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ON LINEARLY MONOTONE CURVES IN THE PROJECTIVE n-SPACE

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In previous papers [3], [4] and [5] the author has studied the connection between two classes of curves in the projective space R^n: The *strictly convex* curves and the *linearly monotone* curves. In [3] and [4] it has been proved that any *plane* strictly convex curve is linearly monotone and, conversely, that any linearly monotone curve is strictly convex at any ordinary point, i.e., a point belonging to an open arc of order 2. Moreover, in [5], we have shown that in R^n, $n \geqq 3$, any strictly convex curve is linearly monotone (Theorem 4.2, p. 228). On the following pages we shall show that, conversely, any linearly monotone curve in R^n is strictly convex at every set of $n-1$ points $P_0, P_1, \ldots, P_{n-2}$ such that the osculating plane at each of these has only the point of contact in common with the $(n-2)$-space determined by the $n-1$ points. This condition is satisfied if the $n-1$ points belong to the same arc of order n.

In Section 1 definitions of strict convexity and linear monotonicity are given. Section 2 deals with an auxiliary theorem on plane curves, and in Section 3 we prove the theorem stated above, on linearly monotone curves in R^n.

1. Definitions

1.1. By an open or closed *curve* we shall mean a topological image in R^n of a segment (including the endpoints) or a circle, respectively. The curves will be assumed n times differentiable, oriented and of bounded order. (A curve is said to be of bounded order if any hyperplane has at most finitely many points in common with it.)

1.2. An open or closed *polygon* in R^n, $n \geqq 2$, is a sequence of segments $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m$, resp. $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m, P_mP_1$, its sides. The *order* of the polygon is said to be n, if $m > n$ and no hyperplane has more than n points in common with it. (In case the hyperplane contains sides of the polygon, only their endpoints are counted.) In [5, p. 226–227]

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it was shown that an open polygon Π_n of order n can be changed to a closed polygon $\bar{\Pi}_n$ of order n, and conversely, by adding, resp. removing, a suitably chosen segment.

A polygon of order $n = 1$ is by definition a finite sequence of segments on a straight line such that two consecutive segments have only an endpoint (or both endpoints) in common whereas two non-consecutive segments have no points in common. The polygon is called closed or open according as the segments make up the whole line or a segment.

The vertices $P_1, P_2, \ldots, P_m$ of a polygon of order 1 form a monotone sequence of points on the (projective) line. The ordered set of the vertices of a polygon of order n in the (projective) R^n is called a *monotone sequence* in R^n [5, p. 225].

1.3. Let a hyperplane H have the m points $P_0, P_1, \ldots, P_{m-1}$, $m \geqq n$, in common with a curve c, the points being taken in the order determined by the parametrization of the curve. The *chord* P_iP_{i+1} is defined as that segment P_iP_{i+1} which together with the arc P_iP_{i+1} forms a closed curve of *even* order. The chords $P_0P_1, P_1P_2, \ldots, P_{m-2}P_{m-1}$ and, if c is closed, in addition $P_{m-1}P_0$, form a polygon Π inscribed in the curve and situated in the hyperplane H. If Π has order $n-1$, the vertices $P_0, P_1, \ldots, P_{m-1}$ form a monotone sequence in H, and the curve is called *linearly monotone* along the hyperplane H. The curve, as a whole, is called linearly monotone if it is linearly monotone along any hyperplane having at least n points in common with it.

For $n = 2$ a curve which is linearly monotone along a line H determines an orientation of the line, namely by the cyclical order of the point P_0, a point P_0' of the chord P_0P_1, and the point P_1. If $m \geqq 3$, this orientation is also determined by the monotone sequence $P_0, P_1, \ldots, P_{m-1}$.

With a view to an application below, we observe the following: If a line l through a point P_0 on a plane curve c *intersects* c at a point $P \neq P_0$ (that is, l is not a local supporting line to c at P), and if c is linearly monotone along all lines through P_0 belonging to a neighbourhood of l, then c is also linearly monotone along l.

1.4. We consider a set of $n-1$ linearly independent points $(P_0, P_1, \ldots, P_{n-2})$ on a curve c in R^n. The curve is called *strictly convex* at the set $(P_0, P_1, \ldots, P_{n-2})$ if there exists at least one hyperplane H having these and no other points in common with the curve. The curve, as a whole, is called strictly convex if it is strictly convex at any set of $n-1$ linearly independent points (Barner [1]).

The $n-1$ points $P_0, P_1, \ldots, P_{n-2}$ span a linear space $B = B^{n-2}$. If c is contained in an angular domain bounded by two hyperplanes through B, then c is strictly convex at the set $(P_0, P_1, \ldots, P_{n-2})$ since any hyperplane through B in the complementary angular domain has only the points $P_0, P_1, \ldots, P_{n-2}$ in common with c.

