MATH. SCAND. 15 (1964), 18–28

FØLNER'S CONDITIONS FOR AMENABLE SEMI-GROUPS

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1. Introduction.

In [3], Følner gave interesting necessary and sufficient conditions for a group to be amenable. Naturally one wonders whether these conditions can be generalized to semi-groups. Følner's necessary condition was generalized to semi-groups by Frey in his thesis [4] by means of a massive calculation more or less modelled after the original proof of Følner. In section 3, we give a vastly simpler proof of the Følner–Frey theorem. It turns out that this theorem is a direct consequence of "strong amenability" of amenable semi-groups. The notion of strong amenability was investigated by Day in [1], in which he raised the question whether Følner's necessary condition was a consequence of strong amenability or not. In section 4, we discuss generalizations of Følner's sufficient condition. The general picture is unsatisfactory in that the conditions of section 3 and 4 are different unless the semigroup in question possesses the right cancellation law, in which case all the conditions are equivalent. In section 5, we shall strengthen Følner's necessary condition for amenable groups. This section was motivated by a lemma of Hewitt in [5].

We wish to thank Professor Day for having read our original version of section 3 and suggesting a simplification. The incorporation of his idea resulted in the reduction of the number of lemmas by one. We also wish to thank Professor Hewitt for the conversations we had on the subject of the present paper and for pointing to us the existence of Frey's thesis.

2. Preliminaries.

Let S be a semi-group, that is, a set in which an associative binary operation (denoted by '$\cdot$') is given, and let $l_1(S)$ be the set of all real valued functions f on S such that

$$\sum \{|f(s)| : \ s \in S\} < \infty .$$

For each f in $l_1(S)$, let

Received June 4, 1964.
This research was partly supported by the U.S. National Science Foundation.

$$\|f\| = \sum \{|f(s)|: s \in S\}.$$

Following Day [1], we shall introduce the *convolution* $f_1 \cdot f_2$ of elements f_1 and f_2 of $l_1(S)$ by the formula:

$$f_1 \cdot f_2(t) = \sum \{f_1(s_1) f_2(s_2): s_1 \cdot s_2 = t\}.$$

The convolution turns $l_1(S)$ into a (real) Banach algebra. Now define a map $I: S \to l_1(S)$ by

$$I(s)(s') = \begin{cases} 1 & \text{if } s = s', \\ 0 & \text{if } s \neq s'. \end{cases}$$

Then $I(s_1 \cdot s_2) = I(s_1) \cdot I(s_2)$, and, because of this fact, we shall systematically fail to distinguish s and its image $I(s)$. An element f of $l_1(S)$ is called a *finite mean* if $f(s) \geqq 0$ for each s in S, $\{s: f(s) > 0\}$ is finite and $\|f\| = \sum \{f(s): s \in S\} = 1$. We shall use Φ to denote the set of all finite means. It is obvious that Φ is a convex subset of $l_1(S)$; in fact, Φ is the convex hull of S.

A semi-group S is called "right amenable" if S admits a "right invariant mean". Instead of introducing more notation in order to define right invariant means, we shall adopt the following definition of right amenability, which can be easily seen to be equivalent to the more traditional definition (cf. Day [1, p. 515]).

2.1 Definition. A semi-group S is *right amenable* if and only if there is a net $\{f_\gamma\}$ in Φ such that, for each s in S, the net $\{f_\gamma \cdot s - f_\gamma\}$ converges to 0 weakly in $l_1(S)$.

In the definition above, if "weakly" is replaced by "in the norm", we have a definition for *strong right amenability* of S. In [1], Day proved that right amenability and strong right amenability are actually equivalent. Since this fact is essential for our proof of Følner's theorem, we shall state and prove Day's result for convenience of the readers.

