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CONGRUENCES FOR THE COEFFICIENTS OF THE MODULAR INVARIANT $j(\tau)$

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1.

The modular invariant $j(\tau)$ is defined by

$$j(\tau) \;=\; x^{-1}\prod_{1}^{\infty}(1-x^n)^{-24}\left(1+240\sum_{1}^{\infty}\sigma_3(n)x^n\right)^3, \qquad x \;=\; e^{2\pi i\tau},$$

where

$$\sigma_k(n) \;=\; \sum_{d\,|\,n} d^k, \qquad \sigma_1(n) \;=\; \sigma(n)\;.$$

It is well known that the coefficients in the expansion

$$j(\tau) \;=\; \sum_{-1}^{\infty} c(n)x^n$$

have remarkable divisibility properties. Lehner [7], [8] has shown, $a>0$,

$$c(2^a n) \;\equiv\; 0 \pmod{2^{3a+8}}\;, \tag{1.1}$$

$$c(3^a n) \;\equiv\; 0 \pmod{3^{2a+3}}\;, \tag{1.2}$$

$$c(5^a n) \;\equiv\; 0 \pmod{5^{a+1}}\;, \tag{1.3}$$

$$c(7^a n) \;\equiv\; 0 \pmod{7^{a}}\;. \tag{1.4}$$

The congruences (1.1) and (1.2) have been improved by Kolberg [1], [2]:

$$c(2^a n) \;\equiv\; -\,2^{3a+8}\,3^{a-1}\,\sigma_7(n) \pmod{2^{3a+13}}, \qquad a \geqq 1, \qquad n \text{ odd}\;. \tag{1.5}$$

$$c(3^a n) \;\equiv\; \mp\,3^{2a+3}\,10^{a-1}\,\sigma(n)/n \pmod{3^{2a+6}} \quad \text{if } n \;\equiv\; \pm 1 \pmod{3}\;. \tag{1.6}$$

Kolberg conjectured that (1.3) and (1.4) could be sharpened in a similar way, and in this note we shall deduce the following congruence

$$c(5^a n) \;\equiv\; -\,3^{a-1}\,5^{a+1}\,n\sigma(n) \pmod{5^{a+2}}, \qquad a>0\;. \tag{1.7}$$

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Hence, especially

$$c(5^a) \not\equiv 0 \pmod{5^{a+2}},$$

a conjecture of Lehner.

We shall also give a new proof of the congruence

$$(1.8) \qquad c(n) \equiv 10n\sigma(n) \pmod{5^2}, \qquad (n/5) = -1,$$

where (n/p) is Legendre's symbol. This congruence was proved by Kolberg in [3], where several other congruences for the modular invariant can be found.

2.

The following definitions and lemmas are all taken from Kolberg [4]. We put

$$\varphi(x) = \prod_1^\infty (1-x^n),$$

$$\Phi(x) = \varphi(x)^{k_1}\varphi(x^2)^{k_2}\ldots\varphi(x^n)^{k_n}, \qquad k_j \text{ integral}.$$

Thus the symbol $\Phi(x)$ is not used to denote one particular function, its meaning will usually be different in different sections. A function of this form will be referred to as a Φ-function.

Let $\Phi(x) = \sum P(n)x^n$ be the power series expansion of $\Phi(x)$. Further let q be a given positiv integer. Then we put

$$\Phi_j = \sum P(qn+j)\, x^{qn+j} = \Phi_j\{\Phi(x)\}.$$

It follows that

$$\Phi(x) = \Phi_0 + \Phi_1 + \ldots + \Phi_{q-1};$$

we shall refer to this as the q-dissection of $\Phi(x)$. We define

$$D = \begin{vmatrix} \Phi_0 & \Phi_1 & \ldots & \Phi_{q-1} \\ \Phi_{q-1} & \Phi_0 & \ldots & \Phi_{q-2} \\ \ldots & \ldots & \ldots & \ldots \\ \Phi_1 & \Phi_2 & \ldots & \Phi_0 \end{vmatrix}, \qquad \Delta_j = \begin{vmatrix} \Phi_{-j} & \Phi_{-j+1} & \ldots & \Phi_{-j+q-2} \\ \Phi_{-j-1} & \Phi_{-j} & \ldots & \Phi_{-j+q-3} \\ \ldots & \ldots & \ldots & \ldots \\ \Phi_{-j-q+2} & \Phi_{-j-q+3} & \ldots & \Phi_{-j} \end{vmatrix},$$

Thus Δ_j is the complement of Φ_{-j} in the circulant D.

