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RECURSIVE FUNCTIONS AND REGRESSIVE ISOLS

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1. Introduction.

The reader is assumed to be familiar with the concepts and main results of the papers listed as references. We shall use the following notations:

$$\begin{aligned} \varepsilon &= \text{the set of all non-negative integers } (\textit{numbers}),\\ \Lambda &= \text{the collection of all isols},\\ \Lambda^* &= \text{the collection of all isolic integers}. \end{aligned}$$

Myhill [5] associated with every recursive combinatorial function $f(x)$ a function $F(X)$ from Λ into Λ, called the canonical extension of $f(x)$; we shall write $C_f(X)$ for the canonical extension of $f(x)$. Nerode [8] associated with every recursive function $f(x)$ a function $F(X)$ from Λ into Λ^*; we shall write $D_f(x)$ for Nerode's extension of $f(x)$. In the special case that $f(x)$ is recursive and combinatorial, $D_f(x)=C_f(X)$ for $X \in \Lambda$. It was proved in [7] that for a recursive function $f(x)$, the function $D_f(X)$ maps Λ into Λ if and only if $f(x)$ is eventually combinatorial. Regressive isols were introduced in [1]. Let Λ_R denote the collection of all regressive isols. It is known that $\varepsilon \subset \Lambda_R \subset \Lambda$, where both $\Lambda_R - \varepsilon$ and $\Lambda - \Lambda_R$ have the cardinality of the continuum.

This paper deals with the following problem:

Which recursive functions $f(x)$ have the property that $D_f(X)$ maps Λ_R into Λ_R?

A function $f(x)$ from ε into ε will be called *increasing* if

$$x < y \quad \Rightarrow \quad f(x) \leqq f(y), \qquad \text{for } x,y \in \varepsilon;$$

and *eventually increasing* if for some number n, the function $g(x)=f(x+n)$ is increasing. The main result of this paper is as follows. For a recursive function $f(x)$ the function $D_f(X)$ maps Λ_R into Λ_R if and only if $f(x)$ is eventually increasing.

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Our presentation will make use of the theory of infinite series of isols. Infinite series of isols were introduced and studied in [1].

2. Regressive isols.

We shall here recall the concepts of a regressive function, regressive set and regressive isol.

DEFINITIONS. A function t_n from ε into ε is *regressive* if it is one-to-one and there exists a partial recursive function $p(x)$ such that

$$\varrho t \subset \delta p\,, \tag{1}$$

$$p(t_0) = t_0 \quad \text{and} \quad (\forall n)[p(t_{n+1}) = t_n]\,. \tag{2}$$

A set is *regressive*, if it is finite or the range of a regressive function. An isol T is *regressive*, if it contains at least one regressive set.

For every partial recursive function $f(x)$ with $\varrho f \subset \delta f$, let

$$f^0(x) = x, \quad \text{and} \quad f^{n+1}(x) = f(f^n(x))\,.$$

Let t_n be a regressive function. By [1, pp. 80, 81] there exists a partial recursive function $p(x)$ which satisfies, besides (1) and (2) the conditions

$$\varrho p \subset \delta p\,, \tag{3}$$

$$(\forall x)\,[x \in \delta p \;\Rightarrow\; (\exists n)[p^{n+1}(x) = p^n(x)]]\,. \tag{4}$$

DEFINITION. Every partial recursive function satisfying (1), (2), (3) and (4) is called a *regressing* function of t_n.

NOTATION. Let $p(x)$ be a partial recursive function satisfying (3) and (4). Then

$$p^*(x) = (\mu y)[p^{y+1}(x) = p^y(x)], \qquad \text{for } x \in \delta p\,.$$

We note that in the event $p(x)$ is a regressing function of t_n, then $p^*(x)$ is a partial recursive function with

$$\delta p^* = \delta p, \quad \varrho t \subset \delta p^* \quad \text{and} \quad (\forall n)[p^*(t_n) = n]\,.$$

3. $\sum_T a_n$.

We shall here recall the definition given in [1] of the infinite series $\sum_T a_n$, in case a_n is a recursive function of n and T a regressive isol.

