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ON THE IDEAL STRUCTURE OF CERTAIN BANACH ALGEBRAS

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Introduction.

Let A be a commutative Banach algebra over the complex numbers. In order to characterize the closed ideals in A it is natural to use in the first place the class of all regular maximal ideals by forming for every closed ideal the class of regular maximal ideals which contain it. As is well-known, every regular maximal ideal in A is a closed ideal of co-dimension 1.

In cases where the class of regular maximal ideals is too small to give a satisfactory description of the ideal structure, it is convenient to use a larger class of comparison ideals. It is then natural to choose the class of all closed ideals which have a finite co-dimension. We can then state the following two problems, which correspond to the problems of "spectral analysis" and "spectral syntheses", raised in various contexts in harmonic analysis.

PROBLEM 1: *Let I be a closed ideal in A, and let m be a positive integer, smaller than the co-dimension of I (which may be infinite). Is it then true that there exists a closed ideal I_m of co-dimension m and which contains I?*

PROBLEM 2: *Is every closed ideal I the intersection of all closed ideals of finite co-dimension, which contain I?*

These problems have been discussed in [2] for a special class of Banach algebras. We shall in this paper mainly treat Problem 1 and make the discussion more general than in [2]. Thus we shall obtain n-dimensional analogues of certain results in [2].

After some preliminaries we prove in § 2 our main theorem (Theorem 2.1), which gives an affirmative answer to Problem 1, assuming certain rather implicit conditions on the algebra. We then exhibit two particular classes of Banach algebras where it is possible to verify that the conditions are fulfilled. The first class is studied in § 3. In § 4 we discuss the second class which is more complicated but also more interesting since

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it gives the abovementioned generalizations of results in [2] (Theorems 4.1 and 4.2). § 4 concludes with a remark which shows that the answers to Problems 1 and 2 are not always affirmative.

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1. Preliminaries.

We start by proving a simple lemma of algebraic nature.

LEMMA 1.1. *Let I_1 and I_2 be two different ideals in A such that $I_2 \subset I_1$ and such that no ideal lies properly between I_1 and I_2. Then I_1/I_2 has the dimension* 1.

PROOF. It is easy to see that we can restrict ourselves to the case when A has an identity. We apply corollary 1 on p. 237 of Zarisky and Samuel [4], which states that there exist a maximal ideal I_0 and an element $x \in I_1$ such that $I_0 \cdot I_1 \subset I_2$ and $I_1 = I_2 + x \cdot A$ (with the notations of [4]). Since we have an identity every maximal ideal has the co-dimension 1. I_1 is thus the linear space spanned by x and $I_2 + xI_0$. But $xI_0 \subset I_0 \cdot I_1 \subset I_2$, and hence I_1 is spanned by x and I_2. This proves the lemma.

If I_2 is closed and has a finite co-dimension, then all ideals, containing I_2, are closed. Hence Lemma 1.1 gives the following equivalent formulation of Problem 1.

PROBLEM 1′: *Is it true that every closed ideal of infinite co-dimension is contained in closed ideals of arbitrarily large finite co-dimension?*

Before proceeding we shall prove another very simple lemma:

LEMMA 1.2. *Suppose that A has a unit and a finite number of generators. Then A has exactly one maximal ideal if and only if there exist elements $\alpha_1, \ldots, \alpha_n$ in A such that*

$$(1.1) \qquad \|\alpha_i^m\|^{1/m} \to 0 \quad as \ m \to \infty ,$$

and such that $e, \alpha_1, \alpha_2, \ldots, \alpha_m$ are generators.

PROOF. Suppose first that A has only one maximal ideal I_0. Let h denote the corresponding complex-valued homomorphism. We have for every $a \in A$ that $a - h(a)e \in I_0$. Let $a_1, a_2, \ldots, a_n$ be generators. Put $\alpha_i = a_i - h(a_i)e$. Then $e, \alpha_1, \ldots, \alpha_n$ are generators. Since $h(\alpha_i) = 0$, the representations of α_i on the space of maximal ideals $\bar{\varphi}$ are identically vanishing. By a fundamental theorem of Gelfand this implies that (1.1) holds.