2.2 Theorem. (Day.) *A semi-group S is right amenable if and only if there is a net $\{f_\gamma\}$ in Φ such that, for each s in S,*

$$\lim_\gamma \|f_\gamma \cdot s - f_\gamma\| = 0.$$

Proof. Let E be the product $(l_1(S))^S$; then E is a locally convex linear topological space under the product of the norm topologies. Define a linear map $T: l_1(S) \to E$ as follows: For f in $l_1(S)$ and for s in S, $T(f)(s) = f \cdot s - f$. Now the weak topology on E is the same as the product of the weak topologies (see, for example, [6; p. 160]); therefore, from definition 2.1, S is right amenable if and only if 0 is in the weak closure of $T[\Phi]$. Since

E is locally convex and $T[\Phi]$ is convex, the weak closure of $T[\Phi]$ is identical with the closure $T[\Phi]^-$ of $T[\Phi]$ relative to the given topology, i.e. the product of the norm topologies. Hence S is right amenable if and only if $0 \in T[\Phi]^-$, which is precisely the content of theorem 2.2.

2.3 REMARK. From the proof above, we see that the semi-group S is not right amenable if and only if there is a continuous linear functional on $(l_1(S))^S$ separating 0 and $T[S]$ strongly; more explicitly stated: S is not right amenable if and only if there are a finite number of bounded functions $u_1, \ldots, u_n$ on S and the same number of elements $t_1, \ldots, t_n$ of S such that

$$\inf\left\{\sum_{i=1}^{n}\left(u_i(s\cdot t_i)-u_i(s)\right):\ s \in S\right\} > 0 .$$

This is the well-known criterion of Dixmier [2].

3. Necessary conditions.

For a finite non-empty subset A of S, we define a member μ_A of Φ as follows:

$$\mu_A(s) = \begin{cases} c(A)^{-1} & \text{if } s \in A , \\ 0 & \text{if } s \notin A , \end{cases}$$

where, in general, $c(M)$ denotes the cardinality of the set M. If χ_A denotes the characteristic function of the set A, then $\mu_A = c(A)^{-1}\chi_A$. In the sequel, a "finite subset" always means a "finite non-empty subset".

3.1. LEMMA. *Let f be a member of Φ; then f can be written in the form $f=\sum_{i=1}^{n}\lambda_i\mu_{A_i}$, where each A_i is finite,*

$$A_i \supset A_{i+1},\ i=1,\ldots,n-1, \qquad \lambda_i>0,\ i=1,\ldots,n ,$$

and

$$\sum_{i=1}^{n}\lambda_i = 1 .$$

PROOF. Let $0<a_1<\ldots<a_n$ be the distinct values of the function f. Let $A_i=\{s:\ a_i \leqq f(s)\}$. Then clearly

$$A_1 \supset A_2 \supset \ldots \supset A_n$$

and

$$f = a_1\chi_{A_1}+(a_2-a_1)\chi_{A_2}+\ldots+(a_n-a_{n-1})\chi_{A_n} = \sum_{i=1}^{n}\lambda_i\mu_{A_i} ,$$

for some positive λ's. Finally,

$$1 = \sum \{f(s):\ s \in S\} = \sum_{i=1}^{n} \lambda_i \sum \{\mu_A(s):\ s \in S\} = \sum_{i=1}^{n} \lambda_i .$$

In what follows, for a subset A of S and for a member s of S, $A \cdot s$ denotes the set

$$\{t:\ t = a \cdot s \text{ for some member } a \text{ of } A\} .$$

3.2 Lemma. *Let A be a finite subset of S and let s be a member of S. Then*

$$(\mu_A \cdot s - \mu_A)(t) \begin{cases} \geqq c(A)^{-1} & \text{if } t \in A \cdot s \sim A , \\ < \quad 0 & \text{if } t \in A \sim A \cdot s, \\ \geqq \quad 0 & \text{for all other } t . \end{cases}$$

Proof. Observe that

$$\mu_A \cdot s(t) = \sum \{\mu_A(s'):\ s' \cdot s = t\} = c(A)^{-1} c(A \cap \{s':\ s' \cdot s = t\}) .$$

From this it follows that

$$(\mu_A \cdot s)(t) \begin{cases} \geqq c(A)^{-1} & \text{if } t \in A \cdot s , \\ = 0 & \text{if } t \notin A \cdot s . \end{cases}$$

The lemma is now clear.