NOTATIONS. Let j, k and l denote the well-known recursive functions defined by

$$j(x,y) = x+\tfrac{1}{2}(x+y)(x+y+1),$$
$$j(k(n), l(n)) = n.$$

For any number n, $\nu(n)=\{x \mid x<n\}$.
For any number n and sets δ and σ,

$$j(n,\sigma) = \{j(n,y) \mid y \in \sigma\} \quad \text{and} \quad j(\delta,\sigma) = \{j(x,y) \mid x \in \delta \ \& \ y \in \sigma\}.$$

We note that the function j maps ε^2 one-to-one onto ε.

DEFINITION. Let a_n be any recursive function of n and T any regressive isol. If T is finite, say $T=k$,

$$\textstyle\sum_T a_n = \sum_{n<k} a_n \qquad (0 \text{ for } k=0).$$

If T is infinite,

$$\textstyle\sum_T a_n = \operatorname{Req} \sum_0^\infty j(t_n, \nu(a_n)),$$

where t_n is any regressive function ranging over any set in T.

By [1, Proposition 3] $\sum_T a_n$ depends on the regressive isol T and not on the particular regressive function whose range is in T. Also, $\sum_T a_n$ is always an isol [1, Theorem 1] and in fact we shall prove in § 4 that it is always a regressive isol.

DEFINITION. Let a_n be any function from ε into ε. The function $s(n)$ such that

$$s(0) = 0,$$
$$s(n) = \sum_{i<n} a_i, \qquad \text{for } n>0,$$

is the *partial sum* function of a_n.

One can easily verify that if the function a_n is recursive and combinatorial then so is its partial sum function $s(n)$. The main result of [1] is Theorem A, stated below. It is essentially this theorem which enables us in § 5 to express $D_f(T)$ (where f is a recursive function and T a regressive isol) as the difference of two infinite series each summed with respect to the regressive isol $T+1$.

THEOREM A. [1, p. 86] *Let a_n be a recursive combinatorial function and $s(n)$ its partial sum function. Then, for every regressive isol T,*

$$\textstyle\sum_T a_n = C_s(T).$$

4. Fundamental properties of $\sum_T a_n$.

NOTATION. Parentheses will be omitted according to the rule that association is to the right. Thus $fg(x)$ means the same as $f(g(x))$.

THEOREM 1. *For every recursive function a_n, $\sum_T a_n$ is a function from Λ_R into Λ_R.*

PROOF. Let T be any regressive isol. We have already noted that $\sum_T a_n$ is an isol and therefore we restrict our attention here to proving $\sum_T a_n$ is regressive. If either T is finite or a_n is identically zero from some number on, then $\sum_T a_n$ is finite and therefore regressive. Now assume T is infinite and a_n is positive infinitely often. Let t_n be any regressive function ranging over any set in T. Let $f(x)$ denote the strictly increasing recursive function ranging over the set $\{n \mid a_n > 0\}$. Then

$$\sum_0^\infty j(t_n, \nu(a_n)) \in \sum_T a_n ,$$

$$\sum_0^\infty j(t_n, \nu(a_n)) = \sum_0^\infty j(t_{f(n)}, \nu(a_{f(n)})) .$$

We wish to prove $\sum_T a_n$ is regressive. We proceed to show that

$$j(t_{f(0)}, 0), \ldots, j(t_{f(0)}, a_{f(0)} - 1), j(t_{f(1)}, 0), \ldots, j(t_{f(1)}, a_{f(1)} - 1), \ldots$$

represents a regressive enumeration of the set $\sum_0^\infty j(t_n, \nu(a_n))$.

Since t_n is a regressive function of n and $f(x)$ is a strictly increasing recursive function it readily follows that $t_{f(n)}$ is also a regressive function of n. Let $p(x)$ be any regressing function of $t_{f(n)}$. We observe here that for every number n, $p^*(t_{f(n)}) = n$. Let $\sigma = j(\delta p, \varepsilon)$ and for $x \in \sigma$ define $q(x)$ by,

$$q(x) = \begin{cases} x, & \text{for} \quad p^*k(x) = 0 \ \& \ l(x) = 0 , \\ j(pk(x), a_{f(p^*k(x)-1)} - 1), & \text{for} \quad p^*k(x) > 0 \ \& \ l(x) = 0 , \\ j(k(x), l(x) - 1), & \text{for} \quad l(x) > 0 . \end{cases}$$

Clearly σ is r.e. and therefore $q(x)$ is a partial recursive function. Since $\varrho t_f \subset \delta p$ it follows that $\sum_0^\infty j(t_n, \nu(a_n)) \subset \delta q$. It is readily seen that $q(x)$ is a regressing function for the above enumeration of $\sum_0^\infty j(t_n, \nu(a_n))$.

