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TEMPERED DISTRIBUTIONS IN INFINITELY MANY DIMENSIONS II, DISPLACEMENT OPERATORS

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1. Introduction.

Summary of results.

In a previous paper [5] (in the sequel quoted as I) the theory of tempered distributions was generalized to certain spaces of infinitely many dimensions. Explicit representations were given for two spaces of testing elements for such generalized tempered distributions, viz. the minimal complete space $\tilde{\mathfrak{S}}$ and the maximal (complete) space $\mathfrak{S}$. These two spaces are carrier spaces for representations of a pair $a, a^\star$ of canonical field operators (precisely speaking, a pair of operator valued distributions). In the terminology of the physics literature, a and $a^\star$ are, respectively, annihilation and creation operators for neutral spin zero bosons.

In the present work we continue the mathematical study of tempered distributions in infinitely many dimensions. In particular, we study displacements of the canonical pair, whereby we understand transformations of the form

$$a \to a-f \equiv a_f ,$$
$$a^\star \to a^\star - f^\star \equiv a_f{}^\star ,$$

where f is a tempered distribution over R, viz. $f \in \mathscr{S}^\star$. Here $f^\star$ denotes the conjugate of f in the sense of the natural conjugation in the space $\mathscr{S}^\star$ of tempered distributions.

Following some preliminary work in Section 2, we prove in Section 3 that such *a displacement can always be represented by means of an intertwining operator $D(f)$, called the displacement operator, satisfying*

$$D(f)a = (a-f)D(f) ,$$
$$D(f)a^\star = (a^\star - f^\star)D(f) .$$

Here $D(f)$ is a continuous linear mapping from $\tilde{\mathfrak{S}}$ into $\tilde{\mathfrak{S}}^\star$, the latter space

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being the dual of $\widetilde{\mathfrak{S}}$. For all $f \in \mathscr{S}^\star$, *$D(f)$ is determined uniquely up to a numerical factor.*

Thus, D is a mapping from $\mathscr{S}^\star$ into $L(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{S}}^\star)$ (the space of continuous operators from $\widetilde{\mathfrak{S}}$ into $\widetilde{\mathfrak{S}}^\star$ provided with the topology of uniform convergence on bounded sets). In Section 4 we prove that *D is continuous and differentiable from $\mathscr{S}^\star$ into $L(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{S}}^\star)$.*

In general a displacement is not a unitary equivalence. It is implied by the work of Gårding and Wightman [3] that unitary equivalence is obtained if and only if $f \in \mathscr{H}$, where the Hilbert-space $\mathscr{H}$ is the completion of $\mathscr{S}$ in the scalar product norm $\|\cdot\|$. For the sake of completeness we rederive this known result in Section 5. We remark that the pair $a_f, a_f^\star$ belongs to one of the continuous representations (cf. [6]) whenever $f \in \mathscr{S}^\star$ but $f \notin \mathscr{H}$. The main result of Section 5 is: Subject to a suitable normalization, *the displacement operator $D(f)$ may be extended to a continuous linear mapping from $\mathfrak{S}$ onto $\mathfrak{S}$ if and only if $f \in \mathscr{S}$. This extension is unique and preserves the scalar product.* Thus the maximal space $\mathfrak{S}$ is invariant under displacements in $\mathscr{S}$.

Scattering of neutral spin zero bosons on an external source.

The remaining part of this introduction is devoted to an application of the above results to quantum physics.

The scattering of neutral spin zero bosons on an external source offers the simplest non-trivial example of an application to quantum field theory of the theory of tempered distributions in infinitely many variables. Let t be the time variable, let $f(t)$ be a one parameter family of elements in $\mathscr{S}^\star$ such that

$$\lim_{t \to -\infty} f(t) = 0 ,$$

and let $a, a^\star$ be a pair of time independent canonical field operators. The scattering process is now described by the pair of time dependent fields operators $a(t)$, $a^\star(t)$ defined as

$$\begin{aligned} a(t) &= a - f(t) , \\ a^\star(t) &= a^\star - f^\star(t) . \end{aligned} \tag{1.1}$$

In particular, the total scattering is given by the field operators for the so-called outgoing field, viz.

$$\begin{aligned} a_{\text{out}} &= a(\infty) &= a - f(\infty) , \\ a_{\text{out}}^\star &= a^\star(\infty) &= a^\star - f^\star(\infty) . \end{aligned}$$

The axiom of unitarity requires that this total scattering be a unitary equivalence, i.e. that a unitary mapping S, called Heisenbergs S-matrix, exists such that

$$a_{\text{out}} = S^\star a S \quad \text{and} \quad a^\star_{\text{out}} = S^\star a^\star S .$$

Hence we have the well-known result that $f(\infty) \in \mathscr{H}$ is required by unitarity. From now on we assume that this be the case.

However, according to the axioms of canonical quantum theory, unitarity is required not only for the total scattering, but for each time instant separately. This assumption is weaker than that of the existence of a Hamiltonian. Thus $f(t) \in \mathscr{H}$ for all values of the time variable is a necessary condition that canonical quantum theory hold. This fact was first pointed out by van Hove [4] and Friedrichs [2]. If (1.1) is a unitary equivalence for all values of t, we may write

$$\begin{aligned} a(t) &= U^\star(t)\, a U(t) , \\ a^\star(t) &= U^\star(t)\, a^\star U(t) , \end{aligned}$$

where $U(t)$, the U-matrix of Dyson, is unitary.

If, further, $f(t)$ is differentiable, then $U(t)$ satisfies the equation of motion

$$i\dot{U}(t) = i\big(a^\star(\dot{f}(t)) - a(\dot{f}(t)^\star)\big)\, U(t) .$$

Under suitable conditions on the family $f(t)$ as a function of t, one has

$$\begin{aligned} \lim_{t\to-\infty} U(t) &= 1 , \\ \lim_{t\to\infty} U(t) &= S , \end{aligned}$$

where 1 denotes the identity mapping and S is Heisenbergs S-matrix.

The results of the present paper show that even in the case that there exist values of the time variable t such that $f(t) \notin \mathscr{H}$, it is possible to save a continuous interpolation in time between the identity mapping and S provided only that the family $f(t)$ of tempered distributions converges pointwise on testing functions to an element $f(\infty)$ of $\mathscr{H}$. The precise result is this:

Let $f(\infty) \in \mathscr{H}$ and let $f(t) \in \mathscr{S}^\star$, $-\infty < t < \infty$, be such that $\lim_{t\to\infty} f(t) = f(\infty)$ *in $\mathscr{S}^\star$. Then the corresponding family of displacement operators may be chosen such that, in the topology of $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^\star)$,*

$$\lim_{t\to\infty} D\big(f(t)\big) = D\big(f(\infty)\big) .$$

Further, $D\big(f(\infty)\big)$ has a unique continuous extension to a unitary operator S in the Hilbert-space $\mathfrak{H}$ obtained by completion of $\tilde{\mathfrak{S}}$ in the scalar product norm $|||\cdot|||$ (cf. Corollary 1, p. 144, and Lemma 6, p. 147).

Of course S becomes identical with Heisenberg's S-matrix for the scattering process considered.

If, in particular, $f(t)$ is differentiable, it follows from Corollary 2, p. 145, that $D(f(t))$ satisfies the equation of motion

$$i\dot{D}(f(t)) = ia^\star(\dot{f}(t))D(f(t)) - iD(f(t))a(\dot{f}(t)^\star) - \dot{\theta}(t)D(f(t)),$$

where θ is a numerical function of t. According to the definition of D, this equation may be written in the form

$$i\dot{D} = HD \quad \text{iff} \quad \langle \dot{f}, f\rangle \text{ exists}.$$

2. Survey of the theory of tempered distributions in infinitely many variables and an extension theorem.

In this section we first give a brief summary of some of the notions and main results of I, and then we add a general extension theorem (Theorem 1).

General notions.

All vector spaces are assumed complex and provided with a locally convex topology.

If S_1 and S_2 are locally convex spaces, then the space of all continuous linear mappings from S_1 into S_2 is denoted $L(S_1, S_2)$ and provided with the topology of uniform convergence on bounded sets. In particular, the space $L(S, C)$ is denoted $S^\star$; its topology is the so-called strong topology.

