MATH. SCAND. 14 (1964), 174–178

ON SUITABLE MANIFOLDS

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1. Introduction.

Let M be a manifold (locally Euclidean connected separable metric space) and let $G(M)$ denote the group of all homeomorphisms of M onto itself with the compact-open topology. Pick a point $e \in M$. Fadell and Neuwirth [2] call M *suitable* if there exists a continuous map $\theta : M \to G(M)$ such that $\theta(x)(x) = e$ and $\theta(e) = \text{identity}$. We shall show that when M is compact, suitability is equivalent to the existence on M of a continuous multiplication which has many of the properties of a group multiplication. The paper concludes with a definition of suitability for differentiable manifolds and a proof that such manifolds are parallelizable.

2. Equivalent definitions.

The map $q : G(M) \to M$ given by $q(h) = h(e)$ is a fibre space [2] and in fact a principal fibre bundle [3] with fibre

$$G_e(M) = \{h \in G(M) \mid h(e) = e\}.$$

THEOREM 1. *A manifold M is suitable if, and only if, the bundle $q : G(M) \to M$ is trivial.*

PROOF. Suppose first that $q : G(M) \to M$ is trivial, i.e., there is a homeomorphism ψ' such that the diagram

$$\begin{array}{ccc} M \times G_e(M) & \xrightarrow{\quad \psi' \quad} & G(M) \\ & \text{proj.} \searrow \qquad \swarrow q & \\ & M & \end{array}$$

Received December 30, 1963.

This research was supported in part by the U.S. National Science Foundation and by the U.S. Air Force Office of Scientific Research.

commutes. Hence there exists $g \in G_e(M)$ such that $\psi'(e,g)=1$ where $1 : M \to M$ denotes the identity homeomorphism. Define $\psi : M \times G_e(M) \to G(M)$ by $\psi(x,h)=\psi'(x,hg)$, then ψ is a well-defined homeomorphism and the preceding diagram commutes with ψ in place of ψ'. Furthermore, $\psi(e,1)=1$ so if we define $\theta : M \to G(M)$ by $\theta(x)=[\psi(x,1)]^{-1}$, then $\theta(e)=1$ and since

$$[\theta(x)]^{-1}(e) = \big(\psi(x,1)\big)(e) = q\psi(x,1) = x\,,$$

we have $\theta(x)(x)=e$. Conversely, suppose that $\theta : M \to G(M)$ exists making M suitable, then define $\sigma : M \to G(M)$ by $\sigma(x)=\big(\theta(x)\big)^{-1}$. Now

$$q\sigma(x) = \sigma(x)(e) = \big(\theta(x)\big)^{-1}(e) = x$$

since $\theta(x)(x)=e$ so $q\sigma=1$ and σ is a cross section of the principal bundle $q : G(M) \to M$. Therefore, by the Cross Section Theorem of [4, p. 36], the bundle is trivial. This completes the proof of the theorem.

We recall from [2] that

$$F_{0,2} = F_{0,2}(M) = \{(x,y) \in M \times M \mid x \neq y\}\,.$$

Define $\pi^1,\pi^2 : M \times M \to M$ to be the projections on the first and second factor respectively.

THEOREM 2. *A manifold M is suitable if, and only if, there exists $\varphi \in G(M \times M)$ such that $\varphi\big(M \times (M-e)\big)=F_{0,2}$ and $\pi^1\varphi=\pi^1$.*

PROOF. Necessity follows from Theorem 4 of [2] if we observe that the homeomorphism defined in the proof of that theorem can be extended to $M \times M$. For the proof of sufficiency we define $\sigma : M \to G(M)$ by $\sigma(x)(y)=\pi^2\varphi(x,y)$. Since $\pi^1\varphi=\pi^1$, it follows that $\big(x,\pi^2\varphi(x,y)\big)=\varphi(x,y)$ so $[\sigma(x)]^{-1}(y)=\pi^2\varphi^{-1}(x,y)$ and σ is well-defined. Furthermore,

$$q\sigma(x) = \sigma(x)(e) = \pi^2\varphi(x,e) = \pi^2(x,x) = x$$

so $q\sigma=1$ and σ is a cross section of $q : G(M) \to M$. Again applying the Cross Section Theorem we have that the bundle $q : G(M) \to M$ is trivial and by Theorem 1 our result is proved.

3. Continuous multiplication.

THEOREM 3. *A compact manifold M is suitable if, and only if, there exists a map $f : M \times M \to M$ $\big($write $f(x,y)=xy\big)$ such that*

(1) *$xe=x$ for all $x \in M$,*
(2) *given $a,b \in M$ there exists $x \in M$ such that $ax=b$,*
(3) *$xy=xz$ implies $y=z$ for all x, y, $z \in M$.*

PROOF. Suppose that M is suitable and let $\varphi \in G(M \times M)$ be a homeomorphism satisfying the hypotheses of Theorem 2. Define $f(x,y) = xy = \pi^2 \varphi(x,y)$. Let Δ denote the diagonal in $M \times M$, then since $\varphi(M \times e) = \Delta$,

$$xe = \pi^2 \varphi(x,e) = \pi^2(x,x) = x$$

and (1) holds. Given $a, b \in M$, let $x = \pi^2 \varphi^{-1}(a,b)$, then

$$ax = \pi^2 \varphi(a,x) = \pi^2 \varphi\big(a, \pi^2 \varphi^{-1}(a,b)\big) = \pi^2(a,b) = b$$

which verifies (2). If $xy = xz$, then

$$\pi^2\varphi(x,y) = \pi^2\varphi(x,z) \qquad \text{and} \qquad \big(x, \pi^2\varphi(x,y)\big) = \big(x, \pi^2\varphi(x,z)\big)$$

so $\varphi(x,y) = \varphi(x,z)$ and since φ is one-to-one, $y = z$. Thus (3) holds.