Conversely, let us assume that $\alpha_1, \alpha_2, \ldots, \alpha_n$ have the announced properties. Let h be a complex-valued homomorphism. Then we have for every i

$$|h(\alpha_i)^m| = |h(\alpha_i^m)| \leqq \|\alpha_i^m\| .$$

Hence

$$|h(\alpha_i)| \leqq \|\alpha_i^m\|^{1/m} ,$$

and (1.1) gives $h(\alpha_i)=0$. We put

$$\alpha^p = \alpha_1^{p_1} \ldots \alpha_n^{p_n}, \qquad p = (p_1, \ldots, p_n) ,$$

where p_i are non-negative integers. With $|p| = p_1 + \ldots + p_n$, the multiplicativity of h gives

$$h(\alpha^p) = 0, \quad \text{if} \quad |p| \geqq 0 .$$

If h is not identically vanishing we must have

$$h(\alpha^0) = h(e) = 1 .$$

The linear closure of the set of elements α^p is dense in A. A bounded linear functional is therefore determined by its values on the elements α^p. Thus we have at most one non-trivial complex-valued homomorphism. By a fundamental theorem in the theory of Banach algebras, we have at least one homomorphism of this kind.

We assume from now on that A is a commutative Banach algebra with a unit, finitely many generators and exactly one maximal ideal. By Lemma 1.2 we can assume that the generators are $e, \alpha_1, \ldots, \alpha_n$, where (1.1) holds.

B denotes the dual space of A. As mentioned in the proof of Lemma 1.2, its elements b are completely characterized by the complex numbers

$$b_p = b(\alpha^p) .$$

As is well known, it is possible to introduce an operation between elements in A and elements in B, corresponding to an interpretation of B as a module over A. For every $a_0 \in A$ and $b \in B$ we denote by $a_0 \circ b$ the linear functional with the value $b(a_0 a)$ for every $a \in A$. It is easy to see that the operation is bilinear and satisfies

$$(a_1 a_2) \circ b = a_1 \circ (a_2 \circ b)$$

for any elements a_1 and a_2 in A and $b \in B$.

Let the norm in A satisfy

$$\|a_1 a_2\| \leqq C \|a_1\| \, \|a_2\| ,$$

for any a_1 and a_2 in A, where C is a constant, and let $\| \|^*$ denote the norm in B. Then, if a and $a_0 \in A$ and $b \in B$

$$|(a_0 \circ b)(a)| = |b(a_0 a)| \leqq \|b\|^* \|aa_0\| \leqq C \|b\|^* \|a\| \|a_0\|$$

and hence $a_0 \circ b$ is a bounded linear functional and

$$\|a_0 \circ b\|^* \leqq C \|a_0\| \|b\|^* . \tag{1.2}$$

The operations $\alpha^{p_0} \circ b$, where p_0 is an arbitrary multi-index, are of particular interest. We have

$$(\alpha^{p_0} \circ b)_p = (\alpha^{p_0} \circ b)(\alpha^p) = b(\alpha^{p+p_0}) = b_{p+p_0} ,$$

which makes it convenient to call $\alpha^{p_0} \circ b$ a translation of b.

An element $b \in B$ is said to have the degree q if $b_p = 0$, whenever $|p| > q$, and if q is the smallest non-negative integer with this property. A closed proper linear subspace of A is said to have the co-degree q if its annihilator subspace of B only contains elements of degree $\leqq q$ and at least one element with the degree q. Infinite degrees are defined in a similar way.

LEMMA 1.3. *The co-degree of a closed proper ideal I of A is finite if and only if the co-dimension of I is finite.*

PROOF. If the co-degree is finite $= q$, then the annihilator subspace contains only elements of degree $\leqq q$ and these form a finite-dimensional subspace.

Conversely, let I be a closed ideal with a finite co-dimension k. Let b be any element in the annihilator subspace. Since $b(a) = 0$ for every $a \in I$, we have $(a_0 \circ b)(a) = 0$ for every a and $a_0 \in I$. Thus $a_0 \circ b$ belongs to the annihilator subspace. In particular, every translate $\alpha^{p_0} \circ b$ belongs to the annihilator subspace.

Since this space has the dimension k, we have coefficients $c_0, c_1, \ldots, c_k$, not all vanishing, such that

$$\sum_{\nu=0}^{k} c_\nu {\alpha_1}^\nu \circ b = 0 .$$

Hence

$$\sum_{\nu=0}^{k} c_\nu ({\alpha_1}^\nu \circ b)({\alpha_1}^m) = \sum_{\nu=0}^{k} c_\nu b({\alpha_1}^{m+\nu}) = 0 , \tag{1.3}$$

if $m = 0, 1, 2, \ldots$. Using (1.1) and (1.2) it is easy to see that

$$|b({\alpha_1}^l)|^{1/l} \to 0 \quad \text{as } l \to \infty . \tag{1.4}$$

The elementary theory of difference equations shows that a solution of an equation of type (1.3), subject to the condition (1.4), has to satisfy the condition

$$b(\alpha_1^{l}) = 0, \quad \text{if} \quad l > k-1 .$$

We now exchange b and α_1 in the discussion above to $\alpha_1^{l_1} \circ b$ and α_2, respectively, where l_1 is arbitrary $\geqq 0$. We then obtain

$$b(\alpha_1^{l_1}\alpha_2^{l_2}) = 0, \quad \text{if} \quad l_1 \text{ or } l_2 > k-1 .$$