LEMMA. *Let q be a prime. Further, in the expression for $\Phi(x)$ let $k_j = 0$ whenever $q \mid j$. Then we have*

$$(2.1) \qquad D = \frac{\Phi(x^q)^{q+1}}{\Phi(x^{q^2})}.$$

LEMMA. *Let $\Phi(x)^{-1} = \Phi_0' + \Phi_1' + \ldots + \Phi'_{q-1}$ be the q-dissection of $\Phi(x)^{-1}$, where $\Phi(x)$ is an arbitrary Φ-function. Then we have*

$$\Phi_j' = (-1)^{(q-1)j} D^{-1} \Delta_j . \tag{2.2}$$

We shall use 5-dissection on $\varphi(x)$. From [5] we have

$$\varphi(x) = \varphi_0 + \varphi_1 + \ldots + \varphi_4 ,$$

$$\varphi_3 = \varphi_4 = 0, \qquad \varphi_1 = -x\varphi(x^{25}), \qquad \varphi_0\varphi_2 = -\varphi_1^2 , \tag{2.3}$$

which follows from the well-known identities

$$\varphi(x) = \sum_{-\infty}^{\infty} (-1)^n x^{1/2n(3n+1)} \qquad \text{(Euler)} ,$$

$$\varphi(x)^3 = \sum_{-\infty}^{\infty} (4n+1) x^{n(2n+1)}$$

$$= \sum_{0}^{\infty} (-1)^n (2n+1) x^{\frac{1}{2}n(n+1)} \qquad \text{(Jacobi)} .$$

In [5] Kolberg also proved

$$\varphi_0^5 + \varphi_2^5 = \varphi(x^5)^6 \varphi(x^{25})^{-1} + 11x^5\varphi(x^{25})^5 . \tag{2.4}$$

Elimination of φ_0 and φ_2 between (2.3) and (2.4) gives

$$x\varphi(x^5)^6\varphi(x)^{-6} = V + 5V^2 + 15V^3 + 25V^4 + 25V^5 , \tag{2.5}$$

where $V = x\varphi(x)^{-1}\varphi(x^{25})$. For the function $\varphi(x)^3$ we have

$$\varphi(x)^3 = (\varphi_0 + \varphi_1 + \varphi_2)^3 = \Phi_0 + \Phi_1 + \Phi_3 .$$

$$\Phi_0\Phi_1 = -3x\varphi(x^5)^6 - 25x^6\varphi(x^{25})^6, \qquad \Phi_3 = +5x^3\varphi(x^{25})^3 . \tag{2.6}$$

$$\Phi_0^5 + \Phi_1^5 = \varphi(x^{25})^{-3}\varphi(x^5)^{18} - 9\cdot 5^2 x^5 \varphi(x^{25})^3 \varphi(x)^{12} - \tag{2.7}$$
$$- 9\cdot 5^4 x^{10}\varphi(x^{25})^9\varphi(x^5)^6 - 11\cdot 5^5 x^{15}\varphi(x^{25})^{15} .$$

Later on, when we write Φ_0, Φ_1, Φ_3 it will always refer to the 5-dissection of $\varphi(x)^3$. We find it convenient to use the notation

$$\varphi(x)^{6k} = \Phi_{0k} + \Phi_{1k} + \ldots + \Phi_{4k}, \qquad \Phi_{jk} = \Phi_j\{\varphi(x)^{6k}\} .$$

3.

Our starting point is the following lemma (see Kolberg [6]): Let p be one of the primes 2, 3, 5, 7, 13 and put

$$\Phi_p(\tau) = x\big(\varphi(x^p)/\varphi(x)\big)^{24/(p-1)} .$$

Then there exist constants A_{kp} such that

$$j(\tau) = \sum_{k=-1}^{p} A_{kp} \Phi_p(\tau)^k .$$

From this lemma we easily get the identity

$$(3.1) \qquad j(\tau) = f^{-1} + 6\cdot 5^3 + 63\cdot 5^5 f + 52\cdot 5^8 f^2 + 63\cdot 5^{10} f^3 + 6\cdot 5^{13} f^4 + 5^{15} f^5 ,$$