THEOREM 2. *Let a_n and b_n be recursive functions. Then for every regressive isol T,*

$$\sum_T (a_n + b_n) = \sum_T a_n + \sum_T b_n .$$

PROOF. If T is finite the statement is clear. Now assume that T is infinite. Let t_n be any regressive function ranging over a set in T. If for each n we let $\sigma_n = \{x + a_n \mid x < b_n\}$, then

$$\sum_0^\infty j(t_n,\nu(a_n)) \in \textstyle\sum_T a_n ,$$

$$\sum_0^\infty j(t_n,\nu(b_n)) \in \textstyle\sum_T b_n ,$$

$$\sum_0^\infty j(t_n,\nu(a_n+b_n)) \in \textstyle\sum_T (a_n+b_n) ,$$

$$\sum_0^\infty j(t_n,\nu(a_n+b_n)) = \sum_0^\infty j(t_n,\nu(a_n))+\sum_0^\infty j(t_n,\sigma_n) .$$

Therefore in order to complete the proof, it suffices to prove

$$\sum_0^\infty j(t_n,\nu(a_n)) \;\Big|\; \sum_0^\infty j(t_n,\sigma_n) , \tag{5}$$

$$\sum_0^\infty j(t_n,\nu(a_n)) \cong \sum_0^\infty j(t_n,\sigma_n) . \tag{6}$$

Let $p(x)$ be a regressing function of t_n. Set

$$\delta = \{j(x,y) \mid x \in \delta p \;\;\&\;\; y < a_{p*(x)}\}$$

and

$$\eta = \{j(x,y) \mid x \in \delta p \;\;\&\;\; y \geqq a_{p*(x)}\} .$$

Then δ and η are disjoint r.e. sets with

$$\sum_0^\infty j(t_n,\nu(a_n)) \subset \delta \quad \text{and} \quad \sum_0^\infty j(t_n,\sigma_n) \subset \eta .$$

This verifies (5). Concerning (6), let

$$q(x) = j(k(x),l(x)+a_{p*k(x)}), \qquad \text{for } x \in j(\delta p,\varepsilon) .$$

Clearly $q(x)$ is a partial recursive function. Also, if $q(x)=q(y)$ for some x and y belonging to δq, then

$$\begin{aligned} q(x) = q(y) \quad &\Rightarrow \quad k(x) = k(y) \;\;\&\;\; l(x)+a_{p*k(x)} = l(y)+a_{p*k(y)} \\ &\Rightarrow \quad k(x) = k(y) \;\;\&\;\; l(x) = l(y) \\ &\Rightarrow \quad x = y . \end{aligned}$$

Therefore $q(x)$ is also one-to-one. It now readily follows that

$$\sum_0^\infty j(t_n,\nu(b_n)) \subset \delta q \quad \text{and} \quad q\colon \sum_0^\infty j(t_n,\nu(b_n)) \to \sum_0^\infty j(t_n,\sigma_n) .$$

This verifies (6) and completes the proof of Theorem 2.

We obtain the following as an immediate corollary of Theorem 2.

COROLLARY 1. *Let a_n and b_n be two recursive functions such that for all $n, a_n \leqq b_n$. Then for all regressive isols T,*

$$\textstyle\sum_T (b_n - a_n) = \sum_T b_n - \sum_T a_n .$$

NOTATION. Let

$$\underbrace{a_0 + a_1 + a_2 + \ldots}_{T} \quad \text{denote } \textstyle\sum_T a_n .$$

PROPOSITION 1. *Let a_n be a recursive function and u any number. Then for every regressive isol T, $T+1$ is also a regressive isol and*

$$\underbrace{u + a_0 + a_1 + a_2 + \ldots}_{T+1} = u + \underbrace{a_0 + a_1 + a_2 + \ldots}_{T} .$$

PROOF. Left to the reader.