Continuing in this way we finally obtain that

$$b(\alpha_1^{l_1}\alpha_2^{l_2}\ldots\alpha_n^{l_n}) = 0, \qquad \text{if some } l_\nu > k-1 .$$

This shows that

$$b_p = b(\alpha^p) = 0, \quad \text{if} \quad |p| > n(k-1) ,$$

and hence the co-degree of the ideal is $\leqq n(k-1)$, i.e. finite. (It is easy to show that the co-degree is, in fact, $\leqq k-1$.)

Lemma 1.3 shows that we have under our assumptions another equivalent formulation of Problem 1 and 1′:

PROBLEM 1″: *Is it true that every closed ideal of infinite co-degree is included in closed ideals of arbitrarily large finite co-degree?*

2. The main theorem.

We shall give a supplementary condition on A which suffices to give a positive answer to Problems 1, 1′ and 1″.

Let us first mention that A can be obtained by a completion of the algebra of complex polynomials in $e, \alpha_1, \ldots, \alpha_n$ in the uniform structure which is determined by the norm. We shall say that a Banach algebra A° is larger than A if it can be interpreted as a similar completion, but using a smaller norm. The dual space B° of A° can then in an obvious way be interpreted as a subspace of B equipped with a larger norm. For, if $b^\circ \in B^\circ$, then $b^\circ(a)$ is well-defined for all $a \in A$ and

$$\sup_{a \,|\, \|a\| \leqq 1} |b^\circ(a)| \leqq \sup_{a^\circ \,|\, \|a^\circ\|^\circ \leqq 1} |b^\circ(a^\circ)| = \|b^\circ\|^{\circ *} < \infty ,$$

where $\| \|^\circ$ and $\| \|^{\circ *}$ denote the norms in A° and B°. Hence b° is a bounded linear functional on A.

SUPPLEMENTARY ASSUMPTION: *Suppose that for every finite integer $q \geqq 0$ and for every closed ideal of infinite co-degree there exists an element β in its annihilator subspace, a larger Banach algebra A° with dual space B° and a constant C, such that*

1° *the degree of* β *is* $\geqq q$,

2° $\beta \in B^\circ$,

3° *for every linear combination* b *of translates of* β,

$$\|b+b'\|^* \leqq C \sum_{p_0 \mid |p_0|=q} \|\alpha^{p_0} \circ b\|^{\circ *}$$

for some $b' \in B$ *of degree* $\leqq q-1$.

We shall make a comment concerning this assumption. It follows from (1.2) that there exists, for every non-negative integer q, a constant C_q such that

$$\sum_{|p_0|=q} \|\alpha^{p_0} \circ b\|^* \leqq C_q \|b\|^* , \tag{2.1}$$

if $b \in B$. The converse inequality is, however, not true in non-trivial situations. The inequality in 3° can be interpreted as a modified converse which is true in certain interesting cases.

THEOREM 2.1. *If the supplementary assumption is fulfilled, then the answer to Problem* 1 *is affirmative.*

PROOF. If an element $b \in B$ has a finite degree q, then we can form the finite-dimensional subspace spanned by the elements $\alpha^p \circ b$. It is easy to see that the subspace of A, consisting of all a such that

$$(\alpha^p \circ b)(a) = 0$$

for every p, is a closed ideal of co-degree q. Hence, using the formulation in Problem 1″, it is enough to prove that, for any closed ideal I of infinite co-degree and for any finite $q \geqq 0$, the annihilator subspace of I contains an element of degree q.