where

$$f = \Phi_5(\tau) = x\varphi(x^5)^6 \varphi(x)^{-6} .$$

We define the operator L by

$$L \sum a_n x^n = \sum a_{5n} x^n .$$

From (2.6) we get

$$(3.2) \qquad Lf^{-1} = 6 - 5^2 f .$$

If q is a prime, we have the obvious congruence

$$\varphi(x)^q \equiv \varphi(x^q) \pmod{q} ,$$

and using this and the well-known congruence (see [3])

$$(3.3) \qquad \sum_1^\infty n\sigma(n) x^n \equiv x \prod_1^\infty (1-x^n)^{24} \pmod{5} ,$$

we obtain from (3.1) and (3.2)

$$(3.4) \qquad \sum_1^\infty c(5n) x^n \equiv -5^2 \sum_1^\infty n\sigma(n) x^n \pmod{5^3} .$$

The congruences refer to the coefficients of the power series in x. The last congruence (3.4) proves (1.7) for $a=1$.

To complete the proof of (1.7) it remains to show

$$(3.5) \qquad c(5^a n) \equiv 3^{a-1} 5^{a-1} c(5n) \pmod{5^{a+2}} .$$

To do so we shall obtain a formula for Lf^k, $k>0$. By direct computation we find, using the results in section 2,

$$\begin{aligned} \Phi_0\{f\} &= x\varphi(x^{25})^6 \varphi(x^5)^{-30} \Delta_{-1} \\ &= 63\cdot 5x^5 R^6 S^{-6} + 52\cdot 5^4 x^{10} R^{12} S^{-12} + 63\cdot 5^6 x^{15} R^{18} S^{-18} + \\ &\quad + 6\cdot 5^9 x^{20} R^{24} S^{-24} + 5^{11} x^{25} R^{30} S^{-30} , \end{aligned}$$

where

$$R = \varphi(x^{25}), \qquad S = \varphi(x^5) .$$

Hence

$$(3.6) \qquad Lf = 63\cdot 5f + 52\cdot 5^4 f^2 + 63\cdot 5^6 f^3 + 6\cdot 5^9 f^4 + 5^{11} f^5 .$$

Next, we will prove the following expressions for Lf^k, $k=2,3,4$:

$$(3.7)\qquad Lf^2 = \sum_1^{10} 5^l a f^l, \qquad Lf^3 = \sum_1^{15} 5^{l-1} a f^l, \qquad Lf^4 = \sum_1^{20} 5^{l-1} a f^l .$$

Here, and in the following a denotes an unspecified integer. From (2.2) we get

$$\Phi_0\{f^2\} = x^2 R^{12} S^{-60} \Delta_{-2} ,$$

where

$$\begin{aligned}\Delta_{-2} = {} & \Phi_{22}{}^4 - 3\Phi_{22}{}^2(\Phi_{12}\Phi_{32} + \Phi_{02}\Phi_{42}) - \Phi_{02}\Phi_{12}\Phi_{32}\Phi_{42} + \\ & + \Phi_{12}{}^2\Phi_{32}{}^2 + \Phi_{02}{}^2\Phi_{42}{}^2 + \\ & + 2\Phi_{22}(\Phi_{02}\Phi_{32}{}^2 + \Phi_{12}{}^2\Phi_{42} + \Phi_{32}\Phi_{42}{}^2 + \Phi_{02}{}^2\Phi_{12}) - \\ & - (\Phi_{12}\Phi_{42}{}^3 + \Phi_{02}\Phi_{12}{}^3 + \Phi_{32}\Phi_{02}{}^3 + \Phi_{42}\Phi_{32}{}^3) ,\end{aligned}$$

and where

$$\begin{aligned}\Phi_{02} &= \Phi_0{}^4 + 4\Phi_3{}^3\Phi_1 + 12\Phi_0\Phi_1{}^2\Phi_3, & \Phi_{32} &= 6\Phi_1{}^2\Phi_3{}^2 + 4\Phi_1{}^3\Phi_0 + 4\Phi_0{}^3\Phi_3 ,\\ \Phi_{12} &= 6\Phi_0{}^2\Phi_3{}^2 + 4\Phi_0{}^3\Phi_1 + 4\Phi_1{}^3\Phi_3, & \Phi_{42} &= \Phi_1{}^4 + 4\Phi_3{}^3\Phi_0 + 12\Phi_0{}^2\Phi_1\Phi_3 ,\\ \Phi_{22} &= 6\Phi_0{}^2\Phi_1{}^2 + 12\Phi_0\Phi_1\Phi_3{}^2 + \Phi_3{}^4 .\end{aligned}$$