If S is a locally convex space, the completion of S is denoted compl S. More specifically, we write $\mathrm{compl}_{\mathscr{T}}\, S$ resp. $\mathrm{compl}_{\|\cdot\|}\, S$ for the completion of S in the topology $\mathscr{T}$ resp. in the topology determined by the norm $\|\cdot\|$.

If $T \in L(S_1, S_2)$, then the *dual* operator $T^\star \in L(S_2{}^\star, S_1{}^\star)$ is defined by

$$\langle T^\star f_2, x_1\rangle = \langle f_2, Tx_1\rangle .$$

If $\langle\cdot,\cdot\rangle$ is a scalar product on S, and if T and $T^\star$ are linear operators on S such that

$$\langle T^\star x_2, x_1\rangle = \langle x_2, Tx_1\rangle ,$$

then T and $T^\star$ are said to be *adjoint*.

The space $\mathscr{S}^n$.

The space $\mathscr{S}^n$ may be identified with Schwartz' space of testing functions in R^n for tempered distributions. In $\mathscr{S}^n$ there is defined a continuous scalar product $\langle\cdot,\cdot\rangle$ with the corresponding norm $\|\cdot\|$, and continuous linear mappings $b_i, b_i^\star$, $i=1,2,\ldots,n$, which are pairwise adjoint with respect to the scalar product and satisfy the well known commutation relations

$$[b_i, b_j^\star] = \delta_{ij}, \qquad [b_i, b_j] = [b_i^\star, b_j^\star] = 0 .$$

The topology of $\mathscr{S}^n$ is determined by the system of norms $\|\cdot\|_r$, $r=0,1,\ldots$, where $\|\varphi\|_r^2 = \langle \varphi, (h^{(n)})^r \varphi \rangle$, $\varphi \in \mathscr{S}^n$. Here

$$h^{(n)} = \sum_{i=1}^{n} b_i b_i^\star$$

has an inverse $(h^{(n)})^{-1} \in L(\mathscr{S}^n, \mathscr{S}^n)$. Thus $\|\cdot\|_r$ may be given a meaning also for negative integers r. The norms $\|\cdot\|_r$, $r=0, \pm 1, \pm 2, \ldots$, satisfy the fundamental norm inequality

$$\text{(2.1)} \qquad \|\cdot\|_{r+1}^2 \geqq n\|\cdot\|_r^2 \quad \text{in} \quad \mathscr{S}^n, \qquad r = 0, \pm 1, \ldots .$$

Note that $\|\cdot\|_0 = \|\cdot\|$. The space $\mathscr{S}^n$ contains a normed element ψ_0, called the cyclic element, which is characterized uniquely up to a numerical factor of modulus 1 by $b_i \psi_0 = 0$, $i=1,2,\ldots,n$.

There exists a unique projection $\mathrm{sym}_n \in L(\mathscr{S}^n, \mathscr{S}^n)$ (sometimes simply denoted sym) satisfying $\mathrm{sym}_n \psi_0 = \psi_0$ and

$$\mathrm{sym}_n b_j = \frac{1}{n} \sum_{i=1}^{n} b_i \,\mathrm{sym}_n, \qquad \mathrm{sym}_n b_j^\star = \frac{1}{n} \sum_{i=1}^{n} b_i^\star \,\mathrm{sym}_n .$$

In Schwartz' representation, sym_n projects on the subspace of symmetric functions. The symmetric part $\mathscr{S}_+^n = \mathrm{sym}_n \mathscr{S}^n$ of $\mathscr{S}^n$ is given the same topology as $\mathscr{S}^n$. The space $\mathscr{S}^1 = \mathscr{S}_+^1$ is often denoted by $\mathscr{S}$. As a matter of convention, for $n=0$ we identify $\mathscr{S}^0 = \mathscr{S}_+^0$ with the complex field C and define $\|c\|_r = \delta_{0r}|c|$ for $r \geqq 0$, $c \in C$.

The spaces $\mathscr{S}^n(\mathscr{S}_+^n)$ are complete metrizable perfect spaces and hence Montel spaces.

We define

$$\mathscr{H}_r^n = \mathrm{compl}_{\|\cdot\|_r} \mathscr{S}_+^n .$$

Obviously $\mathscr{H}_r^n$ is a Hilbert space, and in virtue of the basic norm inequality (2.1) we have $\mathscr{H}_{r+1}^n \subset \mathscr{H}_r^n$, $r=0, \pm 1, \ldots$. For $n=1$, or (and) $r=0$ we often omit the corresponding index. Thus, in particular, $\mathscr{H} = \mathscr{H}_0^1$. Algebraically,

$$\mathscr{S}_+^n = \bigcap_{r=0}^{\infty} \mathscr{H}_r^n ,$$

while for the dual spaces we have

$$\mathscr{S}_+^{n\star} = \bigcup_{r=0}^{\infty} \mathscr{H}_{-r}^n .$$

It may be shown that $\mathscr{S}_+^n$ is the projective limit of the sequence of contracting space $\mathscr{H}_r^n$, $r=0,1,\ldots$, and that $\mathscr{S}_+^{n\star}$ is the inductive limit of the expanding sequence $\mathscr{H}_{-r}^n$, $r=0,1,\ldots$. For the proof of the non trivial parts of this statement we refer to I, Appendix B.

The complex conjugate of an element $\varphi \in \mathscr{S}^n$ is denoted by $\varphi^\star$. Then $(b_i\varphi)^\star = -b_i\varphi^\star$ and $(b_i^\star\varphi)^\star = -b_i^\star\varphi^\star$.

The space $\mathfrak{S}$.

The space $\mathfrak{S}$ may be identified with a space of sequences (the *Fock representation* of $\mathfrak{S}$) of the form

$$\Psi = \{\psi_0, \ldots, \psi_n, \ldots\} = \{\psi_n\},$$

where $\psi_n \in \mathscr{S}_+^n$. The topology of $\mathfrak{S}$ is determined by the system of semi-norms $|||\cdot|||_r$, $r=0,1,\ldots$, where

$$|||\Psi|||_r^2 = \sum_{n=0}^{\infty} \|\psi_n\|_r^2,$$

and $\mathfrak{S}$ consists of all sequences for which all these semi-norms are finite. In particular, $\mathfrak{S}$ is a complete metrizable space, and perfect, and hence $\mathfrak{S}$ is a Montel space. We often write $|||\cdot|||$ instead of $|||\cdot|||_0$; this semi-norm is the norm corresponding to the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ given by

$$\langle\langle\Phi, \Psi\rangle\rangle = \sum_{n=0}^{\infty} \langle\varphi_n, \psi_n\rangle,$$

where $\Phi = \{\varphi_n\}$, $\Psi = \{\psi_n\}$ are elements of $\mathfrak{S}$.

On occasion it is convenient to exhibit the relationship between an element $\Psi \in \mathfrak{S}$ and its representing sequence $\{\psi_n\}$ by the symbol

$$\Psi = \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} a^{\star n \otimes}(\psi_n)\,\Psi_0\,. \tag{2.2}$$

It will appear in the sequel that the right hand side admits an interpretation as a convergent sequence in $\mathfrak{S}$, a fact which for the moment is of no importance.

We define a pair of continuous linear mappings a and $a^\star$ (called canonical field operators) from $\mathscr{S}$ into $L(\mathfrak{S},\mathfrak{S})$ by the formulas

$$\begin{aligned} a(\varphi^\star)\Psi &= \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} a^{\star n\otimes}\big((n+1)^{\frac{1}{2}} \langle\varphi, \psi_{n+1}\rangle_{(1)}\big)\Psi_0\,, \\ a^\star(\varphi)\Psi &= \sum_{n=1}^{\infty} (n!)^{-\frac{1}{2}} a^{\star n\otimes}\big(n^{\frac{1}{2}}\, \mathrm{sym}_n(\varphi\otimes\psi_{n-1})\big)\Psi_0\,, \end{aligned} \tag{2.3}$$

where Ψ is considered given by (2.2). Here we have used the following notation: For $\varphi \in \mathscr{S}$ and $\psi_n \in \mathscr{S}^n$, $\varphi \otimes \psi_n$ denotes that element of $\mathscr{S}^{n+1}$, which in Schwartz' representation is given by

$$(\varphi \otimes \psi_n)(t_1,\ldots,t_{n+1}) = \varphi(t_1)\psi_n(t_2,\ldots,t_{n+1}) ,$$

and $\langle\varphi,\psi_n\rangle_{(1)}$ denotes that element of $\mathscr{S}^{n-1}$, which is given by

$$\int \varphi^\star(t)\psi_n(t,t_1,t_2,\ldots,t_{n-1})\, dt .$$

Later we shall use the symbols $\varphi_m \otimes \psi_n$ and $\langle\varphi_m,\psi_n\rangle_{(k)}$, $k \leqq \min(m,n)$, defined analogously.