Let q be arbitrary and finite, and let I be a closed ideal of infinite co-degree. Choose β, A°, B° and C as in the supplementary assumption. The norm of A° is smaller than the norm of A. Hence by (1.1),

$$(\|\alpha_i^m\|^\circ)^{1/m} \to 0 \quad \text{as } m \to \infty . \tag{2.2}$$

Since A° is a Banach algebra, we obtain from (2.2)

$$\left(\sup_{|p|=m} \|\alpha^p\|^\circ\right)^{1/m} \to 0 \quad \text{as } m \to \infty ,$$

and hence by the analogue of (1.2),

$$\left(\sup_{|p|=m} \|\alpha^p \circ \beta\|^{\circ *}\right)^{1/m} \to 0 \quad \text{as } m \to \infty .$$

Since the degree of β is $\geqq q$ there exists for every $\delta > 0$ an integer $m_0 \geqq q$ such that

$$\sup_{|p|=m_0+1} \|\alpha^p \circ \beta\|^{0*} \leqq \delta \sup_{|p|=m_0} \|\alpha^p \circ \beta\|^{0*} \neq 0 . \tag{2.3}$$

The element β is then of degree $\geqq m_0 \geqq q$. We can assume that $m_0 = q$, for otherwise we can exchange β to a suitable $\alpha^r \circ \beta$, $|r| = m_0 - q$, and this can be done in such a way that this new element satisfies (2.3) with m_0 exchanged to q. Furthermore $\alpha^r \circ \beta$ belongs to the annihilator subspace and fulfils all the conditions which were laid on β in the supplementary assumption.

We can also assume that

$$\sup_{|p|=q} \|\alpha^p \circ \beta\|^{0*} = 1 , \tag{2.4}$$

hence, by (2.3),

$$\sup_{|p|=q+1} \|\alpha^p \circ \beta\|^{0*} \leqq \delta . \tag{2.5}$$

Suppose that $p = p_0$ realizes the supremum in (2.4). We use the definition of the norm in B° and the fact that the elements α^p span a dense linear subspace of A°. This shows that there exists a linear combination a of elements α^p such that

$$\|a\|^0 = 1 \tag{2.6}$$

and

$$|(\alpha^{p_0} \circ \beta)(a)| \geqq \tfrac{1}{2} \|\alpha^{p_0} \circ \beta\|^{0*} = \tfrac{1}{2} . \tag{2.7}$$

We finally form the element

$$b_\delta = a \circ \beta \in B^\circ .$$

(2.4), (2.5), (2.6) and (2.7) give

$$\|\alpha^p \circ b_\delta\|^{0*} \leqq 1, \quad \text{when } |p| = q \tag{2.8}$$

$$\|\alpha^p \circ b_\delta\|^{0*} \leqq \delta, \quad \text{when } |p| = q+1 \tag{2.9}$$

$$|b_\delta(\alpha^{p_0})| \geqq \tfrac{1}{2} . \tag{2.10}$$

From (2.8) and the supplementary assumption, we see that there exists for every $\delta > 0$ an element $b_\delta \in B$ of degree $\leqq q-1$, such that

$$\|b_\delta + b_\delta'\|^*$$

is bounded, considered as a function of δ. Now we take, for every $\delta > 0$, the infimum $K(\delta)$ of

$$\|b_\delta + c\|^* ,$$

taken over the subspace of all elements c of B of degree $\leqq q-1$ and which are *included in the annihilator subspace* of the given ideal. Since these elements form a finite-dimensional subspace, there exists for every $\delta > 0$, an element b_δ'' in the subspace such that

$$K(\delta) = \|b_\delta + b_\delta''\|^* .$$

We shall show that $K(\delta)$ is bounded as $\delta \to 0$. If this were not true, we would have a sequence $\{\delta_m\}_1^\infty$, tending to 0 as $m \to \infty$, and such that $K(\delta_m) \to \infty$ as $m \to \infty$. The element

$$\beta_m = \frac{1}{K(\delta_m)} (b_{\delta_m} + b''_{\delta_m})$$

belongs to the annihilator subspace and has the norm 1. We write

$$\beta_m = \frac{1}{K(\delta_m)} (b_{\delta_m} + b'_{\delta_m}) + \frac{1}{K(\delta_m)} (b''_{\delta_m} - b'_{\delta_m}) .$$

Hence β_m is the sum of an element, the norm of which tends to 0, and an element of degree $\leqq q-1$, the norm of which tends to 1. The second element lies in a finite-dimensional subspace, and hence we can extract a subsequence $\{m_\nu\}_1^\infty$ of $\{m\}_1^\infty$, such that β_{m_ν} converges strongly to an element $\beta \in B$, which then naturally is of degree $\leqq q-1$ and norm $=1$. β belongs to the annihilator subspace since this subspace is closed. We now have

$$\left\| \frac{1}{K(\delta_{m_\nu})} (b_{\delta_{m_\nu}} + b''_{\delta_{m_\nu}}) - \beta \right\|^* \to 0 \quad \text{as } m_\nu \to \infty ,$$

hence, if m_ν is sufficiently large,

$$\left\| b_{\delta_{m_\nu}} + \left(b''_{\delta_{m_\nu}} - K(\delta_{m_\nu})\beta\right) \right\|^* < K(\delta_{m_\nu}) .$$

But since $b''_{m_\nu} - K(\delta_{m_\nu})\beta$ is of degree $\leqq q-1$ and contained in the annihilator subspace, this inequality contradicts the definition of $K(\delta_{m_\nu})$.

