{-1}\int_t^b |g-f|\varrho|y|\,d\tau , \tag{30}$$

yielding

$$\varrho\xi'-\varrho'\xi = -\int_t^b |g-f|\varrho\,|y|\,d\tau , \tag{31}$$

$$\varrho\xi''-\varrho''\xi = |g-f|\varrho|y| \leqq |g-f|\varrho\xi . \tag{32}$$

Putting in (32) $\xi=\psi\zeta$ and $|g-f|\leqq\varphi+\chi$, one obtains (the functions ψ, ζ, φ and χ to be chosen suitably)

$$\psi\zeta''+2\psi'\zeta'+\psi''\zeta \leqq (\varrho^{-1}\varrho''+\varphi+\chi)\psi\zeta . \tag{33}$$

We choose

$$\psi^{-1}\psi'' = \varrho^{-1}\varrho''+\chi , \tag{34}$$

and obtain

$$\psi\zeta''+2\psi'\zeta' \leqq \varphi\psi\zeta \tag{35}$$

or

$$(\psi^2\zeta')' \leqq \varphi\psi^2\zeta , \tag{36}$$

further

$$\psi^2\zeta'(b)-\psi^2\zeta'(\tau) \leqq \int_\tau^b \varphi\psi^2\zeta\,ds , \tag{37}$$

whence

$$(38)\qquad \begin{aligned}\zeta(t) &\leqq \zeta(b) - \psi^2\zeta'(b)\int_t^b \psi^{-2}d\tau + \int_t^b \psi^{-2}(\tau)\left(\int_\tau^b \varphi\psi^2\zeta\,ds\right)d\tau \\ &= \zeta(b) - \psi^2\zeta'(b)\int_t^b \psi^{-2}d\tau + \int_t^b \varphi\psi^2\zeta\left(\int_t^\tau \psi^{-2}ds\right)d\tau\,.\end{aligned}$$

Choosing now $\psi^{-1} \in L^2$,

$$\varphi\psi^2\int_a^\tau \psi^{-2}ds \;\in\; L\,,$$

and applying Bellman's lemma to (38) we find $\zeta(t) \leqq k$, hence

$$(39)\qquad |y(t)| \;\leqq\; \xi(t) \;=\; \psi\zeta \;\leqq\; k\psi\,.$$

Lastly, choosing $\psi \in L^2$ implies $y \in L^2$. Hence we have proved

THEOREM 3. *The equation* (10) *will be of limit circle type at a* (*a finite*) *provided*

$$(40)\qquad |g-f| \;\leqq\; \varphi+\chi\,,$$

where

$$(41)\qquad \varphi\psi^2\int_a^\tau \psi^{-2}ds \in L, \qquad \psi^{-1} \in L^2, \qquad \psi \in L^2\,,$$

and

$$(42)\qquad \chi \;=\; \psi^{-1}\psi''-\varrho^{-1}\varrho''\,,$$

or, using (20)

$$(43)\qquad \chi \;=\; \psi^{-1}\psi''+f-\varrho^{-4}\,,$$

ϱ *being defined by* (17) *and* (15).

As an example we may take $\psi = t^{-\frac{1}{2}(1-\delta)}$ (choosing $a=0$), whence

$$(44)\qquad 4\psi^{-1}\psi'' \;=\; (3-2\delta+\delta^2)t^{-2}\,,$$

$$(45)\qquad t\varphi \in L(0,b)\,.$$

When f is chosen fairly smooth, the present method cannot yield results comparable to those obtained in [5]. However, theorem 3 is generally more flexible, applying easier to a not so smooth f. This remark is equally valid for Theorems 2, 4 and 5 (below).

4. Non-oscillatory equations.

Further general results may be deduced for non-oscillatory equations i.e. equations having no solutions with an infinite number of zeros in the considered interval. Theorem 3 is restricted to the finite interval case because in (41), $\psi^{-1} \in L^2$ and $\psi \in L^2$ are mutually exclusive on an infinite interval. In what follows, the restriction to finite intervals is necessitated by the fact that an equation (2) which is non-oscillatory on an infinite interval is of limit point type at infinity. This was established by P. Hartman [6, p. 698 and 703]; see also [4, p. 12].

Assuming now that (2) is non-oscillatory on the finite interval (a,b) and taking a real, positive solution $x_1(t)$ satisfying $x_1^{-1} \in L^2(a,c)$ (which may always be found, see [4, pp. 14–15]), another linearly independent solution is

$$x_2 = x_1 \int_a^t x_1^{-2} d\tau , \tag{46}$$

and $x_1x_2' - x_1'x_2 = 1$. Substituting this in the integral equation (16), we get

$$y(t) = x_c(t) + \int_t^c [g(\tau)-f(\tau)]x_1(t)x_1(\tau)\left(\int_t^\tau x_1^{-2}ds\right) y(\tau)\, d\tau , \tag{47}$$

hence

$$|y(t)| \leqq \xi(t) = kx_1(t) + \int_t^c |g-f|\, x_1(t)x_1(\tau)\left(\int_t^\tau x_1^{-2}ds\right) |y(\tau)|\, d\tau . \tag{48}$$