3.3 Lemma. *Let a function f in Φ be expressed as in lemma* 3.1. *Then, for each s in S,*

(*) $$\|f \cdot s - f\| \geqq \sum_{i=1}^{n} \lambda_i c(A_i \cdot s \sim A_i)/c(A_i) .$$

Proof. If $f \in \Phi$ is expressed as in lemma 3.1, then

$$f \cdot s - f = \sum_{i=1}^{n} \lambda_i (\mu_{A_i} \cdot s - \mu_{A_i}) .$$

Let

$$B = \bigcup \{(A_j \sim A_j \cdot s):\ j = 1, 2, \ldots, n\};$$

then, by lemma 3.2 each $\lambda_i(\mu_{A_i} \cdot s - \mu_{A_i})$ is non-negative on $S \sim B$. Now, for any i and j, $1 \leqq i,\ j \leqq n$,

$$(A_i \cdot s \sim A_i) \cap (A_j \sim A_j \cdot s) = \emptyset ,$$

because either $A_j \subset A_i$ or $A_j \supset A_i$ (hence $A_j \cdot s \supset A_i \cdot s$) holds. Consequently each $A_i \cdot s \sim A_i$ is contained in $S \sim B$. Hence,

$$\begin{aligned}\|f\cdot s-f\| &\geqq \sum\{(f\cdot s-f)(t):\ t\in S\sim B\}\\ &= \sum_{i=1}^{n}\lambda_i\sum\{(\mu_{A_i}\cdot s-\mu_{A_i})(t):\ t\in S\sim B\}\\ &\geqq \sum_{i=1}^{n}\lambda_i\sum\{(\mu_{A_i}\cdot s-\mu_{A_i})(t):\ t\in A_i\cdot s\sim A_i\}\,.\end{aligned}$$

Finally, using lemma 3.2 again, we obtain (*).

3.4 Remark. A semi-group S is said to have the *right cancellation law* if $s_1\cdot s=s_2\cdot s$ implies $s_1=s_2$. In the lemma above, if S is assumed to have the right cancellation law, then one can actually prove the equality

$$\begin{aligned}\|f\cdot s-f\| &= \sum_{i=1}^{n}\lambda_i\|\mu_{A_i}\cdot s-\mu_{A_i}\|\\ &= 2\sum_{i=1}^{n}\lambda_i c(A_i\cdot s\sim A_i)/c(A_i)\,.\end{aligned}$$

3.5 Theorem. (Følner–Frey.) *Let S be a right amenable semigroup. Then for a given finite subset F of S and for a given positive number ε, there exists a finite subset A of S such that, for each s in F,*

$$c(A\cdot s\sim A) < \varepsilon\cdot c(A)\,.$$

Proof. Let $F=\{s_1,\ldots,s_k\}$; then by theorem 2.2, there exists a member f of Φ such that

$$\|f\cdot s_j-f\| < \varepsilon/k,\qquad j = 1,2,\ldots,k\,.$$

By lemma 3.1, f can be written in the form $f=\sum_{i=1}^{n}\lambda_i\mu_{A_i}$, where λ's and A's satisfy the conditions stated in lemma 3.1. Let $N=\{1,2,\ldots,n\}$, and we define a measure m on the family of all subsets of N as follows: For a subset K of N,

$$m(K) = \sum\{\lambda_i:\ i\in K\}\,.$$

If K is empty, we agree $m(K)=0$. Clearly $m(N)=1$. Now for each $j=1,2,\ldots,k$, let

$$K_j = \{i:\ c(A_i\cdot s_j\sim A_i)/c(A_i) < \varepsilon\}\,.$$

Then for each i in $N\sim K_j$,

$$c(A_i\cdot s_j \sim A_i)/c(A_i) \geqq \varepsilon\,.$$

By lemma 3.3, we know that, for each j ($j=1,2,\ldots,k$),

$$\varepsilon/k > \|f\cdot s_j-f\| \geqq \sum_{i=1}^{n}\lambda_i c(A_i\cdot s_j\sim A_i)/c(A_i)\,.$$