Thus $\|b_\delta + b_\delta''\|^*$ is bounded, and we can therefore choose a sequence $\{\delta_\nu\}_1^\infty$, where $\delta_\nu \to 0$ as $\nu \to \infty$, such that

$$b_{\delta_\nu}(\alpha^p) + b''_{\delta_\nu}(\alpha^p)$$

converges for every p. Since the elements α^p span a dense subspace, this means that $\{b_{\delta_\nu} + b''_{\delta_\nu}\}$ is weakly convergent. The limit element b is in the annihilator subspace. We shall show that b has the degree q.

The coefficients of $b_{\delta_\nu} + b''_{\delta_\nu}$ coincide, if $|p| \geqq q$, with the coefficients of b_{δ_ν}. By (2.10) the coefficient with index p_0 of b_{δ_ν} has an absolute value $\geqq \frac{1}{2}$. Hence the same is true of b. Thus b has a degree $\geqq q$. It remains to show that all the coefficients of b with $|p| > q$ vanish. Take any such

p and split it in the form $p = p_1 + p_2$, where $|p_1| = q+1$. By (2.9) and the B° analogue of (1.2) we obtain

$$\begin{aligned} |b_{\delta_\nu} + b''_{\delta_\nu})(\alpha^p)| &= |b_{\delta_\nu}(\alpha^p)| \\ &= |(\alpha^{p_1} \circ b_{\delta_\nu})(\alpha^{p_2})| \\ &\leqq \text{const.} \cdot \|\alpha^{p_1} \circ b_{\delta_\nu}\|^{0*} \leqq \text{const.} \cdot \delta_\nu , \end{aligned}$$

which tends to 0 as $\delta_\nu \to 0$. Hence $b_p = 0$, if $|p| > q$, and the theorem is proved.

3. First application.

Suppose that the positive numbers $S_i(m)$, $i = 1, 2, \ldots, n$; $m = 0, 1, \ldots$, are defined in such a way that $\log S_i(m)$ is convex and monotonically increasing in m, and satisfies

$$\lim_{m \to \infty} \frac{\log S_i(m)}{m} = \infty . \tag{3.1}$$

We put

$$S(p) = S(p_1, p_2, \ldots, p_n) = \prod_{i=1}^{n} S_i(p_i) .$$

Let A be the Banach space of all multi-sequences $a = \{a_p\}$ of complex numbers with the norm

$$\|a\| = \textstyle\sum_p |a_p|/S(p) ,$$

where the sum is taken over all non-negative multi-indices p.

We define for any pair $a' = \{a_p'\}$ and $a'' = \{a_p''\}$ in A, $a'a''$ as the sequence

$$\left\{ \sum_{p'+p''=p} a'_{p'} a''_{p''} \right\} .$$

Using the properties of $\log S_i$ it is easy to see that, for any p' and p''

$$S(0) \cdot S(p' + p'') \geqq S(p') \cdot S(p'') .$$

This has the immediate consequence that $a'a'' \in A$ and

$$\|a' \cdot a''\| \leqq S(0) \cdot \|a'\| \|a''\| ,$$

i.e. that A is a commutative Banach algebra. The element e with $a_p = 1$ if $p = (0, 0, \ldots, 0)$ and $a_p = 0$ elsewhere is the identity. α_i denotes the element with $a_p = 1$ if $p = (0, 0, \ldots, 1, \ldots, 0)$ (1 in the i-th place), and $a_p = 0$ elsewhere. It is easy to see that $e, \alpha_1, \ldots, \alpha_n$ are generators. (3.1) gives easily that (1.1) holds. Hence, by Lemma 1.2, A has exactly one maximal ideal.