Replacing first in (48) $\int_t^\tau x_1^{-2}ds$ with $\int_a^\tau x_1^{-2}ds$, an application of Bellman's lemma as in the deduction of Theorem 2 leads to

$$|y| \leqq kx_1 \exp \int_t^c |g-f|\, x_1^2 \left(\int_a^\tau x_1^{-2}ds\right) d\tau . \tag{49}$$

Hence Theorem 2 is improved for a finite interval in the special case when (10) is compared to an equation (9) which is non-oscillatory on (a,b), and we have

THEOREM 4. *When on a finite interval equation* (9) *is non-oscillatory and of limit circle type at a, and x_1 denotes a solution of* (9) *satisfying*

$$x_1^{-1} \in L^2(a,c) , \tag{50}$$

and further

$$x_1 \exp \int_t^c |g-f|\, x_1^2 \left(\int_a^\tau x_1^{-2}ds\right) d\tau \in L^2 , \tag{51}$$

then (10) *is of limit circle type as well.*

A last result is obtained by treating (48) by the same method as (29), which lead to Theorem 3. The results will be analogous, i.e. $|y| \leqq k\psi$ when

$$\psi^{-1}\psi'' = x_1^{-1}x_1'' + \chi = \chi - f, \qquad \varphi\psi^2 \int_a^t \psi^{-2} d\tau \in L, \qquad |g-f| \leqq \varphi + \psi .$$

Hence we have

THEOREM 5. *Under conditions* (40) *and* (41) *of Theorem* 3, *but where*

$$\chi = \psi^{-1}\psi'' + f , \tag{52}$$

with the restriction on f that equation (9), $x'' + fx = 0$, *is non-oscillatory near the singular point a, then equation* (10), $y'' + gy = 0$, *is of limit circle type at a.*

Putting $g = f$ in Theorem 5, one obtains the following

COROLLARY. *When* (9) *is non-oscillatory, and further*

$$f(t) \geqq -\psi^{-1}\psi'' - \varphi , \tag{53}$$

φ and ψ being given by (41), *then the equation will be of limit-circle type at a.*

The presence of φ in (53) constitutes an improvement from the comparison criterion annonced in [4, p. 26].

Condition (52) in Theorem 5 clearly constitutes, comparing it with (43) in Theorem 3, a certain improvement on the latter theorem in the special case of a non-oscillatory comparison equation (9). In fact, the inequality

$$|g-f| \leqq \psi^{-1}\psi'' + f + \varphi ,$$

resulting from (40) and (52), is in a sense the best possible insofar as the comparison is restricted to non-oscillatory equations, since we have gained through this inequality a natural lower bound for $g(t)$, independent of $f(t)$, namely $g_0(t) = -\psi^{-1}\psi''$, where the main restrictions on ψ are $\psi^{-1} \in L^2$ (see remark concerning x_1^{-1} just above (46)) and $\psi \in L^2$. This means that in $y'' + gy = 0$, $g(t)$ may be chosen as "near" as we wish to a function yielding (non-oscillatory) solutions $y(t)$ which are not all of class L^2. The following example will illustrate the last statements.

EXAMPLE. Taking $a = 0$ as the singular point, and choosing in (9)

$$f(t) = \tfrac{1}{4}t^{-2}[1 + l^{-2} + (ll_2)^{-2} + \ldots + (ll_2 \ldots l_n)^{-2}] , \tag{54}$$

where we have put

$$l = \log(1/t), \qquad l_2 = \log\log(1/t), \qquad l_k = \log l_{k-1} ,$$

two solutions are

$$x_1 = x_2 l_{n+1}, \qquad x_2 = (tll_2 \dots l_n)^{\frac{1}{2}}. \tag{55}$$

When the coefficient of the last member in the braces of (54) gets an increase from 1 to $1+\delta$, $\delta>0$, equation (9) will be oscillatory (see for example [3, p. 121]). Thus, with (54) we are near the limit of applicability of the last theorem. Further, choosing

$$\psi = (tll_2 \dots l_N)^{-\frac{1}{2}} l_N^{-\frac{1}{2}\delta}, \qquad \delta>0 \tag{56}$$

we get

$$\psi^{-1}\psi'' = t^{-2}[\tfrac{3}{4}-l^{-1}-(ll_2)^{-1}- \dots -(1+\delta)(ll_2 \dots l_N)^{-1}+O(l^{-2})]. \tag{57}$$

Hence, from Theorem 5, the lower bound for g is given (neglecting φ, which is restricted by $t\varphi \in L(0,b)$) by

$$g(t) \geqq t^{-2}[-\tfrac{3}{4}+l^{-1}+(ll_2)^{-1}+ \dots +(1+\delta)(ll_2 \dots l_N)^{-1}]. \tag{58}$$

On the other hand, $y=(tll_2 \dots l_N)^{-\frac{1}{2}} \notin L^2(0,b)$ yields

$$-y^{-1}y'' = t^{-2}[-\tfrac{3}{4}+l^{-1}+(ll_2)^{-1}+ \dots +(ll_2 \dots l_N)^{-1}+O(l^{-2})]. \tag{59}$$

The upper bound for $g(t)$ obtained here is far from being best possible; neglecting the logarithmic refinements we find $g(t) \leqq \frac{5}{4}t^{-2}$, thus obtaining only one half of the range given in (6) above.

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