It follows that

$$\varepsilon/k > \varepsilon \cdot \sum \{\lambda_i \colon i \in N \sim K_j\} = \varepsilon \cdot m(N \sim K_j) .$$

Therefore, $m(N \sim K_j) < k^{-1}$ for each $j = 1, 2, \ldots, k$. Hence,

$$\begin{aligned} 1 - m\left(\bigcap_{j=1}^{k} K_j\right) &= m\left(N \sim \bigcap_{j=1}^{k} K_j\right) \\ &= m\left(\bigcup_{j=1}^{k} (N \sim K_j)\right) \\ &\leqq \sum_{j=1}^{k} m(N \sim K_j) < k \cdot k^{-1} = 1 . \end{aligned}$$

This implies that $m(\bigcap_{j=1}^{k} K_j) > 0$; hence, $\bigcap_{j=1}^{k} K_j$ is not empty. Choose i in $\bigcap_{j=1}^{k} K_j$ and let $A = A_i$; then clearly A satisfies the condition of the theorem.

The following corollary is an immediate consequence of theorem 3.5.

3.6. Corollary. *Let S be a right amenable semi-group with the right cancellation law. Then given a finite subset F of S and a number k, $0 < k < 1$, there exists a finite subset A of S such that, for each s in F.*

$$c(A \cdot s \cap A) \geqq k \cdot c(A) .$$

We conclude the section with a discussion concerning locally compact groups. Let G be a locally compact topological group, let μ be a right invariant Haar measure and let L_1 be the real L_1-space constructed with the measure μ. For each f in L_1 and g in G define f^g to be the right translate of f by g^{-1}, i.e. $f^g(x) = f(xg^{-1})$, and let R_g be the continuous linear transformation $L_1 \to L_1$ defined by $R_g(f) = f^g$. Since L_1 is a Banach lattice, so is L_1^{**} (see [6; 239]). We call a member φ of L_1^{**} a *mean over* L_1^* (or *mean over* L_∞ using a suitable identification of L_1^* and L_∞) if $\varphi \geqq 0$ and $\|\varphi\| = 1$. We call G a *right amenable topological group* if and only if there is a mean φ over L_1^* such that $R_g^{**}(\varphi) = \varphi$ for each g in G, where R_g^{**} denotes the second adjoint transformation $L_1^{**} \to L_1^{**}$ of R_g. Since a right invariant Haar measure is unique up to a constant multiple this notion of right amenability is indeed independent of the choice of μ. If G is given the discrete topology, then G is a right amenable topological group if and only if G is right amenable as an abstract group. It can be seen easily that if G is a locally compact topological group and G is right amenable as an abstract group then G is a right amenable topological group. In particular Abelian locally compact groups are right amenable topological groups. Also, compact groups are right amenable topological groups.

With the definitions introduced above all the material of the last two sections can be adapted to right amenable topological groups with, of course, some obvious modifications. For instance, the set Φ should be replaced by the convex subset Ψ of L_1 consisting of all non-negative simple functions of norm 1. Then it is true that G is a right amenable topological group if and only if there is a net $\{f_\gamma\}$ in Ψ such that, for each g in G, the net $\{f_\gamma{}^g - f_\gamma\}$ converges weakly to 0 in L_1, and by repeating the argument of theorem 2.2, we can conclude that *a locally compact group G is a right amenable topological group if and only if there is a net $\{f_\gamma\}$ in Ψ such that* $\lim_\gamma \|f_\gamma{}^g - f_\gamma\| = 0$ *for each g in G.*

We can then follow the method of the present section to establish the following theorem, which is of interest because it is precisely the converse of the assertion of Dixmier in [2; § 4, 3(a)].