We shall show that the supplementary assumption in § 2 is fulfilled. Because of the special structure of the "weight-function" $S(p)$ it is possible to choose A° independently of the ideal in question. We take as A° the space of all $a=\{a_p\}$ with a finite norm

$$\|a\|^0 = \sum_p \frac{|a_p|}{S_1(p+q)S_2(p+q)\cdot \ldots \cdot S_n(p_n+q)}.$$

This gives a new Banach algebra, larger than A. The dual space has the norm

$$\|b\|^{0*} = \sup_p |b_p| S_1(p+q)\cdot \ldots \cdot S_n(p_n+q)\,.$$

Using this definition it is easy to see that, starting from an element $b \in B$ of degree $\geqq q$, it is possible to form a suitable translate $p=\alpha^p \circ b$, such that β fulfils the properties 1° and 2° in the supplementary assumption. We can also prove that 3° is true for any $b \in B^\circ$. For the right hand member of the inequality is then

$$\sum_{p_0 \,|\, |p_0|=q} \|\alpha^{p_0}\circ b\|^{0*} = \sum_{p_0 \,|\, |p_0|=q} \sup_p |b_{p+p_0}|\cdot S_1(p_1+q)\ldots S_n(p_n+q)$$
$$\geqq \sum_{p_0 \,|\, |p_0|=q} \sup_p |b_{p+p_0}| S(p+p_0) \geqq \sup_{|p|\geqq q} |\, b_p| S(p)\,.$$

By choosing b' of degree $\leqq q-1$ such that its coefficients for $|p|\leqq q-1$ coincide with the corresponding coefficients of $-b$ we see that the left hand member of the inequality 3° gets the value

$$\sup_{|p|\geqq q} |b_p| S(p)\,.$$

Hence the inequality is true and the supplementary assumption is fulfilled.

The answer to Problem 1 *is thus affirmative for this Banach algebra.*

It is possible to treat more general weight-functions $S(p)$ in the same way. It should also be observed, that, if $n=1$, then also Problem 2 has an affirmative answer. This is a trivial consequence of the fact that the space of all elements in B with a finite degree is weakly dense in B.

4. Second application.

Let α satisfy $0<\alpha<1$. We form the class of all Lebesgue-measurable complex-valued function f on R^n, such that

$$\|f\| = \int e^{|x|^\alpha} |f(x)|\, dx < \infty\,,$$

where $x=(x_1,\ldots,x_n)$ and $|x|^2=\sum x_i^2$. It is easy to see that we obtain a commutative Banach algebra K with convolution as operation. Algebras of this kind have been studied in [1]. The space of regular maximal ideals is homeomorphic to the dual group R^n and the Gelfand representation of an element f is given by its Fourier transform

$$\hat{f}(t) = \int e^{-ix\cdot t} f(x)\, dx ,$$

where $t=(t_1,t_2,\ldots,t_n)$ and $x\cdot t=\sum x_i t_i$. We form the ideal I_0 which is the closure of the ideal of all functions f for which $\hat{f}$ vanishes in a neighbourhood of $t=0$. Let A be K/I_0. K is regular, and hence A is a commutative Banach algebra with a unit e and only one maximal ideal.

It is easy to show that there exists for every j a function in K for which the Fourier transform coincides with it_j in a neighbourhood of the origin. Let α_j be the corresponding element in A. It is included in the maximal ideal, and hence $\|\alpha_j{}^m\|^{1/m} \to 0$, as $m \to \infty$.

We shall show that $e,\alpha_1,\ldots,\alpha_n$ are generators of A. Let a be any element of A and let $\varepsilon > 0$ be arbitrary. We choose a representative $f \in K$ of a. There exists a $f_0 \in K$ with compact support such that

$$\|f-f_0\| < \varepsilon .$$

The Fourier transform $\hat{f}_0$ of f_0 can then be represented as a series

$$\sum_p c_p t_1{}^{p_1} t_2{}^{p_2} \cdot \ldots \cdot t_n{}^{p_n} .$$

The coefficients c_p tend to 0, as $|p| \to \infty$. Thus

$$\sum_p c_p \alpha_1{}^{p_1} \alpha_2{}^{p_2} \cdot \ldots \cdot \alpha_n{}^{p_n}$$

converges strongly and has as sum the element $a_0 \in A$ for which f_0 is a representative. We can find a finite partial sum a_1 of the series such that

$$\|a_1-a_0\| < \varepsilon .$$

Hence

$$\|a_1-a\| \leqq \|a_1-a_0\|+\|f-f_0\| < 2\varepsilon ,$$

which proves that $e,\alpha_1,\ldots,\alpha_n$ are generators.