The mappings $\varphi\otimes$ and $\langle\varphi,\cdot\rangle_{(1)}$ are continuous, and the latter is uniquely determined on $\mathscr{S}_+{}^n$ by its values on elements of the particular form $\psi^{n\otimes}$ as follows from I, Corollary (2.23), which we here reproduce in the form of

Lemma 1. *The elements of the form* $\psi \otimes \psi \otimes \ldots \otimes \psi = \psi^{n\otimes}$, $\psi \in \mathscr{S}$, *span a dense subset* $\mathscr{S}_+{}^{n\otimes}$ *of* $\mathscr{S}_+{}^n$.

Let $k \in L(\mathscr{S},\mathscr{S})$. We define an operator $K \in L(\mathfrak{S},\mathfrak{S})$, called the *normalized bi-quantization* of k, by

$$K\Psi = \sum_{n=1}^{\infty} (n!)^{-\frac{1}{2}} a^{\star n\otimes}(k^{(n)}\psi_n)\,\Psi_0 ,$$

where the operator $k^{(n)} \in L(\mathscr{S}_+{}^n,\mathscr{S}_+{}^n)$ is characterized by

$$k^{(n)}\psi^{n\otimes} = n \operatorname{sym}_n(k\psi \otimes \psi^{(n-1)\otimes}) .$$

The element in $\mathfrak{S}$ corresponding to the sequence $\{1,0,0,\ldots\}$ is denoted Ψ_0 and called the (normalized) *vacuum element*. Up to a numerical factor of modulus 1, it is characterized uniquely by the equations $|||\Psi_0|||=1$, $a(\varphi)\Psi_0=0$, $\varphi \in \mathscr{S}$. Observe that $K\Psi_0=0$ for all $k \in L(\mathscr{S},\mathscr{S})$.

As an immediate consequence of these definitions we have: $a(\varphi^\star)$ is the adjoint of $a^\star(\varphi)$, and for all $\varphi,\psi \in \mathscr{S}$, the canonical commutation relations

(2.4)
$$[a(\varphi^\star),a^\star(\psi)] = \langle\varphi,\psi\rangle ,$$
$$[a(\varphi^\star),a(\psi^\star)] = [a^\star(\varphi),a^\star(\psi)] = 0$$

hold. If k is self-adjoint (with respect to the scalar product), then so is its bi-quantization K, and

$$[K,a^\star(\varphi)] = a^\star(k\varphi) .$$

Let P_0 be the projection on Ψ_0, viz. $P_0\Psi = \Psi_0\langle\langle\Psi_0,\Psi\rangle\rangle$, and let H be the normalized bi-quantization of $h = bb^\star \in L(\mathscr{S}, \mathscr{S})$. Then $|||\Psi|||_r^2 = \langle\langle \Psi, H^r \Psi \rangle\rangle$, and hence $|||\Psi_0|||_r = 0$, $r = 1, 2, \ldots$. However, it follows from (2.1) that $(H + P_0)^{-1} \in L(\mathfrak{S}, \mathfrak{S})$, and that the norms $|||\cdot|||_{(r)}$ defined by

$$|||\Psi|||_{(r)}^2 = \langle\langle \Psi, (H + P_0)^r \Psi \rangle\rangle, \qquad r = 0, \pm 1, \pm 2, \ldots$$

are increasing. It is easily seen that $(H + P_0)^r = H^r + P_0$ for $r > 0$, and hence the family $|||\cdot|||_{(r)}$, $r = 0, \pm 1, \pm 2, \ldots$, determines the topology of $\mathfrak{S}$.

We define

$$\mathfrak{H}_r = \text{compl}_{|||\cdot|||_{(r)}} \mathfrak{S}, \qquad r = 0, \pm 1, \ldots .$$

The space $\mathfrak{S}$ is now the projective limit of the contracting sequence of Hilbert-spaces $\{\mathfrak{H}_r\}$, $r = 0, 1, \ldots$, while the dual space $\mathfrak{S}^\star$ is the inductive limit of the expanding sequence $\{\mathfrak{H}_{-r}\}$, $r = 0, 1, \ldots$, In particular, a subset of $\mathfrak{S}^\star$ is bounded if and only if it is a bounded subset of some Hilbert-space $\mathfrak{H}_{-r}$.

The space $\tilde{\mathfrak{S}}$.

Obviously the subspace of $\mathfrak{S}$ which consists of elements

$$(n!)^{-\frac{1}{2}} a^{\star n \otimes}(\varphi_n) \Psi_0$$

with a fixed n is a copy of $\mathscr{S}_+{}^n$. Hence the direct sum $\tilde{\mathfrak{S}}$ of these spaces may be considered a subspace of $\mathfrak{S}$. Algebraically $\tilde{\mathfrak{S}}$ is the space of all sequences in (the Fock representation of) $\mathfrak{S}$, for which all but a finite number of coordinate functions vanish. The space $\tilde{\mathfrak{S}}$ is given the direct sum topology. Operators a and $a^\star$, as well as bi-quantizations, are defined by restriction from $\mathfrak{S}$. The properties of these restricted operators relative to $\tilde{\mathfrak{S}}$ are similar to their properties relative to $\mathfrak{S}$. For further details we refer to I; let us only note that the dual space $\tilde{\mathfrak{S}}^\star$ admits a Fock representation in which elements $\boldsymbol{T} \in \tilde{\mathfrak{S}}^\star$ have the form $\boldsymbol{T} = \{T_0, T_1, \ldots, T_n, \ldots\}$ with $T_n \in \mathscr{S}_+{}^{n\star}$, and all such sequences belong to $\tilde{\mathfrak{S}}^\star$.

An extension theorem.

The mapping $a^{\star n \otimes}$ from elements $\varphi^{n \otimes} \in \mathscr{S}_+{}^n$ into $L(\mathfrak{S}, \mathfrak{S})$ defined by $a^{\star n \otimes}(\varphi^{n \otimes}) = a^\star(\varphi)^n$ can by linearity be extended to a mapping from $\mathscr{S}_+{}^{n \otimes}$, and further, in a standard fashion (cf. Lemma 1 and the proof of the Theorem (3.27) in I), to a continuous linear mapping from $\mathscr{S}_+{}^n$ into $L(\mathfrak{S}, \mathfrak{S})$. In a similar way we introduce $a^{n \otimes}$. Obviously $a^{n \otimes}(\varphi_n{}^\star)$ and $a^{\star n \otimes}(\varphi_n)$ are adjoint for all $\varphi_n \in \mathscr{S}_+{}^n$. The restrictions to $\tilde{\mathfrak{S}}$ of $a^{\star n \otimes}(\varphi_n)$ and of $a^{n \otimes}(\varphi_n{}^\star)$ belong to $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}})$.

THEOREM 1. *The mappings $a^{\star n}\otimes$ and $a^{n}\otimes$ have unique continuous extensions as given in the following table:*

	$a^{\star n}\otimes$	$a^{n}\otimes$
$\mathscr{S}_+^{\,n} \to L(\mathfrak{S}^\star, \mathfrak{S}^\star)$	yes	yes
$\mathscr{S}_+^{\,n\star} \to L(\mathfrak{S}, \mathfrak{S})$	no	yes
$\mathscr{S}_+^{\,n\star} \to L(\mathfrak{S}^\star, \mathfrak{S}^\star)$	yes	no
$\mathscr{S}_+^{\,n\star} \to L(\mathfrak{S}, \mathfrak{S}^\star)$	yes	yes

For every $f_n \in \mathscr{S}_+^{\,n\star}$ with $f_n \notin \mathscr{S}_+^{\,n}$ and for every $\Psi \in \mathfrak{S}$ $(\Psi \neq 0)$,

$$a^{\star n}\otimes(f_n)\Psi \notin \mathfrak{S}\,.$$

A similar theorem holds when $\mathfrak{S}$ is substituted by $\tilde{\mathfrak{S}}$ everywhere.