We obtain

$$\Delta_{-2} = \sum_{\substack{t,u\geqq 0\\ 5u+2t\leqq 16}} a\Phi_0{}^t\Phi_1{}^t\Phi_3{}^{16-5u-2t}(\Phi_0{}^5 + \Phi_1{}^5)^u .$$

(2.7) and (2.8) yields

$$\Phi_0{}^t\Phi_1{}^t = \sum_{l=0}^{t} a 5^{2t-2l} x^{6t-5l} R^{6t-6l} S^{6l}, \qquad \Phi_3 = 5x^3R^3 ,$$

$$(\Phi_0{}^5 + \Phi_1{}^5)^u = R^{-3u}S^{18u} + a\cdot 5^2 x^5 R^{6-3u} S^{18u-6} + \sum_{v=2}^{3u} a 5^{v+2} x^{5v} R^{-3u+6v} S^{18u-6v} .$$

Hence

$$\begin{aligned}\Phi_0\{f^2\} = {} & \sum_{\substack{t,u\geqq 0\\ 5u+2t\leqq 16}} a\cdot 5^{16-5u-2t} x^{50-15u-5t} R^{60-18u-6t} S^{-60+18u+6t} + \\ & + \sum_{\substack{t\geqq 0\\ u>0\\ 5u+2t\leqq 16}} a\cdot 5^{18-5u-2t} x^{55-15u-5t} R^{66-18u-6t} S^{-66+18u+6t} + \\ & + \sum_{\substack{t\geqq 0\\ u>0\\ 5u+2t\leqq 16\\ 3u\geqq v\geqq 2}} a\cdot 5^{18-5u-2t+v} x^{50-15u-5t+5v} R^{60-18u-6t+6v} S^{-60+18u+6t-6v} .\end{aligned}$$

We have thus got a polynomial in $f_1=f(x^5)$ of degree $\leqq 10$.

It remains to consider the exponent of 5. We see at once that it is sufficient to consider the first of the three sums in the last expression for $\Phi_0\{f^2\}$. Putting

$$u = 0,\ t = 8 \quad \text{we get} \quad af_1^2\,,$$
$$u = 0,\ t = 7 \quad - \quad - \quad 5^2 af_1^3\,,$$
$$u = 0,\ t = 6 \quad - \quad - \quad 5^4 af_1^4\,.$$

Hence

$$Lf^2 = \sum_{l=1}^{3} af^l + \sum_{4}^{10} 5^l af^l\,. \tag{3.8}$$

We easily find

$$\begin{cases} LV = 5f\,, \\ LV^2 = 2\cdot 5f + 5^3 f^2\,, \\ LV^3 = 9f + 3\cdot 5^3 f^2 + 5^5 f^3\,, \\ LV^4 = 4f + 22\cdot 5^2 f^2 + 4\cdot 5^5 f^3 + 5^7 f^4\,, \\ LV^5 = f + 20\cdot 5^2 f^2 + 40\cdot 5^4 f^3 + 5\cdot 5^7 f^4 + 5^9 f^5\,, \\ LV^6 = 63\cdot 5f^2 + 52\cdot 5^4 f^3 + 63\cdot 5^6 f^4 + 6\cdot 5^9 f^5 + 5^{11} f^6\,. \end{cases} \tag{3.9}$$

Comparing now (2.5), (3.8) and (3.9) we obtain

$$Lf^2 = \sum_{l=1}^{10} 5^l af^l\,.$$

By the same method we find

$$Lf^3 = \sum_{1}^{4} af^l + \sum_{5}^{15} 5^{l-1} af^l\,. \tag{3.10}$$

And again from (2.5), (3.9) and (3.10)

$$Lf^3 = \sum_{l=1}^{15} 5^{l-1} af^l\,.$$

Similarly we find

$$Lf^4 = \sum_{l=1}^{6} a_l f^l + \sum_{7}^{20} 5^{l-1} af^l\,. \tag{3.11}$$

The simplest way to investigate the six first coefficients in this case is to use the power series expansion of (3.11). Using Watson's table [10] of $\tau(n)$, and Newman's table [9] of $\eta(\tau)$, we find

$$Lf^4 = \sum_{l=1}^{20} 5^{l-1} af^l\,.$$

We have thus proved (3.7).