Conversely, if a continuous multiplication is defined on M satisfying (1), (2), and (3), define $\varphi : M \times M \to M \times M$ by $\varphi(x,y) = (x, xy)$. If $x \neq x'$, then $\varphi(x,y) \neq \varphi(x',y')$ for all $y, y' \in M$. If $\varphi(x,y) = \varphi(x,z)$ then $xy = xz$ and by (3), $y = z$ so φ is one-to-one. Given $(x,y) \in M \times M$, there is a $z \in M$ with $\varphi(x,z) = (x,y)$, namely that z such that $xz = y$ (property (2)) and thus φ is onto. Since M is compact, φ is therefore a homeomorphism. By (1), $\varphi(M \times e) = \Delta$ so

$$\varphi\big(M \times (M - e)\big) = F_{0,2}$$

since φ is a homeomorphism. Finally, it is obvious that $\pi^1 \varphi = \pi^1$ so we can apply Theorem 2 to complete the proof.

REMARKS.

(a) The proof of necessity in Theorem 3 does not require that M be compact.

(b) When M is suitable, it follows from Theorem 3 that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1} = e$.

(c) It is easy to see that the function $i : M \to M$ given by $i(x) = x^{-1}$ is continuous since $x^{-1} = \pi^2 \varphi^{-1}(x,e)$.

(d) It was noted in [2] that every suitable manifold is an H-space. However, this does not imply that every homoeomorphism $\varphi \in G(M \times M)$ satisfying the conditions of Theorem 2 gives rise to an H-space multiplication by setting $xy = \pi^2\varphi(x,y)$. On the other hand, if any $\varphi \in G(M \times M)$ satisfying the conditions of that theorem does exist, then we can obtain another homeomorphism $\psi \in G(M \times M)$ such that, in addition, for all $x \in M$, $\psi(e,x) = (e,x)$ and $xy = \pi^2\psi(x,y)$ will be an H-space multiplication as well as having all the properties previously described.

(e) The example of the 7-sphere shows that not all suitable manifolds admit a group multiplication.

4. Differentiable manifolds.

We define a *differentiable* manifold M to be *suitable* if there is a diffeomorphism $\varphi : M \times M \to M \times M$ such that $\varphi(M \times (M-e)) = F_{0,2}$ and $\pi^1\varphi = \pi^1$. This definition makes the multiplication defined in Theorem 3 differentiable.

An example due to Milnor [1] shows that the classical result that all Lie groups are parallelizable can not be extended to differentiable manifolds which admit an H-space multiplication. A Lie group is a suitable differentiable manifold since we can set $\varphi(x,y) = (x,xy)$ and $\varphi^{-1}(x,y) = (x,x^{-1}y)$ (compare [2]). On the other hand, a suitable differentiable manifold is an H-space by Remark (d).

Theorem 4. *A suitable differentiable manifold is parallelizable.*

Proof. Let $(\tau(M),p,M)$ denote the tangent (plane) bundle of a suitable differentiable manifold M. We must exhibit maps α and β such that the diagram

$$\begin{array}{ccc} \tau(M) & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & M \times p^{-1}(e) \\ & p \searrow \quad \swarrow \text{proj.} & \\ & M & \end{array}$$

commutes and α and β are inverses of one another. Let $x \in M$ and let f be a real function defined on a neighborhood of e and differentiable at e. We define a real function $(x \vee f)$ in a neighborhood of x by

$$(x \vee f)(y) = f(\pi^2\varphi^{-1}(x,y)) .$$

Since φ is a diffeomorphism and $(x \vee f)(x) = f(e)$, the function $(x \vee f)$ is differentiable at x. For $T_x \in p^{-1}(x)$, we set

$$\alpha(T_x) = (x,\bar{\alpha}T_x), \quad \text{where} \quad (\bar{\alpha}T_x)(f) = T_x(x \vee f) .$$

Similarly, for $x \in M$ and g a real function defined on a neighborhood of x and differentiable at x, we define $(x \wedge g)$, a real function defined on a neighborhood of e and differentiable at e by $(x \wedge g)(z) = g(\pi^2\varphi(x,z))$. Then for $T_e \in p^{-1}(e)$, let

$$\beta(x,T_e)(g) = T_e(x \wedge g) .$$

Now take $x \in M$, $T_x \in p^{-1}(x)$, f a real function defined on a neighborhood of x and differentiable at x, and y in the domain of $(x \vee f)$, then

$$\begin{aligned}\beta[\alpha T_x(f)(y)] &= \beta[x, T_x(x \vee f)(y)] \\ &= \beta\big[x, T_x(f)\big(\pi^2\varphi^{-1}(x,y)\big)\big] \\ &= T_x(x \wedge f)\big(\pi^2\varphi^{-1}(x,y)\big) \\ &= T_x(f)\big(\pi^2\varphi\big(x,\pi^2\varphi^{-1}(x,y)\big)\big) = T_x(f)(y)\end{aligned}$$

so $\beta\alpha =$ identity. By a similar computation, $\alpha\beta =$ identity. It is obvious from the definition that α makes the diagram commute and in order to see that β also makes the diagram commute, we observe that $T_e(x\wedge\ )$ is a vector tangent to the manifold at x.

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