PROOF. The proof is essentially of the same type as the proofs given of similar (and in part equivalent) extension theorems in Section 4 of I. We shall only consider the topological part of the argument in detail.

An elementary calculation based on (2.3), Lemma 1, and simple continuity properties leads to the formula

$$\langle\langle\Phi, a^{\star n}\otimes(\omega_n)\Psi\rangle\rangle = \sum_{t=0}^{\infty}\left(\frac{(t+n)!}{t!}\right)^{\frac{1}{2}} \langle\langle\omega_n, \varphi_{t+n}\rangle_{(n)}, \psi_t\rangle\,,$$

valid for all elements Φ, $\Psi \in \mathfrak{S}$, and all elements $\omega_n \in \mathscr{S}_+^{\,n}$. Writing

$$\omega_n = (h^{(n)})^{\frac{1}{2}r}\,(h^{(n)})^{-\frac{1}{2}r}\,\omega_n, \qquad \psi_t = (h^{(t)})^{\frac{1}{2}s}\,(h^{(t)})^{-\frac{1}{2}s}\,\psi_t\,,$$

and using the estimate (3.23) of I (or Cauchy–Schwarz' inequality) and (2.1), we now find the (not best possible) estimate

$$|\langle\langle\Phi, a^{\star n}\otimes(\omega_n)\Psi\rangle\rangle| \leqq \|\omega_n\|_{-r}\;|||\Phi|||_{(r+s+n)}\;|||\Psi|||_{(-s)}\,, \tag{2.5}$$

valid for all pairs of integers $r, s \geqq 0$.

For fixed $\omega_n \in \mathscr{S}_+^{\,n}$ and $r=0$ this inequality shows that $a^{\star n}\otimes(\omega_n)$ is continuous from $\mathfrak{S}$ with the topology of $\mathfrak{H}_{-s}$ into $\mathfrak{S}$ with the topology of $\mathfrak{S}^\star$. Since $\mathfrak{S}^\star$ is the inductive limit of the sequence $\{\mathfrak{H}_{-s}\}$, $s \to \infty$, it follows that $a^{\star n}\otimes(\omega_n)$ is continuous from $\mathfrak{S}$ into $\mathfrak{S}$, when $\mathfrak{S}$ is given the topology of $\mathfrak{S}^\star$. Since $\mathfrak{S}$ is dense in $\mathfrak{S}^\star$ and $\mathfrak{S}^\star$ is complete, it follows that $a^{\star n}\otimes(\omega_n)$ has a unique continuous extension (also denoted $a^{\star n}\otimes(\omega_n)$) from $\mathfrak{S}^\star$ into $\mathfrak{S}^\star$.

Now let r be arbitrary. By continuity, the estimate (2.5) is valid for $\Psi \in \mathfrak{H}_{-s} \subset \mathfrak{S}^\star$, and since bounded sets in $\mathfrak{S}^\star$ are contained in and bounded in some $\mathfrak{H}_{-s}$, it follows that $a^{\star n}\otimes$ is continuous from $\mathscr{S}_+^{\,n}$ with the topology of $\mathscr{H}^n_{-r}$ into $L(\mathfrak{S}^\star, \mathfrak{S}^\star)$. Since $\mathscr{S}_+^{n\star}$ is the inductive limit of the sequence $\mathscr{H}^n_{-r}$, $r \to \infty$, and since $\mathscr{S}_+^n$ is dense in $\mathscr{S}_+^{n\star}$ and $L(\mathfrak{S}^\star, \mathfrak{S}^\star)$ is complete, we conclude that $a^{\star n \otimes}$ has a unique continuous extension from $\mathscr{S}_+^{n\star}$ into $L(\mathfrak{S}^\star, \mathfrak{S}^\star)$.

We have now proved the yes-part of the third line of the table and a fortiori also the assertion concerning $a^{\star n \otimes}$ in the first line of the table.

Since $a^{\star n \otimes}(\omega_n)$ and $a^{n \otimes}(\omega_n{}^\star)$ are adjoint in $L(\mathfrak{S}, \mathfrak{S})$, the mapping $a^{\star n \otimes}(\omega_n) \in L(\mathfrak{S}^\star, \mathfrak{S}^\star)$ is dual to the mapping $a^{n \otimes}(\omega_n{}^\star) \in L(\mathfrak{S}, \mathfrak{S})$. Now, since $\mathfrak{S}$ is reflexive, passage to the dual mapping is an algebraic and topological isomorphism from $L(\mathfrak{S}, \mathfrak{S})$ onto $L(\mathfrak{S}^\star, \mathfrak{S}^\star)$, and the yes-part of the second line of the table follows.

Since $\mathfrak{S} \subset \mathfrak{S}^\star$ algebraically and topologically, we have

$$L(\mathfrak{S}, \mathfrak{S}) \subset L(\mathfrak{S}, \mathfrak{S}^\star)$$

and

$$L(\mathfrak{S}^\star, \mathfrak{S}^\star) \subset L(\mathfrak{S}, \mathfrak{S}^\star)$$

algebraically and topologically, and the last line of the table follows from the second and third.

The assertion concerning $a^{n \otimes}$ in the first line is by duality equivalent to the continuity of $a^{\star n \otimes}$ from $\mathscr{S}_+^n$ into $L(\mathfrak{S}, \mathfrak{S})$, and this follows in a straightforward manner (cf. the proof of I, Lemma (3.36)), from the identity

$$H^s\, a^{\star n \otimes}(\omega_n) = \sum_{i=0}^{s} \binom{s}{i}\, a^{\star n \otimes}\, (h^{(n) i} \omega_n)\, H^{s-i}$$

and the estimate

$$|||a^{\star n \otimes}(\omega_n)\Psi|||^2 \leqq (n+1)^n\, \|\omega_n\|^2\, |||\Psi|||_n{}^2 .$$

The yes-part of the theorem is now proved as far as $\mathfrak{S}$ and $\mathfrak{S}^\star$ are concerned. That also the no-part holds is a trivial consequence of the definitions (2.3). Finally, for the case of $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathfrak{S}}^\star$, the proof proceeds along similar lines for each summand- and factor-space separately, and by the properties of the direct sum topology this suffices to verify the said extension properties, and we have proved the theorem.

We conclude this section by the remark that with the literal interpretation now possible, the right hand side of (2.2) converges unconditionally to the element $\Psi \in \mathfrak{S}$. If in particular Ψ is an element of $\widetilde{\mathfrak{S}}$, we sometimes write

$$\Psi = \sum_n{}' \, (n!)^{-\frac{1}{2}} a^{\star n \otimes}(\psi_n) \Psi_0 .$$

Here, as well as in the sequel, we let a primed summation sign denote the sum of a series in which but a finite number of terms are different from the zero element.

3. Displacement operators in $\widetilde{\mathfrak{S}}$.

Let $f \in \mathscr{S}^\star$ and $\varphi \in \mathscr{S}$. In the sequel $\langle \varphi, f \rangle$ denotes the complex conjugate of $\langle f, \varphi \rangle$.

Definition. Let $f \in \mathscr{S}^\star$. An operator $D(f) \in L(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{S}}^\star)$ is called a *displacement operator* associated with f iff

$$D(f)a(\varphi^\star) = \big(a(\varphi^\star) - \langle \varphi, f \rangle\big)D(f) , \tag{3.1}$$

$$D(f)a^\star(\varphi) = \big(a^\star(\varphi) - \langle f, \varphi \rangle\big)D(f) \tag{3.2}$$

for all $\varphi \in \mathscr{S}$.

In this section we prove the existence and essential uniqueness of displacement operators. We remark that in our framework the "boundary condition" $D(f) \in L(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{S}}^\star)$ roughly speaking is the weakest possible one.

Before stating the main theorem of this section we introduce some further notation.

If $f \in \mathscr{S}^\star$, then $f^{n\otimes}$ denotes that element of $\mathscr{S}_+^{n\star}$ which on elements of the form $\varphi^{n\otimes}$ has the value

$$\langle f^{n\otimes}, \varphi^{n\otimes} \rangle = \langle f, \varphi \rangle^n$$

(cf. Lemma 1).