5. The extensions $C_f(X)$ and $D_f(X)$.

With every recursive function $f(x)$, Myhill [6] associates two specific recursive combinatorial functions $f^+(x)$ and $f^-(x)$, called the *positive* and *negative* parts of $f(x)$, respectively. These functions have the property that for all x, $f(x) = f^+(x) = f^-(x)$. The Nerode extension of $f(x)$ to Λ may then be given by

$$D_f(X) = C_{f^+}(X) - C_{f^-}(X), \qquad \text{for } X \in \Lambda .$$

It can be proven that if $f^+(x)$ and $f^-(x)$ are any two recursive combinatorial functions, such that for all x

$$f(x) = f^+(x) - f^-(x) ,$$

then

$$D_f(X) = C_{f^+}(X) - C_{f^-}(X), \qquad \text{for } X \in \Lambda . \tag{7}$$

In this section we wish to give a representation of $D_f(T)$ (for f a recursive function and T a regressive isol) as the difference of two infinite series, each summed with respect to the regressive isol $T+1$. In view of (7) it will be sufficient to describe a method whereby $C_f(T)$ (for f a recursive combinatorial function and T a regressive isol) can be expressed as an infinite series summed with respect to $T+1$.

DEFINITION. Let $f(x)$ be a function from ε into ε. The functions

$$\begin{aligned} d(n) &= f(n+1) \dot{-} f(n) , \\ e(0) &= f(0) , \\ e(n+1) &= d(n) , \end{aligned}$$

are the *d-difference* and *e-difference* functions of $f(x)$, respectively. We shall also write d_n for $d(n)$ and e_n for $e(n)$.

It is an easy consequence of the above definition that the d-difference and e-difference functions of recursive functions are also recursive. In addition, if $f(x)$ is recursive and combinatorial, then so is $d(x)$ and we have for all x,

$$f(x) = f(0) + \sum_{n<x} d_n .$$

THEOREM 3. *Let $f(x)$ be a recursive combinatorial function. Let $d(x)$ be the d-difference function of $f(x)$. Then for all regressive isols T,*

$$C_f(T) = f(0) + \textstyle\sum_T d_n .$$

PROOF. Let $s(n)$ be the partial sum function of $d(n)$. Since $d(n)$ is recursive and combinatorial, so is $s(n)$. Also, we have for all x

$$s(x) = \sum_{n<x} d_n , \tag{8}$$

$$f(x) = f(0) + \sum_{n<x} d_n , \tag{9}$$

$$f(x) = f(0)+s(x) . \tag{10}$$

Identity (10) concerns only recursive combinatorial functions and therefore yields by a well-known theorem of Myhill

$$C_f(X) = f(0)+C_s(X), \qquad \text{for } X \in \Lambda . \tag{11}$$

By applying Theorem A to (8) we obtain

$$C_s(T) = \textstyle\sum_T d_n, \qquad \text{for } T \in \Lambda_R . \tag{12}$$

Combining (11) and (12) we obtain the desired result

$$C_f(T) = f(0) + \textstyle\sum_T d_n, \qquad \text{for } T \in \Lambda_R .$$

COROLLARY 2. *Let $f(x)$ be a recursive combinatorial function and $e(x)$ its e-difference function. Then for all regressive isols T,*

$$C_f(T) = \textstyle\sum_{T+1} e_n .$$

PROOF. Use Theorem 3, Proposition 1 and the definition of e_n.

COROLLARY 3. *Let $f(x)$ be a recursive function. Let $f^+(x)$ and $f^-(x)$ be any two recursive combinatorial functions such that, for all x, $f(x) = f^+(x) - f^-(x)$. Let $e^+(x)$ and $e^-(x)$ be the e-difference functions of $f^+(x)$ and $f^-(x)$, respectively. Then for all regressive isols T,*

$$D_f(T) = \textstyle\sum_{T+1} e_n^+ - \sum_{T+1} e_n^- .$$

PROOF. Use (7) and Corollary 2.