2. Plane curves.

2.1. Let c denote an open or closed curve in a projective plane. It is assumed to be linearly monotone along any line l_P which connects a fixed point P_0 on c with a variable point $P \neq P_0$ on c. The lines l_P are then oriented in accordance with the orientation of the curve, as described above. We denote by l_{P_0} the tangent to c at P_0 oriented in accordance with the orientation of c. Let e be a conic having P_0 as an interior point, and let u denote the polar of P_0 with respect to e. The line l_P intersects u in a point U_P and e in two points of which we choose that point E_P for which the triple (P_0, E_P, U_P) is in accordance with the orientation of the line. The point E_P is called the *point of orientation* of the oriented line l_P. Without restricting the generality the conic e may be regarded as a circle with centre P_0.

To each point $P \in c$ corresponds in this manner a point $E_P \in c$. The mapping φ of c into e thus defined is continuous. Consequently, the image $\varphi(c)$ is closed and connected, hence a closed arc of e or the whole circle e. However, if P and Q are points on c both different from P_0, the corresponding points E_P and E_Q cannot be diametrically opposite. This implies that the image $\varphi(c)$ is an arc contained in a semicircle, and that it cannot be a semicircle unless E_{P_0} is one if its endpoints.

If the arc $\varphi(c)$ is smaller than a semicircle, the curve c is contained in one of the angular domains bounded by the lines joining P_0 with the endpoints of $\varphi(c)$. Since a line through P_0 in the complementary angular domain has only P_0 in common with c, it follows that c is strictly convex at P_0 in this case.

Now, if P_0 is an *ordinary* point of c, that is, an interior point of a convex subarc of c, then E_{P_0} is an interior point of the arc $\varphi(c)$, and hence this arc is smaller than a semicircle. The lines joining P_0 with the endpoints of the arc $\varphi(c)$ are local supporting lines to c at points different from P_0. If c is open, these lines may pass through the endpoints of c.

2.2. The above result has been deduced under the assumption that c is linearly monotone along every line l_P. However, because of the fact observed at the end of 1.3, it is still true if, for a *finite* number of lines, we replace the assumption of linear monotonicity by the assumption that each of these lines *intersects* c in at least one point different from P_0.

Thus we have shown

THEOREM 1. *If a curve c is linearly monotone along the lines which connect an ordinary point $P_0 \in c$ with an arbitrary other point $P \in c$, with the exception of at most finitely many lines which intersect the curve outside P_0, then the curve c is contained in an angular domain with P_0 as vertex and, hence, is strictly convex at P_0.*

2.3. If P_0 is a point of inflection of c, the point E_{P_0} may be an interior point of $\varphi(c)$, and the above conclusion that c is strictly convex at P_0 then still holds. This is also the case if E_{P_0} is an endpoint of $\varphi(c)$ and the diametrically opposite point on e is not a point of orientation. However, it may happen that E_{P_0} is an endpoint of $\varphi(c)$ and the diametrically opposite point is a point of orientation. Then c is not strictly convex at P_0 since the lines l_P then cover the whole plane. This can only be the case if the line l_{P_0} has at least one point different from P_0 in common with c and is a local supporting line to c at each of these points.

These remarks show that a curve linearly monotone as a whole is strictly convex not only at the ordinary points but in general also at the points of inflection, that is, it is strictly convex as a whole.

3. Curves in R^n, $n \geqq 3$.

3.1. Let c denote an open or closed curve in R^n. We assume that c is linearly monotone along a hyperplane H, that is, there exists in H a polygon Π_{n-1} (or $\bar{\Pi}_{n-1}$) with vertices $P_0, P_1, \ldots, P_{m-1}$, $m \geqq n$, which is inscribed in c. If c is open we replace the open polygon Π_{n-1} by the corresponding closed polygon $\bar{\Pi}_{n-1}$ (§ 1.2).

We consider the projection of c from a vertex P_i onto a hyperplane H_1 different from H. The intersection $H_1 \cap H$ is an $(n-2)$-space H'. We prove the following

LEMMA. *If c is linearly monotone along H, its projection c' is linearly monotone along H'.*

To prove the lemma we shall show that the projection of $\bar{\Pi}_{n-1}$ into H' is a polygon of order $n-2$ and inscribed in c'.

Derry [2] has shown that the projection of a closed polygon $\bar{\Pi}_{n-1}$ into a hyperplane from a vertex P_i is an open polygon Π'_{n-2} whose sides are the projections of the sides of $\bar{\Pi}_{n-1}$, with the exception of $P_{i-1}P_i$ and P_iP_{i+1}. The polygon Π'_{n-2} may be closed without increasing its order by adding the projection $\sigma' = P'_{i-1}P'_{i+1}$ of that segment $\sigma = P_{i-1}P_{i+1}$

which together with the sides $P_{i-1}P_i$ and P_iP_{i+1} forms a polygon (a triangle) of *odd* order [2, p. 51].