We introduce the operators a_f and $a_f^\star$ (called the *displaced field operators*) by

$$\begin{aligned} a_f(\varphi^\star) &= a(\varphi^\star) - \langle \varphi, f \rangle , \\ a_f^\star(\varphi) &= a^\star(\varphi) - \langle f, \varphi \rangle , \end{aligned} \tag{3.3}$$

for $\varphi \in \mathscr{S}$, $f \in \mathscr{S}^\star$, and their "tensor powers"

$$\begin{aligned} a_f^{n\otimes}(\psi_n^\star) &= \sum_{r=0}^{n} (-1)^r \binom{n}{r} a^{(n-r)\otimes}\big(\langle \psi_n, f^{r\otimes} \rangle_{(r)}\big) , \\ a_f^{\star n\otimes}(\psi_n) &= \sum_{r=0}^{n} (-1)^r \binom{n}{r} a^{\star(n-r)\otimes}\big(\langle f^{r\otimes}, \psi_n \rangle_{(r)}\big) , \end{aligned} \tag{3.4}$$

for $\psi_n \in \mathscr{S}_+^n$. Obviously these operators are adjoint with respect to the scalar product. Further, $a_f^{n\otimes}$ and $a_f^{\star n\otimes}$ are continuous from $\mathscr{S}_+^n$ into $L(\widetilde{\mathfrak{S}}^\star, \widetilde{\mathfrak{S}}^\star)$.

Theorem 2. *Let $f \in \mathscr{S}^\star$. Then $D(f)$ is a displacement operator associated with f iff it is of the form*

$$(3.5)\qquad D(f)\Psi = \sum_n{}' (n!)^{-\frac{1}{2}} a_f^{\star n\otimes}(\psi_n)\Psi_0[f]$$

for all $\Psi = \{\psi_n\} \in \tilde{\mathfrak{S}}$, *with*

$$(3.6)\qquad \Psi_0[f] = c\sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} a^{\star n\otimes}\big((n!)^{-\frac{1}{2}} f^{n\otimes}\big)\Psi_0\,,$$

where c is some complex number.

PROOF. First note that if $D(f)$ is a displacement operator associated with f, then it follows from (3.1) that $D(f)\Psi_0$ satisfies the equation

$$(3.7)\qquad a(\varphi^\star)\boldsymbol{T} = \langle\varphi, f\rangle \boldsymbol{T}$$

for all $\varphi \in \mathscr{S}$. We shall prove below (Lemma 2) that the complete solution to (3.7) is the one-dimensional manifold defined in (3.6).

Next, it follows from (3.2) that

$$D(f)\,a^{\star n\otimes}(\varphi^{n\otimes}) = a_f^{\star n\otimes}(\varphi^{n\otimes})D(f)$$

for all $\varphi \in \mathscr{S}$, and hence that

$$D(f)\,a^{\star n\otimes}(\psi_n) = a_f^{\star n\otimes}(\psi_n)D(f)$$

for all $\psi_n \in \mathscr{S}_+^{\,n}$. Consequently, $D(f)$ must have the form (3.5). Now consider the mapping $D(f)$ defined by (3.5) and (3.6). Since the mapping $a_f^{\star n\otimes}(\cdot)\Psi_0[f]$ is continuous from $\mathscr{S}_+^{\,n}$ into $\tilde{\mathfrak{S}}^\star$, it follows from the properties of the topology of $\tilde{\mathfrak{S}}$ that $D(f) \in L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^\star)$.

It is trivial that $D(f)$ satisfies (3.2). To prove (3.1) we first remark that by the commutation relations (2.4) we get

$$[a(\varphi^\star), a^{\star n\otimes}(\psi^{n\otimes})] = n\langle\varphi,\psi\rangle\, a^{\star(n-1)\otimes}(\psi^{(n-1)\otimes})\,.$$

Using this and Lemma 2 below, we find for $\Psi = \{\psi_n\} \in \tilde{\mathfrak{S}}$

$$\begin{aligned} a(\varphi^\star)D(f)\Psi &= \sum_n{}' (n!)^{-\frac{1}{2}} a_f^{\star n\otimes}\big((n+1)^{\frac{1}{2}}\langle\varphi,\psi_{n+1}\rangle_{(1)}\big)\Psi_0[f] + \\ &\qquad + \sum_n{}' (n!)^{-\frac{1}{2}} a_f^{\star n\otimes}(\psi_n) a(\varphi^\star)\Psi_0[f] \\ &= D(f)a(\varphi^\star)\Psi + D(f)\langle\varphi, f\rangle\Psi\,. \end{aligned}$$

LEMMA 2. *The complete solution in* $\tilde{\mathfrak{S}}^\star$ *to the equations*

$$(3.7)\qquad a(\varphi^\star)\boldsymbol{T} = \langle\varphi, f\rangle\boldsymbol{T}, \qquad \varphi \in \mathscr{S}\,,$$

is the one-dimensional manifold (3.6).

PROOF. Define $\langle\varphi,\cdot\rangle_{(1)} \in L(\mathscr{S}^{(n+1)\star}, \mathscr{S}^{n\star})$ by

$$\langle\langle\varphi, T_{n+1}\rangle_{(1)}, \psi_n\rangle = \langle T_{n+1}, \varphi\otimes\psi_n\rangle\,,$$

and let $\boldsymbol{T}=\{T_n\}\in\tilde{\mathfrak{S}}^\star$. It is easily seen that

$$\boldsymbol{T} = \sum_{n=0}^{\infty}(n!)^{-\frac{1}{2}}a^{\star n\otimes}(T_n)\Psi_0$$

and

$$a(\varphi^\star)\boldsymbol{T} = \sum_{n=0}^{\infty}(n!)^{-\frac{1}{2}}a^{\star n\otimes}\big((n+1)^{\frac{1}{2}}\langle\varphi,T_{n+1}\rangle_{(1)}\big)\Psi_0 .$$

Hence (3.7) may be written in the form

$$\sum_{n=0}^{\infty}(n!)^{-\frac{1}{2}}a^{\star n\otimes}\big((n+1)^{\frac{1}{2}}\langle\varphi,T_{n+1}\rangle_{(1)}\big)\Psi_0 = \sum_{n=0}^{\infty}(n!)^{-\frac{1}{2}}a^{\star n\otimes}(\langle\varphi,f\rangle T_n)\Psi_0 ,$$

that is, (3.7) is equivalent to the recursion formula

$$(n+1)^{\frac{1}{2}}\langle\varphi,T_{n+1}\rangle_{(1)} = \langle\varphi,f\rangle T_n, \qquad n = 0,1,2,\ldots ,$$

for which the complete solution is given by

$$T_n = c(n!)^{-\frac{1}{2}}f^{n\otimes} .$$

This proves the lemma.

For use in the next section we remark that for all $f\in\mathscr{S}^\star$ the displacement operator $D(f)$ (with constant c in (3.6)) may be factorized

(3.8) $$D(f) = cD_-(f)D_+(f) ,$$

where

$$D_+(f) = e^{-a(f^\star)} = \sum_{n=0}^{\infty}(-1)^n(n!)^{-1}a(f^\star)^n ,$$

$$D_-(f) = e^{a^\star(f)} = \sum_{n=0}^{\infty}(n!)^{-1}a^\star(f)^n .$$

Thus, for all $\Psi=\{\psi_n\}\in\tilde{\mathfrak{S}}$,

$$D_+(f)\Psi = \sum_n{}'\,(n!)^{-\frac{1}{2}}a_f{}^{\star n\otimes}(\psi_n)\Psi_0 ,$$

$$D_-(f)\Psi = \sum_n{}'\,(n!)^{-\frac{1}{2}}a^{\star n\otimes}(\psi_n)\Psi_0[f] .$$

Obviously, $D_+(f)\in L(\tilde{\mathfrak{S}},\tilde{\mathfrak{S}})$ and $D_-(f)\in L(\tilde{\mathfrak{S}}^\star,\tilde{\mathfrak{S}}^\star)$, and the dual of $D_+(-f)$ is $D_-(f)$, viz.