LEMMA 1. *Let $f(x)$ be an increasing recursive function. Let $f^+(x)$ and $f^-(x)$ be any two recursive combinatorial functions such that, for all x, $f(x)=f^+(x)-f^-(x)$. Let $e(x)$, $e^+(x)$ and $e^-(x)$ denote the e-difference functions of $f(x)$, $f^+(x)$ and $f^-(x)$, respectively. Then for all x,*

$$e^+(x) \geqq e^-(x) ,$$
$$e(x) = e^+(x) - e^-(x) .$$

PROOF. Left to the reader.

PROPOSITION 2. *Let $f(x)$ be an increasing recursive function and $e(x)$ its e-difference function. Then for all regressive isols T,*

$$D_f(T) = \textstyle\sum_{T+1} e_n .$$

PROOF. Since every recursive function can be expressed as the difference of two recursive combinatorial functions, the desired conclusion will follow from Lemma 1, Corollary 3 and Theorem 2.

COROLLARY 4. *Let $f(x)$ be an increasing recursive function. Then for all regressive isols T,*

$$D_f(T) \in \Lambda_R .$$

PROOF. Use Proposition 2 and Theorem 1.

6. The principal theorem.

Let $f(x)$ and $g(x)$ be any two recursive combinatorial functions. Then it is well-known that both of the functions $[f+g](x)$ and $fg(x)$ are also recursive combinatorial and moreover for all isols X,

$$C_{f+g}(X) = C_f(X) + C_g(X) , \tag{13}$$

$$C_{fg}(X) = C_f(C_g(X)) . \tag{14}$$

We shall use (13) and (14) in proving the following two lemmas.

LEMMA 2. *Let $f(x)$ be a recursive function and $h(x)$ a recursive combinatorial function. Then*

$$D_{fh}(X) = D_f(C_h(X)), \qquad \text{for } X \in \Lambda . \tag{15}$$

PROOF. Let $f^+(x)$ and $f^-(x)$ be any two recursive combinatorial functions such that for all x,

$$f(x) = f^+(x) - f^-(x) .$$

Since $h(x)$ is recursive and combinatorial, so are $f^+h(x)$ and $f^-h(x)$, and for all x,

$$fh(x) = f^+h(x) - f^-h(x)\,.$$

From (7) it now follows that for all isols X,

$$D_f(X) = C_{f^+}(X) - C_{f^-}(X)\,, \tag{16}$$

$$D_{fh}(X) = C_{f^+h}(X) - C_{f^-h}(X)\,. \tag{17}$$

In view of (14), (17) yields

$$D_{fh}(X) = C_{f^+}\big(C_h(X)\big) - C_{f^-}\big(C_h(X)\big), \qquad \text{for } X \in \Lambda\,. \tag{18}$$

We can now obtain (15) from (16) and (18). This completes the proof of Lemma 2.

Lemma 3. *Let $f(x)$ be a recursive function. Let $f^{\wedge}(x)$ and $f^{\vee}(x)$ be any two recursive functions such that*

$$f(x) = f^{\wedge}(x) - f^{\vee}(x), \qquad \text{for } x \in \varepsilon\,.$$

Then

$$D_f(X) = D_{f^{\wedge}}(X) - D_{f^{\vee}}(X), \qquad \text{for } X \in \Lambda\,. \tag{19}$$

Proof. Let $g^+(x)$ and $g^-(x)$, and $h^+(x)$ and $h^-(x)$ be any two pairs of recursive combinatorial functions such that, for all x

$$f^{\wedge}(x) = g^+(x) - g^-(x) \quad \text{and} \quad f^{\vee}(x) = h^+(x) - h^-(x)\,. \tag{20}$$

It readily follows that $[g^+ + h^-](x)$ and $[g^- + h^+](x)$ are also recursive combinatorial functions and such that, for all x

$$f(x) = [g^+ + h^-](x) - [g^- + h^+](x)\,. \tag{21}$$

In view of (7), (20) and (21) imply that for all isols X,

$$D_f(X) = C_{g^+}(X) - C_{g^-}(X)\,, \tag{22}$$

$$D_f(X) = C_{h^+}(X) - C_{h^-}(X)\,, \tag{23}$$

$$D_f(x) = C_{g^+ + h^-}(X) - C_{g^- + h^+}(x)\,. \tag{24}$$

Application of (13) to (24) yields

$$D_f(X) = [C_{g^+}(X) + C_{h^-}(X)] - [C_{g^-}(X) + C_{h^+}(X)], \qquad \text{for } X \in \Lambda\,. \tag{25}$$

Algebraic rearrangement of (25) together with (22) and (23) give (19). This completes the proof of Lemma 3.