3.7 THEOREM. *Let G be a locally compact topological group and let μ be a right invariant Haar measure. If G is a right amenable topological group, then for a given finite subset F of G and a positive number ε, there exists a measurable subset A of G such that $0 < \mu(A) < \infty$ and $\mu(A \cdot g \sim A) < \varepsilon\mu(A)$ for each g in F.*

4. Sufficient conditions.

The converse of theorem 3.5 is false, because if S is finite then $c(S \cdot s \sim S) = 0$ for any s in S, but not all finite semi-groups are right amenable. In order to see to what extent the necessary conditions (for a semi-group to be right amenable) obtained in the last section are also sufficient, we state the following generalization of the second part of Følner's main theorem [3; p. 245]. We omit the proof, because it is a straight forward modification of Følner's proof based on Dixmier's criterion (see remark 2.3).

4.1 THEOREM. *For a semi-group S to be right amenable it is sufficient that there exists a number r, $0 < r < 1$, such that, for a given choice of elements $s_1, \dots, s_n$; $s_1', \dots, s_n'$ of S (not necessarily distinct), there is a finite subset A of S which satisfies*

$$n^{-1} \sum_{i=1}^{n} c(A \cdot s_i \cap A \cdot s_i') \geqq r \cdot c(A) .$$

If S is a group, then $c(A \cdot s_i \cap A \cdot s_i') = c(A \cap A s_i' s_i^{-1})$. Therefore in this case the condition of theorem 4.1 becomes equivalent to the necessary condition given by Følner.

4.2 Corollary. *For a semi-group S to be right amenable it is sufficient that there exists a number ε, $0<\varepsilon<\frac{1}{2}$, such that for a given choice of elements $s_1,\ldots,s_n$ of S (not necessarily distinct), there is a finite subset A of S which satisfies*

$$n^{-1}\sum_{i=1}^{n} c(A \sim A\cdot s_i) \leqq \varepsilon\cdot c(A)\,.$$

Proof. Suppose the elements $s_1,\ldots,s_n$; $s_1',\ldots,s_n'$ of S are given. Then from the assumption, one can find a finite subset A of S such that

$$(2n)^{-1}\sum_{i=1}^{n}\big(c(A \sim A\cdot s_i)+c(A \sim A\cdot s_i')\big) \leqq \varepsilon c(A)\,.$$

Now observe that

$$\begin{aligned} c(A \sim A\cdot s_i)+c(A \sim A\cdot s_i') &\geqq c\big(A \sim (A\cdot s_i \cap A\cdot s_i')\big)\\ &= c(A)-c(A\cdot s_i \cap A\cdot s_i')\,. \end{aligned}$$

It follows that

$$2\varepsilon\cdot c(A) \geqq n^{-1}\sum_{i=1}^{n}\big(c(A)-c(A\cdot s_i\cap A\cdot s_i')\big) = c(A)-n^{-1}\sum_{i=1}^{n}c(A\cdot s_i\cap A\cdot s_i')\,.$$

Therefore,

$$n^{-1}\sum_{i=1}^{n}c(A\cdot s_i\cap A\cdot s_i') \geqq (1-2\varepsilon)c(A)\,.$$

Since $0<1-2\varepsilon<1$, our semi-group S satisfies the condition of theorem 4.1; hence, S is right amenable.

The range of ε given in corollary 4.2 is the best possible. For, consider the semi-group $S=\{a,b\}$ with the following multiplication table:

	a	b
a	a	b
b	a	b

Here we see that $c(S\sim S\cdot a)=c(S\sim S\cdot b)=1$, and therefore the condition of the corollary is satisfied with $\varepsilon=\frac{1}{2}$. On the other hand it is easy to check that this semi-group is not right amenable.