(3.9) $$D_-(f) = D_+(-f)^\star .$$

Furthermore,

(3.10) $$D_+(f)D_+(g) = D_+(f+g), \qquad D_-(f)D_-(g) = D_-(f+g)$$

for all $f,g\in\mathscr{S}^\star$.

The proofs of these assertions are straightforward.

4. Continuity and differentiability of the family of displacement operators.

In the first part of this section we assume a common value of the constant c in (3.6) for all displacement operators, and without loss of generality we may choose $c=1$. Then D is a non-linear mapping of $\mathscr{S}^\star$ into $L(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{S}}^\star)$. We prove in this section that D is continuous and even differentiable.

Let $\varphi_n \in \mathscr{S}_+^n$. By $\Psi_n[\varphi_n]$ we denote the element

$$\Psi_n[\varphi_n] = (n!)^{-\frac{1}{2}} a^{\star n \otimes}(\varphi_n)\Psi_0$$

of $\widetilde{\mathfrak{S}}$. The topology of $L(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{S}}^\star)$ is determined by the system of semi-norms

$$q_{B^{(1)}, B^{(2)}}(\cdot) = \sup_{\Phi \in B^{(1)}} \sup_{\Psi \in B^{(2)}} |\langle\langle \Phi, \cdot \Psi \rangle\rangle| ,$$

where $B^{(1)}, B^{(2)}$ run through all bounded sets in $\widetilde{\mathfrak{S}}$. By the properties of $\widetilde{\mathfrak{S}}$ as a direct sum, the topology of $L(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{S}}^\star)$ is already determined by the system

$$q_{B_n, B_m}(\cdot) = \sup_{\varphi_n \in B_n} \sup_{\psi_m \in B_m} |\langle\langle \Psi_n[\varphi_n], \cdot \Psi_m[\psi_m] \rangle\rangle| ,$$

where B_n, B_m, $n,m = 0,1,\ldots$, run through all bounded sets of $\mathscr{S}_+^n$ and $\mathscr{S}_+^m$, respectively. By (3.8) and (3.10) we have

$$D(f+g) - D(f) = D_-(f)\big(D(g)-1\big)D_+(f)$$

and hence

$$\begin{aligned} &\langle\langle \Psi_n[\varphi_n], \big(D(f+g)-D(f)\big)\Psi_m[\psi_m] \rangle\rangle \\ &\qquad = \langle\langle D_+(-f)\Psi_n[\varphi_n], \big(D(g)-1\big)D_+(f)\Psi_m[\psi_m] \rangle\rangle . \end{aligned}$$

Since $D_+(f) \in L(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{S}})$, we have that if $B_m \subset \mathscr{S}_+^m$ is bounded, then $D_+(f)B_m$ is bounded in $\widetilde{\mathfrak{S}}$, and hence it suffices to prove that D is continuous at zero in $\mathscr{S}^\star$. We first prove a lemma.

If $B_n \subset \mathscr{S}^n$, then $\overline{\mathrm{conv}}(B_n)$ denotes the closed convex hull of B_n, and if $B \subset \mathscr{S}$, then B^n denotes the set of all elements in $\mathscr{S}^n$ of the form $\varphi_1 \otimes \ldots \otimes \varphi_n$, $\varphi_i \in B$ for $i = 1,\ldots,n$. Then we have

Lemma 3. *A subset $B_n \subset \mathscr{S}^n$ is bounded iff there exists a bounded subset $B \subset \mathscr{S}$ such that $B_n \subset \overline{\mathrm{conv}}(B^n)$.*

Proof. Assume first that B is bounded in $\mathscr{S}$. To prove that $\overline{\mathrm{conv}}(B^n)$ is bounded it is enough to prove that B^n is bounded (I, Lemma (A.4)). If $h_i = b_i b_i^\star$ we have

$$\|\varphi_1 \otimes \ldots \otimes \varphi_n\|_r^2 = \langle \varphi_1 \otimes \ldots \otimes \varphi_n, (h_1 + \ldots + h_n)^r \varphi_1 \otimes \ldots \otimes \varphi_n \rangle$$
$$\leqq n^r \|\varphi_1\|_r^2 \ldots \|\varphi_n\|_r^2 ,$$

which verifies the boundedness of B^n.

Next, assume that B_n is bounded in $\mathscr{S}^n$. Let $\{\psi_i\}$ denote the orthonormal basis of Hermite elements in $\mathscr{S}$. Then the elements $\psi_\nu = \psi_{\nu_1} \otimes \ldots \otimes \psi_{\nu_n}$ form an orthonormal basis in $\mathscr{S}^n$, and if $\varphi \in \mathscr{S}^n$, we have the development

$$\varphi = \sum_{\nu \in N^n} c_\nu \psi_\nu ,$$

the norms of φ being given by

$$\|\varphi\|_r^2 = \sum_{\nu \in N^n} (|\nu| + n)^r |c_\nu|^2 ,$$

where $|\nu| = \nu_1 + \ldots + \nu_n$. By Hölders inequality it follows that

$$M = \sup_{\varphi \in B_n} \sum_{\nu \in N^n} |c_\nu|^{1/(n+1)}$$

is finite, and if we define

$$t_\nu = M^{-1} |c_\nu|^{1/(n+1)} ,$$
$$\varphi_{\nu_1} = \operatorname{sgn} c_\nu |c_\nu|^{1/(n+1)} M^{1/n} \psi_{\nu_1} ,$$
$$\varphi_{\nu_i} = |c_\nu|^{1/(n+1)} M^{1/n} \psi_{\nu_i}, \qquad i = 2,3,\ldots,n ,$$

then for all $\varphi \in B_n$ we have

$$\varphi = \sum_{\nu \in N^n} t_\nu \varphi_{\nu_1} \otimes \ldots \otimes \varphi_{\nu_n}, \quad \text{where} \quad \sum_{\nu \in N^n} t_\nu \leqq 1 .$$

To prove the lemma we therefore have to prove that if $\varphi \in B_n$, then the φ_{ν_i} are contained in some bounded set $B \subset \mathscr{S}$, but this is an immediate consequence of the boundedness of B_n and Hölders inequality.

For the sake of completeness we have included the proof of this lemma; essentially the same result can be found in Ehrenpreis [1].

We now turn to the proof of the continuity of D at zero. From (3.8) and (3.9) we get

$$\langle\langle \Psi_n[\varphi_n], (D(f) - 1) \Psi_m[\psi_m] \rangle\rangle = Q_1 + Q_2 + Q_3 ,$$

where

$$Q_1 = \langle\langle (D_+(-f) - 1) \Psi_n[\varphi_n], \Psi_m[\psi_m] \rangle\rangle ,$$
$$Q_2 = \langle\langle \Psi_n[\varphi_n], (D_+(f) - 1) \Psi_m[\psi_m] \rangle\rangle ,$$
$$Q_3 = \langle\langle (D_+(-f) - 1) \Psi_n[\varphi_n], (D_+(f) - 1) \Psi_m[\psi_m] \rangle\rangle .$$

Without loss of generality we may assume $m \geqq n$. An explicit calculation then shows that

$$Q_1 = 0 ,$$

$$Q_2 = \begin{cases} 0 & \text{for } m=n , \\ (-1)^{m-n} c_{mn} \langle f^{(m-n)\otimes}, \langle \varphi_n, \psi_m \rangle_{(n)} \rangle & \text{for } m>n , \end{cases}$$

$$Q_3 = \sum_{t=0}^{n-1} (-1)^{m-t} c_{mnt} \langle f^{\star(n-t)\otimes} \otimes f^{(m-t)\otimes}, \langle \varphi_n, \psi_m \rangle_{(t)} \rangle ,$$

where c_{mn} and c_{mnt} denote positive constants.