Theorem 4. *Let $f(x)$ be a recursive function. Then the following conditions are equivalent.*

(26) *$f(x)$ is an eventually increasing recursive function.*

(27) $$D_f(X) \text{ maps } \Lambda_R \text{ into } \Lambda_R.$$

PROOF. We first prove that (26) implies (27). Let $f(x)$ be an eventually increasing recursive function. Obviously for every finite regressive isol t, $D_f(t)$ is finite and hence also regressive.

Suppose now T is an infinite regressive isol. Let n be a number such that $g(x)=f(x+n)$ is an increasing recursive function. Both $f(x)$ and $g(x)$ are recursive functions. Also, it is well-known that $x+n$ is a recursive combinatorial function of x with canonical extension $X+n$. We can now conclude from Lemma 2, that

(28) $$D_g(X) = D_f(X+n), \qquad \text{for } X \in \Lambda .$$

Since T is an infinite regressive isol, so is $T-n$. Substitution in (28) gives

(29) $$D_g(T-n) = D_f(T) .$$

In view of Corollary 4, $D_f(T)$ is a regressive isol. This gives (27) and completes half of the proof.

We shall prove that (27) implies (26) by proving the following statement.

(30) $\left\{\begin{array}{l}\text{Let } f(x) \text{ be a recursive function which is not eventu-}\\ \text{ally increasing. Then there exists a regressive isol } T\\ \text{such that } D_f(T) \text{ is not an isol, that is } D_f(T) \in \Lambda^* - \Lambda.\end{array}\right.$

Let $f(x)$ be a recursive function which is not eventually increasing. Define the functions

$$\begin{aligned} \Delta(0) &= f(0) , \\ \Delta(n) &= f(n)-f(n-1) \qquad \text{for } n>0 , \\ a_n &= \max\big(0,\Delta(n)\big) , \\ b_n &= \max\big(0, -\Delta(n)\big) . \end{aligned}$$

$\Delta(n)$ will not map ε into ε. However, it readily follows from the recursiveness of $f(x)$ that a_n and b_n are recursive functions of n. Also, it follows that

(31) $$(\forall n)[\min(a_n,b_n) = 0] ,$$

(32) $$(\forall x)\left[f(x) = \sum_{n\leqq x} a_n - \sum_{n\leqq x} b_n\right].$$

Set

$$f^{\wedge}(x) = \sum_{n\leqq x} a_n \quad \text{and} \quad f^{\vee}(x) = \sum_{n\leqq x} b_n .$$

Then $f^{\wedge}(x)$ and $f^{\vee}(x)$ are increasing recursive functions such that, for all x

(33) $$f(x) = f^{\wedge}(x) - f^{\vee}(x) .$$

By Lemma 3, (33) yields

(34) $$D_f(X) = D_{f^{\wedge}}(X) - D_{f^{\vee}}(X), \qquad \text{for } X \in \Lambda .$$

$f^{\wedge}(x)$ and $f^{\vee}(x)$ are increasing recursive functions; in view of their definitions they will have a_n and b_n as their respective e-difference functions. Therefore (34) and Proposition 2 imply

(35) $$D_f(T) = \sum_{T+1} a_n - \sum_{T+1} b_n, \qquad \text{for } T \in \Lambda_R .$$

It now follows that in order to prove statement (30), it suffices to prove the existence of a regressive isol T, such that

(36) $$\text{not}[\sum_{T+1} a_n \geqq \sum_{T+1} b_n] .$$

This will be our approach here and we shall use a technique introduced in the proof of [3, Theorem 95].