If S is a semi-group with the right cancellation law, then, for any finite subset A of S and for any member s of S, it is true that $c(A\cdot s\sim A)=c(A\sim A\cdot s)$. Therefore, in this case, the necessary condition of theorem 3.5 for S to be right amenable implies the condition of corollary 4.2 (hence, of theorem 4.1 also). We have thus proved:

4.3 COROLLARY. *If S is a semi-group with right cancellation law, the conditions of theorem* 3.5, *theorem* 4.1 *and corollary* 4.2 *are both necessary and sufficient for the right amenability of S.*

4.4 REMARK. One can also prove very easily that for finite right amenable semi-groups the conditions of corollary 4.2 and theorem 4.1 are necessary. We doubt that the condition of theorem 4.1 is necessary for an arbitrary right amenable semi-group, but we do not have any example to substantiate our doubts.

5. For groups only.

In theorem 3.5 and corollary 3.6, one does not have much control over A except the finiteness of A. However, if S is a group, we can do much better: in fact, we shall prove in this section that A can be chosen to be symmetric and "arbitrarily large". Let G be a group and let f be a real valued function G. Then define f^* to be the function: $f^*(g)=f(g^{-1})$ for each g in G. A *symmetric function* is a function f such that $f=f^*$. A subset A of G is called symmetric if $A=A^{-1}$. Let Φ_S denote the subset of Φ consisting of all the symmetric finite means on G. Then again Φ_S is convex. Although it is known that right amenable groups are "amenable", we shall keep calling them right amenable.

5.1 LEMMA. *If G is a right amenable group, then there is a net $\{f_\gamma\}$ in Φ_S such that, for each g in G, $\lim_\gamma \|f_\gamma \cdot g - f_\gamma\| = 0$.*

PROOF. Let M_S be a subset of $l_\infty{}^*(G)=l_1{}^{**}(G)$ such that $\mu \in M_S$ if and only if

(a) $\mu(u) \geqq 0$ whenever u is a non-negative member of $l_\infty(G)$ (that is, $u(g) \geqq 0$ for all g in G),

(b) $\mu(\mathbf{1})=1$, where $\mathbf{1}$ is the function identically equal to 1 on G, and

(c) $\mu(u)=\mu(u^*)$ for each u in $l_\infty(G)$.

Let e denote the evaluation map of $l_1(G)$ into $l_\infty{}^*(G)$. Then $e[\Phi_S] \subset M_S$, and we assert that $e[\Phi_S]$ is weak* dense in M_S. For, if not, then there is a member μ_0 in M_S which is not contained in the weak* closure of $e[\Phi_S]$. Then by a standard separation theorem (see, for example corollary 14.4 of [6]), there is a member u in $l_\infty(G)$ such that

$$\mu_0(u) > \sup\{\langle u, f \rangle : f \in \Phi_S\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard pairing of $l_1(G)$ and $l_\infty(G)$. Now since

$$\mu_0(u) = \mu_0(\tfrac{1}{2}(u+u^*)) \quad \text{and} \quad \langle u, f \rangle = \langle \tfrac{1}{2}(u+u^*), f \rangle$$

we can assume u to be symmetric. Now, for each g in G,

$$\tfrac{1}{2}(g+g^{-1}) \in \Phi_S \quad \text{and} \quad \langle u, \tfrac{1}{2}(g+g^{-1})\rangle = \tfrac{1}{2}u(g)+\tfrac{1}{2}u(g^{-1}) = u(g)\,.$$

It follows that

$$\sup\{\langle u,f\rangle :\ f \in \Phi_S\} = \sup\{u(g):\ g \in G\}\,.$$

Consequently

$$\sup\{u(g):\ g \in G\} < \mu_0(u)\,,$$

but this violates properties (a) and (b) for μ_0. Hence $e[\Phi_S]$ must be weak* dense in M_S.

If G is right amenable, then there is a right and left invariant mean λ on G (see § 4 (B) of [1; p. 515]). If one defines λ_S by

$$\lambda_S(u) = \tfrac{1}{2}\lambda(u)+\tfrac{1}{2}\lambda(u^*)$$

for u in $l_\infty(G)$, it is easy to see that λ_S is again right and left invariant and $\lambda_S \in M_S$. Therefore there is a net $\{f_\gamma\}$ in Φ_S such that the net $\{e(f_\gamma)\}$ weak* converges to λ_S. Then by a simple argument (see § 5 (A) and (D) of [1]), we conclude that the net $\{f_\gamma \cdot g - f_\gamma\}$ converges to 0 weakly in $l_1(G)$ for each g in G. To complete the proof of the lemma, it is only necessary to repeat the proof of theorem 2.2 replacing Φ by Φ_S.