Consider one of the terms

$$A_{mnt} = \langle f^{\star(n-t)\otimes} \otimes f^{(m-t)\otimes}, \langle \varphi_n, \psi_m \rangle_{(t)} \rangle$$

of Q_3, and suppose that φ_n and ψ_m run through bounded sets B_n and B_m in $\mathscr{S}_+^n$ and $\mathscr{S}_+^m$ respectively. By Lemma 3 there exists a bounded set B in $\mathscr{S}$ such that φ_n and ψ_m can be represented in the form

$$\varphi_n = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} c_{k_1 \ldots k_n} \omega_{k_1} \otimes \cdots \otimes \omega_{k_n} ,$$

$$\psi_m = \sum_{l_1=1}^{\infty} \cdots \sum_{l_m=1}^{\infty} d_{l_1 \ldots l_m} \chi_{l_1} \otimes \cdots \otimes \chi_{l_m} ,$$

with all ω_{k_i} and χ_{l_j} in B, all c and d non-negative, and $\sum c \leqq 1$, $\sum d \leqq 1$. It follows that

$$|A_{mnt}| \leqq \|f\|_B^{m+n-2t} \sup_{\omega \in B} \|\omega\|^{2t} ,$$

where

$$\|f\|_B = \sup_{\omega \in B} |\langle f, \omega \rangle| .$$

Since this estimate holds also for $t=n<m$, we get

$$q_{B_n, B_m}(D(f)-1) \leqq P(\|f\|_B) \|f\|_B ,$$

where P is a polynomial. We have now proved

THEOREM 3. *The mapping D (with fixed constant c in (3.6)) is continuous from $\mathscr{S}^\star$ into $L(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{S}}^\star)$.*

COROLLARY 1. *Let $f(t)$ be a mapping of the real axis into $\mathscr{S}^\star$ with the property that*

$$\lim_{t \to \infty} f(t) = f(\infty)$$

exists in $\mathscr{S}^\star$, and let $c(t)$ be a complex valued function with the property that

$$\lim_{t \to \infty} c(t) = c(\infty)$$

exists. Let further $D(f(t))$ *be defined with* $c(t)$ *as the constant in* (3.6). *Then*

$$\lim_{t\to\infty} D(f(t)) = D(f(\infty))$$

in $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^\star)$.

Using the same methods as in the proof of Theorem 3, one proves

THEOREM 4. *The mapping* D *(with a fixed constant* c *in* (3.6)) *has for all* $f,g \in \mathscr{S}^\star$ *the property that*

$$D(f+g)-D(f) = a^\star(g)D(f)-D(f)a(g^\star) + o(f,g) ,$$

where for all continuous semi-norms q *on* $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^\star)$ *and for fixed* f, $q(o(f,g))$ *is of the order* $p(g)^2$ *for some semi-norm* p *on* $\mathscr{S}^\star$.

We introduce the definition:

DEFINITION. A mapping F from a locally convex space S_1 into a locally convex space S_2 is called *differentiable* at $f \in S_1$ iff there exists a continuous real-linear mapping $\dot{F}(f;\cdot)$ (called the *differential* of F) from S_1 into S_2 such that

$$F(f+g) = F(f)+\dot{F}(f;g)+o(f,g) ,$$

where $o(f,g)$ has the property that for every continuous semi-norm q on S_2 there exists a continuous semi-norm p on S_1 such that $q(o(f,g)) = o(p(g))$ in the usual sense.

Theorem 4 can then be formulated

THEOREM 4′. *The mapping* D *(with a fixed constant* c *in* (3.6)) *is differentiable at all* $f \in \mathscr{S}^\star$ *with the differential*

$$\dot{D}(f;g) = a^\star(g)D(f)-D(f)a(g^\star) .$$

In particular we have

COROLLARY 2. *Let* $f(t)$ *be a differentiable mapping of the real axis into* $\mathscr{S}^\star$ *and let* $c(t)$ *be a differentiable complex valued function. Let further* $D(f(t))$ *denote the displacement operator with* $c(t)$ *as the constant in* (3.6). *Then* $D(f(t))$ *is differentiable from the real axis into* $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^\star)$, *and the derivative is given by*

$$\dot{D}(f(t)) = a^\star(\dot{f}(t))D(f(t)) - D(f(t))a(\dot{f}(t)^\star) + \frac{d\log c(t)}{dt}D(f(t)) ,$$

where $\dot{f}$ *denotes the derivative of* f.

5. Displacement operators in $\mathfrak{S}$ and in $\mathfrak{H}$.

The displaced field operators a_f and $a_f^\star$ defined by (3.3) satisfy the canonical commutation relations

$$(5.1)\qquad \begin{aligned}[][a_f^\star(\varphi^\star), a_f^\star(\psi)] &= \langle\varphi,\psi\rangle\,,\\ [a_f^\star(\varphi), a_f^\star(\psi)] &= [a_f(\varphi^\star), a_f(\psi^\star)] = 0\,,\end{aligned}$$

and $a_f(\varphi^\star)$ and $a_f^\star(\varphi)$ are adjoint with respect to the scalar product in $\mathfrak{S}$. According to Lemma 2 the equations

$$(5.2)\qquad a_f(\varphi^\star)\Psi_0[f] = 0$$

for all $\varphi \in \mathscr{S}$, characterize a vacuum element $\Psi_0[f]$ for these operators in an essentially unique way. However, in general $\Psi_0[f]$ is not an element of $\mathfrak{S}$ or even of $\mathfrak{S}^\star$.

By $\mathfrak{H}$ we denote the Hilbert space obtained by completion of $\mathfrak{S}$ in the norm $|||\cdot|||$. It is easily seen that $\mathfrak{H}$ admits a Fock representation and that

$$\mathfrak{H} = \left\{\Psi = \{\psi_n\} \;\middle|\; \psi_n \in \mathscr{H}^n,\ |||\Psi|||^2 = \sum_{n=0}^{\infty} \|\psi_n\|^2 < \infty\right\},$$

where $\mathscr{H}^n$ denotes the completion of $\mathscr{S}_+^n$ in the norm $\|\cdot\|$.

In this section we investigate the conditions under which a displacement operator may be extended to $\mathfrak{S}$, respectively to $\mathfrak{H}$, as well as properties of these extensions.

LEMMA 4. *The element $\Psi_0[f]$ belongs to $\mathfrak{H}$ iff $f \in \mathscr{H}$.*

PROOF. In the Fock representation $\Psi_0[f]$ is given by $\Psi_0[f] = \{c(n!)^{-\frac{1}{2}} f^{n\otimes}\}$, and the lemma follows.

If $f \in \mathscr{H}$, then

$$|||\Psi_0[f]|||^2 = \sum_{n=0}^{\infty} |c|^2 (n!)^{-1} \|f\|^{2n} = |c|^2 e^{\|f\|^2}\,.$$

For $f \in \mathscr{H}$ (and hence in particular for $f \in \mathscr{S}$) we shall in the remaining part of this section normalize the factor c in (3.6) to be

$$c = e^{-\frac{1}{2}\|f\|^2}\,,$$

so that

$$|||\Psi_0[f]||| = 1\,.$$

We shall refer to the displacement operator defined in this way as the normalized displacement operator.

LEMMA 5. *The element $\Psi_0[f]$ belongs to $\mathfrak{S}$ iff $f \in \mathscr{S}$.*

PROOF. It is trivial that if $\Psi_0[f] \in \mathfrak{S}$, then $f \in \mathscr{S}$. Now observe that for $f \in \mathscr{S}$,

$$\|f^{n\otimes}\|_r^2 = \langle f^{n\otimes}, (h_1 + \ldots + h_n)^r f^{n\otimes} \rangle = \sum_{s_1+\ldots+s_n=r} \|f\|_{s_1}^2 \cdots \|f\|_{s_n}^2 \leqq n^r \|f\|_r^{2n} ,$$

since the norm system $\|\cdot\|_r$ is not decreasing. Hence,

$$|||\Psi_0[f]|||_r^2 \leqq e^{-\|f\|^2} \sum_{n=0}^{\infty} n^r (n!)^{-1} \|f\|_r^{2n} < \infty ,$$

and the lemma follows.