We first observe that the recursive functions a_n and b_n are each positive infinitely often. This follows from (32) and the fact that the function $f(x)$ is recursive and not eventually increasing. Therefore the sets $\{n \mid a_n > 0\}$ and $\{n \mid b_n > 0\}$ are each infinite and recursive, and by (31) are disjoint. Let $g(n)$ and $h(n)$ denote the strictly increasing recursive functions ranging over $\{n \mid a_n > 0\}$ and $\{n \mid b_n > 0\}$, respectively.

Let $p_i(x)$ be a function of the two variables i and x not necessarily everywhere defined such that every one-to-one partial recursive function, and no other function, appears in the sequence $\{p_i\}$. We shall now define a regressive function t_n, such that if $T+1 = \text{Req}(\rho t)$, then T will be an infinite regressive isol satisfying (36).

Set $t_0 = 1$. Assuming that $t_0, \ldots, t_{k-1}$ with $k \geqq 1$ have already been defined, we define t_k by setting

$$t_k = j(t_{k-1}, u_k) ,$$

where u_k is determined as follows:

Case 1: $k \notin \rho g$ and $k \notin \rho h$. Set $u_k = 0$.

Case 2: $k \in \rho g$, say $k = g(i)$. Let $h(0), \ldots, h(s)$ be all those $h(r)$'s with $h(r) < g(i)$. Now put u_k equal to the smallest number w such that for each v with $0 \leqq v < a_{g(i)}$,

$$j(j(t_{k-1}, w), v) \notin \left(p_0 j(t_{h(0)}, 0), \ldots, p_s j(t_{h(s)}, 0)\right) .$$

Case 3: $k \in \rho h$, say $k = h(i)$. Let $g(0), \ldots, g(s)$ be all those $g(r)$'s with $g(r) < h(i)$. Now put u_k equal to the smallest number w such that either

$$p_i\, j\big(j(t_{k-1}, w), 0\big) \text{ is undefined},$$

or $p_i j\big(j(t_{k-1}, w), 0\big)$ is defined and

$$p_i\, j\big(j(t_{k-1}, w), 0\big) \notin \{j(t_{g(n)}, v) \mid n \leqq s \;\&\; v < a_{g(n)}\}.$$

The existence of u_k in Case 2 follows because j maps ε^2 one-to-one onto ε. We obtain the existence of u_k in Case 3 because each of the functions p_i is one-to-one.

Let

$$\tau = \varrho t, \quad \tau' = \tau - (t_0) \quad \text{and} \quad T = \operatorname{Reg}(\tau').$$

To complete the proof we will verify:

(i) t_n is a regressive function,
(ii) T and $T+1$ are regressive R.E.T.s,
(iii) T satisfies (36),
(iv) T is an isol.

Re (i). It follows from

$$j(x,y) = \tfrac{1}{2}(x+y)(x+y+1)+x$$

that $x<j(x,y)$ for $x>0$. By the definition of t we have

$$t_0 > 0 \quad \text{and} \quad (\forall n)(\exists u)[t_{n+1} = j(t_n, u)]$$

Hence $t_0 < t_1 < t_2 < \ldots$, and therefore t is a one-to-one function. Let

$$g(x) = \begin{cases} t_0, & \text{for } x = t_0, \\ k(x), & \text{for } x \neq t_0. \end{cases}$$

It readily follows that $g(x)$ is a regressing function of t_n. Hence t_n is a regressive function.

Re (ii). Set $t'_n = t_{n+1}$. Then t'_n is obviously also a regressive function. Since $\tau = \varrho t$ and $\tau' = \varrho t'$, τ and τ' are regressive sets. Also, $\tau' \in T$ and $\tau \in T+1$, and therefore T and $T+1$ are regressive R.E.T.s.

Re (iii). Since t_n is a regressive function ranging over a set in $T+1$, we have

$$\sum_0^\infty j\big(t_n, \nu(a_n)\big) \in \textstyle\sum_{T+1} a_n, \tag{37}$$

$$\sum_0^\infty j\big(t_n, \nu(b_n)\big) \in \textstyle\sum_{T+1} b_n. \tag{38}$$

In addition, since the functions $g(n)$ and $h(n)$ range over all the positive a_n and b_n, respectively, it follows that

$$\sum_0^\infty j(t_n, \nu(a_n)) = \sum_0^\infty j(t_{g(n)}, \nu(a_{g(n)})), \tag{39}$$

$$\sum_0^\infty j(t_n, \nu(b_n)) = \sum_0^\infty j(t_{h(n)}, \nu(b_{h(n)})). \tag{40}$$