5.2 Theorem. *Let G be a right amenable group. Then, for given finite subsets F and K of G and a number k, $0<k<1$, there exists a finite symmetric subset A of G such that $A \supset K$ and $c(A\cdot g \cap A) \geqq k\cdot c(A)$ for each g in F.*

Proof. By replacing, if necessary, K by $K \cup (K^{-1})$, we can assume K to be symmetric. If G is finite then the theorem is trivial. Therefore let us assume that G be infinite. Choose n so large that $k(1+c(K)\cdot n^{-1})<1$. Suppose we can find a symmetric finite subset B of G such that

(i) $c(B) \geqq n$, and
(ii) $c(B\cdot g \cap B) \geqq k(1+c(K)\cdot n^{-1})\cdot c(B)$ for each g in F;

then we are through. For, let $A = B \cup K$; then obviously A is symmetric and, for each g in F,

$$\begin{aligned} c(Ag \cap A) \geqq c(Bg \cap B) &\geqq k\cdot\big(1+c(K)\cdot n^{-1}\big)\cdot c(B) \\ &\geqq k\cdot\big(1+c(K)\cdot c(B)^{-1}\big)\cdot c(B) \\ &= k\big(c(B)+c(K)\big) \geqq k\cdot c(A)\,. \end{aligned}$$

Therefore it remains to produce a symmetric finite subset B of G satisfying (i) and (ii) above.

First we make the following observation. In the proof of theorem 3.5, if S happens to be a right amenable group, we can choose f from Φ_S because of lemma 5.1. Then each A_i appearing in that proof is symmetric.

Therefore the conclusion of theorem 3.5 (and hence of corollary 3.6) is valid for some finite symmetric subset A of S, if S is a group.

Now let F_1 be a subset of G such that $F_1 \supset F$ and $c(F_1) \geqq n^2$. (We are assuming G to be infinite!) By the observation of the last paragraph, one can find a finite symmetric subset B of G such that

$$c(B \cdot g \cap B) \geqq k \cdot \left(1 + c(K) \cdot n^{-1}\right) c(B)$$

for each g in F_1 (hence in F). This set B, therefore, satisfies the property (ii). To see (i), notice that $Bg \cap B \neq \emptyset$ for each g in F_1. It follows that $F_1 \subset B^{-1}B$; consequently, $n^2 \leqq c(F_1) \leqq c(B)^2$ or $n \leqq c(B)$.

The following corollary for countable Abelian groups was proved by Hewitt [5].

5.3 Corollary. *Let G be a countable amenable group. Then there is a sequence $\{A_i\colon i=1,2,\ldots\}$ of finite symmetric subsets of G such that $A_i \subset A_{i+1}$, $i=1,2,\ldots,$*

$$\bigcup \{A_i\colon i=1,2,\ldots\} = G \quad \textit{and} \quad \lim_{i\to\infty} c(A_i \cdot g \cap A_i)/c(A_i) = 1$$

for each g in G.

Proof. Let $G = \bigcup\{K_i\colon i=1,2,..\}$, where each K_i is finite and $K_i \subset K_{i+1}$. Let A_1 be a finite symmetric subset of G such that $A_1 \supset K_1$. We choose $A_2, A_3, \ldots$ inductively. Assume that A_{i-1} has been chosen; then by theorem 5.2 there exists a finite symmetric subset A_i of G such that $A_i \supset A_{i-1} \cup K_i$ and

$$c(A_i \cdot g \cap A_i) \geqq (1 - 1/i) c(A_i)$$

for each g in K_i. Clearly the sequence $\{A_i\colon i=1,2,\ldots\}$ so constructed satisfies all the conditions of the corollary.

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