In the proof of Theorem 1 we observed in particular that $a^{\star n\otimes}$ maps $\mathscr{S}_+^n$ into $L(\mathfrak{S}, \mathfrak{S})$. It then follows from (3.4) that $a_f^{\star n\otimes}$ maps $\mathscr{S}_+^n$ into $L(\mathfrak{S}, \mathfrak{S})$, and it is easily seen that $a_f^{\star n\otimes}$ maps $\mathscr{H}$ into $L(\mathfrak{H}, \mathfrak{H})$ provided that $f \in \mathscr{H}$. Hence we have

COROLLARY 3. *The operator $D(f)$ maps $\tilde{\tilde{\mathfrak{S}}}$ into $\mathfrak{H}$ iff $f \in \mathscr{H}$, and $D(f)$ maps $\tilde{\tilde{\mathfrak{S}}}$ into $\mathfrak{S}$ iff $f \in \mathscr{S}$.*

LEMMA 6. *If $f \in \mathscr{H}$ then the normalized displacement operator $D(f)$ has a unique continuous extension to $\mathfrak{H}$, and this extension is a unitary operator in $\mathfrak{H}$.*

PROOF. By exactly the same reasoning as in Section 3 of I it follows from (5.1) and (5.2) that for $f \in \mathscr{S}$ (and hence by continuity also for $f \in \mathscr{H}$) we have

$$\begin{aligned} &\langle\langle D(f)\Psi_n[\psi_n], D(f)\Psi_m[\varphi_m]\rangle\rangle \\ &\quad = \langle\langle (n!)^{-\frac{1}{2}} a_f^{\star n\otimes}(\psi_n)\Psi_0[f], (m!)^{-\frac{1}{2}} a_f^{\star m\otimes}(\varphi_m)\Psi_0[f]\rangle\rangle \\ &\quad = \begin{cases} 0 & \text{for } n \neq m , \\ \langle \psi_n, \varphi_n \rangle & \text{for } n = m . \end{cases} \end{aligned}$$

Thus, for Φ, $\Psi \in \tilde{\tilde{\mathfrak{S}}}$ and $f \in \mathscr{H}$ we have

$$\langle\langle D(f)\Phi, D(f)\Psi \rangle\rangle = \langle\langle \Phi, \Psi \rangle\rangle ,$$

and since $\tilde{\tilde{\mathfrak{S}}}$ is dense in $\mathfrak{H}$, it follows that $D(f)$ has a unique continuous extension to $\mathfrak{H}$, and that this extension (which we also denote $D(f)$) is an isometry. That $D(f)$ is unitary then follows from the fact that $D(f)$ and $D(-f)$ are adjoint.

The unitarity of $D(f)$ also follows from the following lemma, which shows that apart from a numerical factor the operators $D(f)$ constitute a representation of the additive group $\mathscr{H}$.

Lemma 7. *For $f,g \in \mathscr{H}$ the normalized displacement operators $D(f)$ and $D(g)$ satisfy*

$$D(f)D(g) = e^{-i\,\mathrm{Im}\langle f,g\rangle} D(f+g)\,.$$

Proof. We have

$$D(f)D(g)a^\star(\varphi) = a^\star_{f+g}(\varphi)D(f)D(g)\,,$$
$$D(f)D(g)a(\varphi^\star) = a_{f+g}(\varphi^\star)D(f)D(g)\,.$$

It then follows from Theorem 2 that

$$D(f)D(g) = kD(f+g)$$

for some complex number k. To evaluate k we calculate the vacuum expectation value

$$\langle\langle \Psi_0, D(f)D(g)\Psi_0\rangle\rangle = k\langle\langle \Psi_0, D(f+g)\Psi_0\rangle\rangle\,.$$

We find

$$\begin{aligned}\langle\langle \Psi_0, D(f)D(g)\Psi_0\rangle\rangle &= \langle\langle D(-f)\Psi_0, D(g)\Psi_0\rangle\rangle \\ &= \langle\langle \Psi_0[-f], \Psi_0[g]\rangle\rangle \\ &= \exp(-\tfrac{1}{2}\|f\|^2 - \tfrac{1}{2}\|g\|^2) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\langle f,g\rangle^n \\ &= \exp(-\tfrac{1}{2}\|f\|^2 - \tfrac{1}{2}\|g\|^2 - \langle f,g\rangle)\end{aligned}$$

and

$$\langle\langle \Psi_0, D(f+g)\Psi_0\rangle\rangle = \exp(-\tfrac{1}{2}\|f+g\|^2)\,,$$

and the result follows. — In particular, we get

$$D(-f) = D(f)^\star = D(f)^{-1}$$

for $f \in \mathscr{H}$.

Lemma 8. *Let $f \in \mathscr{S}$, and let H denote the normalized bi-quantization of the operator $h = bb^\star$; then*

$$HD(f) = D(f)\big(H + a^\star(hf) + a((hf)^\star) + \langle f, hf\rangle\big)$$

holds from $\tilde{\mathfrak{S}}$ into $\mathfrak{S}$.

Proof. For $\Psi \in \tilde{\mathfrak{S}}$ we have $D_+(f)\Psi \in \tilde{\mathfrak{S}}$ and

$$D_-(f)D_+(f)\Psi = \sum_{n=0}^{\infty} (n!)^{-1} a^\star(f)^n D_+(f)\Psi\,,$$

where the series is convergent in $\mathfrak{S}$ (Corollary 3). Since H is continuous from $\mathfrak{S}$ into $\mathfrak{S}$, we get (pointwise in $\tilde{\mathfrak{S}}$)

$$
\begin{aligned}
HD_-(f)D_+(f) &= \sum_{n=0}^{\infty}(n!)^{-1}Ha^\star(f)^n D_+(f) \\
&= \sum_{n=0}^{\infty}(n!)^{-1}\big(a^\star(f)^n H + na^\star(f)^{n-1}a^\star(hf)\big)D_+(f) \\
&= D_-(f)\big(H+a^\star(hf)\big)D_+(f)\,.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
HD_+(f) &= \sum_{n=0}^{\infty}H(-1)^n(n!)^{-1}a(f^\star)^n \\
&= \sum_{n=0}^{\infty}(-1)^n(n!)^{-1}\big(H-na(f^\star)^{n-1}a\big((hf)^\star\big)\big) \\
&= D_+(f)\big(H+a\big((hf)^\star\big)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
a^\star(hf)D_+(f) &= \sum_{n=0}^{\infty}a^\star(hf)(-1)^n(n!)^{-1}a(f^\star)^n \\
&= \sum_{n=0}^{\infty}(-1)^n(n!)^{-1}\big(a(f^\star)^n a^\star(hf) - na(f^\star)^{n-1}\langle f,hf\rangle\big) \\
&= D_+(f)\big(a^\star(hf)+\langle f,hf\rangle\big)\,,
\end{aligned}
$$

and the lemma follows.

We may now formulate the main result of this section.

THEOREM 5. *If $f\in\mathscr{S}$, then the unitary mapping $D(f)$ defined in Lemma 6 maps $\mathfrak{S}$ onto $\mathfrak{S}$, and its restriction to $\mathfrak{S}$ belongs to $L(\mathfrak{S},\mathfrak{S})$.*

PROOF. Let $f\in\mathscr{S}$, and assume $\Psi\in\mathfrak{S}$. Then it follows from Lemma 8 that

$$
|||D(f)\Psi|||_{2r} = |||(H+a^\star(hf)+a\big((hf)^\star\big)+\|f\|_1{}^2)^r\Psi|||\,.
$$

Since $H+a^\star(hf)+a\big((hf)^\star\big)+\|f\|_1{}^2\in L(\mathfrak{S},\mathfrak{S})$, it follows that $D(f)$ is continuous from $\tilde{\mathfrak{S}}$ with the topology of $\mathfrak{S}$ into $\mathfrak{S}$, and hence that $D(f)$ has a unique extension belonging to $L(\mathfrak{S},\mathfrak{S})$. This extension of course coincides with the unitary extension of $D(f)$ on $\mathfrak{S}$. Since $D(f)^{-1}=D(-f)$ also maps $\mathfrak{S}$ into $\mathfrak{S}$, it follows that $D(f)$ maps $\mathfrak{S}$ onto $\mathfrak{S}$.

Hence we may say that the maximal space $\mathfrak{S}$ is invariant under displacements $f\in\mathscr{S}$. Obviously, $\tilde{\mathfrak{S}}$ is not invariant under any displacement other than $D(0)$. We remark that there exist spaces of type $\mathfrak{S}$ smaller than $\mathfrak{S}$ which are invariant under all displacements $f \in \mathscr{S}$. This is for instance the case for the space of all $\Psi = \{\psi_n\} \in \mathfrak{S}$ satisfying:

For each $r = 0, 1, \ldots$ there exists an $M < \infty$ such that

$$\sum_{n=0}^{\infty} n!\, M^{-n} \langle \psi_n, (h_1 h_2 \ldots h_n)^r \psi_n \rangle < \infty .$$

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