We wish to prove that T satisfies (36). Let us suppose otherwise, namely

$$\textstyle\sum_{T+1} b_n \leqq \sum_{T+1} a_n . \tag{41}$$

Then in view of (37), (38), (39) and (40) it follows that there would exist a one-to-one partial recursive function $p(x)$, such that

$$\sum_0^\infty j(t_{h(n)}, \nu(b_{h(n)})) \subset \delta p , \tag{42}$$

$$p\left(\sum_0^\infty j(t_{h(x)}, \nu(b_{h(n)}))\right) \subset \sum_0^\infty j(t_{g(n)}, \nu(a_{g(n)})). \tag{43}$$

We shall prove that this is not possible. Suppose the index of $p(x)$ in our enumeration of all one-to-one partial recursive functions is i, that is, $p(x) = p_i(x)$. Then from (42) and (43) we would have in particular for some number r,

$$j(t_{h(i)}, 0) \in \delta p_i , \tag{44}$$

$$p_i j(t_{h(i)}, 0) \in (j(t_{g(r)}, 0), \ldots, j(t_{g(r)}, a_{g(r)} - 1)). \tag{45}$$

The functions $g(n)$ and $h(n)$ have disjoint ranges and therefore either $h(i) < g(r)$ or $g(r) < h(i)$.

Suppose that $h(i) < g(r)$. By Case 2 of the construction, $t_{g(n)}$ is so defined that for each v with $0 \leqq v < a_{g(r)}$,

$$j(t_{g(r)}, v) \notin \{p_n j(t_{h(n)}, 0) \mid h(n) < g(r)\} .$$

Since $h(i) < g(r)$, this implies

$$p_i j(t_{h(i)}, 0) \notin (j(t_{g(r)}, 0), \ldots, j(t_{g(r)}, a_{g(r)} - 1)) ,$$

which would be false in view of (45). Therefore $h(i) < g(r)$ would not be possible.

Suppose now that $g(r) < h(i)$. By Case 3 of the construction, $t_{h(i)}$ is so defined that either $p_i j(t_{h(i)}, 0)$ is undefined or $p_i j(t_{h(i)}, 0)$ is defined and

$$p_i j(t_{h(i)}, 0) \notin \{j(t_{g(n)}, v) \mid g(n) < h(i) \ \& \ v < a_{g(n)}\} . \tag{46}$$

In view of (44) we may assume that $p_i j(t_{h(i)}, 0)$ is defined. Since $g(r) < h(i)$, (46) yields

$$p_i j(t_{h(i)}, 0) \notin \left(j(t_{g(r)}, 0), \ldots, j(t_{g(r)}, a_{g(r)} - 1)\right),$$

which would be false in view of (45). Therefore $g(r) < h(i)$ would also not be possible.

We can now conclude that there exists no one-to-one partial recursive function satisfying conditions (42) and (43). Therefore the inequality of (41) is false and T will be a regressive R.E.T. satisfying (36). This completes the proof of (iii).

Re (*iv*). An infinite set is *retraceable*, if it is the range of a strictly increasing regressive function. As a consequence, the function t_n mentioned in the proof of (*i*) is a strictly increasing regressive function, and hence τ is a retraceable set. It is proved in [4] that every retraceable set is either recursive or immune.

We now prove that τ is not a recursive set. Suppose otherwise, then the strictly increasing function ranging over τ, namely t_n, would be recursive. Since $g(n)$ and $h(n)$ are also recursive functions, it would then be an easy consequence that

$$\sum_0^\infty j\left(t_{g(n)}, \nu(a_{g(n)})\right) \quad \text{and} \quad \sum_0^\infty j\left(t_{h(n)}, \nu(b_{h(n)})\right)$$

are infinite r.e. sets. However, then these sets would be recursively equivalent, contrary to our above remarks.

Since τ is retraceable and not recursive, τ is immune. Since $\tau \in T+1$, it follows that $T+1$ and hence also T is an isol. This verifies (iv) and completes the proof of Theorem 4.

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