From Lehner [7] we have

$$\begin{aligned} f^5 = {} & f_1 + 5f(6f_1 + 5^2 f_1^2) + 5f^2(63 f_1 + 6\cdot 5^3 f_1^2 + 5^5 f_1^3) + \\ & + 5^2 f^3(52 f_1 + 63\cdot 5^2 f_1^2 + 6\cdot 5^5 f_1^3 + 5^7 f_1^4) + \\ & + 5^2 f^4(63 f_1 + 52\cdot 5^3 f_1^2 + 63\cdot 5^5 f_1^3 + 6\cdot 5^8 f_1^4 + 5^{10} f_1^5)\,. \end{aligned} \tag{3.12}$$

Noticing that $Lf^k f_1^s = f^s Lf^k$ we obtain from (3.6), (3.7) and (3.12)

$$Lf^5 = \sum_{l=1}^{25} 5^{l-1} a f^l . \tag{3.13}$$

By means of (3.6), (3.7), (3.12) and (3.13) we readily get the following general expressions for Lf^k, $k>0$,

$$\begin{cases} Lf^{5(k-1)+r} = \displaystyle\sum_{l=k}^{25(k-1)+5r} 5^{l+1-k} a f^l, & r = 1,2 , \\ Lf^{5(k-1)+p} = \displaystyle\sum_{l=k}^{25(k-1)+5p} 5^{l-k} a f^l, & p = 3,4,5 . \end{cases} \tag{3.14}$$

Defining

$$T = \sum_{k=1}^{t} 5^k a f^k ,$$

(3.14) implies

$$LT = 5T . \tag{3.15}$$

We obtain by (3.1) and (3.2)

$$Lj = 744 - 5^2 f + 5^3 T .$$

(3.6) and (3.15) yields

$$L^2 j = 744 - (63)5^3 f + 5^4 T ,$$

and generally

$$L^a j = 744 - (63)^{a-1} 5^{a+1} f + 5^{a+2} T ,$$

which implies (3.5).

4.

Proof of (1.8). We put

$$F = x\varphi(x)^{24} .$$

5-dissection on (3.1) yields

$$\sum c(5n+2)x^{5n+2} + \sum c(5n+3)x^{5n+3} \equiv \frac{1}{x}\varphi(x^5)^{-6}\big(\Phi_3\{\varphi(x)^6\} + \Phi_4\{\varphi(x)^6\}\big) \pmod{5^2} . \tag{4.1}$$

We easily get

$$\begin{aligned} \Phi_3\{\varphi(x)^6\} + \Phi_4\{\varphi(x)^6\} &= -10\varphi_1^3(\varphi_0^3+\varphi_2^3) - 30\varphi_1^4(\varphi_0^2+\varphi_2^2) \\ &\equiv 10x^3\varphi(x)^3\varphi(x^{25})^3 \equiv 10x^3\varphi(x)^{78} \pmod{5^2} . \end{aligned} \tag{4.2}$$

(4.1) and (4.2) give

$$\sum c(5n+2)x^{5n+2} + \sum c(5n+3)x^{5n+3} \equiv 10F^2 \pmod{5^2} . \tag{4.3}$$

We have

$$x\varphi(x^5)^4\varphi(x)^4 \equiv x\varphi(x)^{24} \pmod 5 .$$

5-dissection on (3.4) yields

$$\sum c(25n+5)x^{5n+1}+\sum c(25n+20)x^{5n+4} \equiv 10^2x\varphi(x^5)^4\big(\Phi_0\{\varphi(x)^4\}+\Phi_3\{\varphi(x)^4\}\big) \pmod{5^3}, \tag{4.4}$$

and

$$\Phi_0\{\varphi(x)^4\}+\Phi_3\{\varphi(x)^4\} = {\varphi_0}^4+{\varphi_2}^4-8{\varphi_1}^3(\varphi_0+\varphi_2) \equiv \varphi(x)^4-x\varphi(x)^{28} \pmod 5 . \tag{4.5}$$

Hence

$$\sum c(25n+5)x^{5n+1}+\sum c(25n+20)x^{5n+4} \equiv 10^2F-10^2F^2 \pmod{5^3} . \tag{4.6}$$

(4.6) gives the following congruences:

$$\Phi_2\{F\} \equiv \Phi_2\{F^2\}, \quad \Phi_3\{F\} \equiv \Phi_3\{F^2\} \pmod 5 . \tag{4.7}$$

(3.3), (4.3) and (4.7) imply (1.8).

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