The dual space of K can be interpreted as the class of measurable functions b on R^n such that

$$\text{ess sup}\, |b(x)| e^{-|x|^\alpha} < \infty$$

in the sense that if $f \in K$

$$b(f) = \int_{-\infty}^{\infty} f(x) b(-x)\, dx .$$

The dual space B of A is then in the same sense the subspace of functionals on K which vanish on the ideal I_0. It can be proved that if $b \in B$, then it coincides almost everywhere with a function with arbitrarily high differentiability (in fact an entire function). The derivative $D^p b$ is given by $\alpha^p \circ b$. This is shown in the one-dimensional case in [1], Theorem 3.34, and the n-dimensional proof is similar. The derivative $D^p b$ can thus be obtained by taking an arbitrary function $f \in K$, the Fourier transform of which coincides with

$$(it_1)^{p_1}(it_2)^{p_2} \cdot \ldots \cdot (it_n)^{p_n}$$

in a neighbourhood of $t=0$, and then form the function

$$\int b(y) f(x-y)\, dy \, .$$

Since $\alpha^{p_0} \circ b = D^p b$, the functionals b of finite degree are the same as polynomials of the same degree in $x_1, \ldots, x_n$.

We shall now show that the supplementary assumption in § 2 is fulfilled. As in the first application, we can choose A° in such a way that it only depends on q.

We take as A° the class which is defined in the same way as A but with $e^{|x|^\alpha}$ exchanged to

$$\frac{1}{1+|x|^q} e^{|x|^\alpha} .$$

It is easy to see that this weight-function gives a Banach algebra, larger than A.

If $b \in B$ is chosen continuous, we have

$$\|b\|^* = \sup |b(x)| e^{-|x|^\alpha} ,$$

while the norm in B° is

$$\|b\|^{0*} = \sup |b(x)|(1+|x|^q) e^{-|x|^\alpha} .$$

We shall now prove that if $b \in B$ and if j is arbitrary, then

$$\alpha_j{}^m \circ b = \frac{\partial^m b}{\partial x_j{}^m} \in B^\circ ,$$

if m is sufficiently large. Using the elementary inequality

$$|c|^\alpha \leqq 1 + |c-1|^\beta, \quad \text{if} \quad \alpha < \beta < 1, \; c \text{ real} ,$$

with $c = x/x_0$, we see that

$$|x|^\alpha \leqq |x_0|^\alpha + |x-x_0|^\beta |x_0|^{\alpha-\beta} , \tag{4.1}$$

if $x \in R^n$, $x_0 \in R^n$, $x_0 \neq 0$.

We choose f such that

$$\int e^{|x|^\beta} |f| \, dx < \infty ,$$

while

$$\hat{f} = (it_j)^m$$

in a neighbourhood of $t=0$. Then, by (4.1)

$$(4.2) \quad e^{-|x_0|^\alpha} \left| \int b(x)\, |x_0|^{(\alpha-\beta)/\beta} f\big((x_0-x)|x_0|^{(\alpha-\beta)/\beta}\big)\, dx \right|$$

$$\leqq \int |b(x)|\, e^{-|x|^\alpha}\, |x_0|^{(\alpha-\beta)/\beta} \exp\big(|x-x_0|^\beta |x_0|^{\alpha-\beta}\big)\, \big|f\big((x_0-x)|x_0|^{(\alpha-\beta)/\beta}\big)\big|\, dx$$

$$\leqq \sup |b(x)| e^{-|x|^\alpha} \cdot \int e^{|x|^\beta} |f(x)|\, dx = C < \infty ,$$

where C is independent of x_0. Since

$$|x_0|^{(\alpha-\beta)/\beta} f\big((x_0-x)|x_0|^{(\alpha-\beta)/\beta}\big)\, dx$$

belongs to A and has a Fouriertransform which coincides with

$$(it_j |x_0|^{(\beta-\alpha)/\beta})^m$$

in a neighbourhood of $t=0$, the left hand member of (4.2) is

$$e^{-|x_0|^\alpha} |x_0|^{m\cdot(\beta-\alpha)/\beta} \left| \frac{\partial^m b}{\partial x_j{}^m} \right|_{x=x_0} .$$

Thus

$$\frac{\partial^m b}{\partial x_j{}^m} \in B^\circ, \quad \text{if} \quad m\cdot(\beta-\alpha)/\beta \geqq q .$$

If b of infinite degree, then at least for some j, $\alpha_j{}^m b \neq 0$ for every m, and this together with the above result proves 1° and 2° in the supplementary assumptions.

We shall then prove that 3° in these assumptions is true for every $b \in B^\circ$. We choose b' as the polynomial of degree $\leqq q-1$ for which all derivatives of order $\leqq q-1$ coincide with the corresponding derivatives of $-b$. Put $b+b'=b''$. The inequality to prove is then

$$(4.3) \qquad \|b''\|^* \leqq C \sum_{p_0\,|\,|p_0|=q} \|D^{p_0} b''\|^{0*} .$$

But Taylor's theorem shows that

$$|b''(x)| \leqq \frac{1}{q!} (|x_1| + \ldots + |x_n|)^q \sup_{\substack{|p_0|=q \\ |\xi|\leqq|x|}} |D^{p_0} b''(\xi)|$$

$$\leqq \text{const.} \cdot e^{|x|^\alpha} \cdot \sum_{p_0\,|\,|p_0|=q} \|D^{p_0} b''\|^{0*} ,$$

which proves (4.3).