Since a closed curve of *even* order is projected onto a closed curve of even order from a point *outside* the curve, the sides

$$P'_0P'_1,\ldots,P'_{i-2}P'_{i-1},P'_{i+1}P'_{i+2},\ldots,P'_{m-2}P'_{m-1}$$

(and, if c is closed, $P'_{m-1}P'_0$) are chords of c'. The union of the segment σ and the arc $P_{i-1}P_iP_{i+1}$ is a closed curve of *odd* order. For, a hyperplane through a point of σ which intersects the chords $P_{i-1}P_i$ and P_iP_{i+1} has an odd number of points in common with each of the corresponding arcs $P_{i-1}P_i$ and P_iP_{i+1}. Now, a closed curve of odd order is projected onto a closed curve of *even* order from a point *on* the curve, and consequently σ' is a chord of c'. This finishes the proof of the Lemma.

3.2. Now we assume that the curve c is linearly monotone along any hyperplane $H=H(P)$ which connects $n-1$ fixed points $P_0,P_1,\ldots,P_{n-2}\in c$ with a variable point $P\in c$. For any position of H the points $P_0,P_1,\ldots,P_{n-2}$ and P are vertices of a polygon Π_{n-1} (or $\bar{\Pi}_{n-1}$) and consequently linearly independent. The fixed points $P_0,P_1,\ldots,P_{n-2}$ determine a $(n-2)$-space $B=[P_0P_1\ldots P_{n-2}]$.

Let α denote a plane which has only P_0 in common with B. By the projection from the $(n-3)$-space $[P_1,P_2,\ldots,P_{n-2}]$ onto the plane α the curve c is mapped onto a curve c'. The image P' of a point P is the intersection of the $(n-2)$-space $[P_1,P_2,\ldots,P_{n-2},P]$ with α, and the line P_0P' is the intersection of $H(P)$ with α.

The projection of c onto c' may be decomposed into a sequence of projections from single points. First, by the projection from P_{n-2} onto the hyperplane $[\alpha,P_1,P_2,\ldots,P_{n-3}]$ the curve c is mapped onto a curve c_1. Then this is mapped onto a curve c_2 by the projection from P_{n-3} onto the subspace $[\alpha,P_1,P_2,\ldots,P_{n-4}]$. Continuing in this way we end up with the projection from P_1 onto α, by which a certain curve c_{n-4} in a 3-space is mapped onto $c_{n-3}=c'$.

By repeated use of the lemma it is seen that c' is linearly monotone along any line P_0P' where P' is an arbitrary point of c' which is different from P_0 and from the projections $P'_1,P'_2,\ldots,P'_{n-2}$ of the corresponding points of c. If c' is contained in an angular domain V' then c is contained in an angular domain V such that $V'=\alpha\cap V$.

To make the application of Theorem 1 possible we have to add assumptions in order that P_0 be an ordinary point of c' and that the lines $P_0P'_i$ for $i=1,2,\ldots,n-2$ intersect c' at points different from P_0. These conditions will be satisfied if we assume that the *osculating planes*

$\tau^2(P_i)$, $i = 0, 1, \ldots, n-2$, *have only the point of contact in common with the* $(n-2)$*-space* B. It is clear that P_0 will be an ordinary point of c', and since the tangent to c' at a point P'_i, $i \neq 0$, is the intersection of α and the hyperplane $[P_1 P_2 \ldots P_{n-2}, \tau^2(P_i)]$, it cannot pass through P_0, that is, the line $P_0 P'_i$ intersects c' at P'_i.

Using Theorem 1 we have then proved

THEOREM 2. *If a curve c is linearly monotone along every hyperplane which connects* $n-1$ *linearly independent points* $P_0, P_1, \ldots, P_{n-2}$ *on c with an arbitrary point P on c, and the osculating planes* $\tau^2(P_i)$, $i = 0, 1, \ldots, n-2$, *have only the point of contact in common with the* $(n-2)$*-space* $B = [P_0 P_1, \ldots, P_{n-2}]$, *then c is contained in an angular domain V bounded by two hyperplanes through B and is strictly convex at the set* $(P_0, P_1, \ldots, P_{n-2})$.

The boundary hyperplanes of V are locally supporting hyperplanes for c and pass through a tangent or, if c is an open curve, possibly through an endpoint of c.

3.3. If the $n-1$ points $P_0, P_1, \ldots, P_{n-2}$ belong to the same arc b_n of order n, then the condition for the osculating planes $\tau^2(P_i)$ is always satisfied. For, if for instance $\tau^2(P_0)$ had other points than P_0 in common with B there would exist a hyperplane through B and $\tau^2(P_0)$ having the $n-2$ points $P_1, P_2, \ldots, P_{n-2}$ and 3 coinciding points at P_0 in common with b_n. But this is impossible (see [6, p. 174]).

Hence, *if c as a whole is linearly monotone, it is strictly convex at any set of* $n-1$ *points belonging to the same arc of order n.*

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