Hence the supplementary assumptions are fulfilled, and by Theorem 2.1 *the answer to Problem* 1 *is affirmative.*

We shall now apply this to show that the same result is true for the original Banach algebra K. We formulate this as a special theorem.

THEOREM 4.1. *For the Banach algebra K, defined in the beginning of this section, the answer to Problem* 1 *is affirmative.*

PROOF. We turn to the equivalent Problem 1'. If I is included in an infinite number of regular maximal ideals, we can take a sequence of such ideals $I_1, I_2, \ldots$, and then form $I_1, I_1 \cap I_2, I_1 \cap I_2 \cap I_3, \ldots$ which all contain I. The regularity of K shows that these are ideals of the co-dimensions $1, 2, 3, \ldots$.

Then we assume that I is included in only a finite number of regular maximal ideals $I_1, \ldots, I_m$ corresponding to points $t^1, t^2, \ldots, t^m$. Let I_ν, $\nu = 1, \ldots, m$, be the smallest closed ideal, containing I, and not included in any other regular maximal ideal than I_ν. We now once more use the regularity and the fact that Wiener's Tauberian theorem holds in K ([1], Theorem 1.53). From this it can be seen that the co-dimension of I is the sum of the co-dimensions of all I_ν. Hence the problem is reduced to the case when I is included in only one regular maximal ideal. We may assume that the corresponding $t^\circ = 0$, otherwise we modify the ideal I by changing every $f \in I$ to $fe^{-it_\nu x}$. This gives a new ideal which obviously has the same structure and co-dimension as I. Once more using the Wiener Tauberian theorem, it is easily seen that the co-dimension of I in K is the same as the co-dimension of I/I_0 in $A = K/I_0$, where I_0 is defined in the beginning of this section. (I_0 is the smallest closed ideal which is only contained in the regular maximal ideal which corresponds to $t = 0$.) Hence the co-dimension of I/I_0 is infinite and by our earlier results in this section, there is for every $q > 0$ a closed ideal in A of co-dimension q and which contains I/I_0. The corresponding ideal in K contains I, and it is also of co-dimension q. Hence the theorem is proved.

Let L be the class of functions of the form

$$\sum_{\nu=1}^{n} P_\nu(x) e^{it_\nu x} ,$$

where P_ν are polynomials in $x_1, \ldots, x_n$ and t_ν are arbitrary real numbers. The following theorem can be proved as a direct consequence of Theorem 4.1.

THEOREM 4.2. *Let K_0 be a subclass of K. We consider the class C of all φ belonging to the dual space of K and such that*

$$\int \varphi(x-x_0)\, f(x_0)\, dx_0 \equiv 0$$

for every $f \in K_0$. *Then two cases may occur:*

1. C *is finite-dimensional. Then* $C \subset L$.
2. C *is infinite-dimensional. Then* C *contains an infinite-dimensional subspace of* L.

REMARKS. It was demonstrated in [2] that the answer to Problem 2 is affirmative for the algebra A in this section if $n=1$.

If the weight-function $e^{|x|^\alpha}$ in this section in the case $n=1$ is exchanged to the function

$$p(x) = \begin{cases} e^{|x|^{2/3}}, & x \geqq 0 \\ e^{|x|^{1/3}}, & x < 0\,, \end{cases}$$

then it is easy to see, using a Phragmén–Lindelöf theorem, that the class of all polynomials are not weakly dense in B, the dual space of A (cf. a remark in Hačatrjan [3]). This implies that the intersection of all closed ideals in A with a finite co-dimension is not empty. Hence the answer to Problem 2 is negative for this algebra A.

The methods in this section are still valid with this new weight-function. Hence the answer to Problem 1 is affirmative. (This is also a direct result of Theorem 1 in [2].)

If $n=1$ and $e^{|x|^\alpha}$ is exchanged to the function

$$p(x) = \begin{cases} e^{|x|^{2/3}}, & x \geqq 0 \\ 1\;, & x < 0\,, \end{cases}$$

then there are no non-constant polynomials in the dual space of A. However, B contains other functions than the constant function (in fact all entire functions g of exponential type 0 for which g/p is bounded). Hence the answer to Problem 1 is negative (for $m=2$) for this algebra A. The same is true for the corresponding K, and also Theorem 4.2 is no longer true.

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