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DIOPHANTINE EQUATIONS IN RECURSIVE DIFFERENCE

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We determine the general solutions (in natural numbers) of a variety of linear equations involving the recursive difference function $x \dot{-} y$ which has the value 0 when $x \leqq y$ and is equal to the excess of x over y when $x > y$. We assume a familiarity with properties of recursive difference, in particular the properties

$$x+(y \dot{-} x) = y+(x \dot{-} y), \qquad x \dot{-} (x \dot{-} y) = y \dot{-} (y \dot{-} x)\,,$$
$$(x+y) \dot{-} z = (x \dot{-} z)+\{y \dot{-} (z \dot{-} x)\}\,,$$
$$x = (x \dot{-} y)+\{x \dot{-} (x \dot{-} y)\}$$
$$x(y \dot{-} z) = xy \dot{-} xz\,.$$

To illustrate the kind of results to be obtained, we consider first the equation

$$x+(y \dot{-} x) = a\,. \tag{1}$$

The general solution of this equation is

$$\begin{cases} x = a \dot{-} (u \dot{-} v) \\ y = a \dot{-} (v \dot{-} u) \end{cases} \tag{1*}$$

where u,v are arbitrary parameters. For if x,y are given by (1*) then

$$\begin{aligned} x+(y \dot{-} x) &= \{a \dot{-} (u \dot{-} v)\}+\big\{\big(a \dot{-} (v \dot{-} u)\big) \dot{-} \big(a \dot{-} (u \dot{-} v)\big)\big\} \\ &= a \end{aligned}$$

(consider in turn the cases $u \leqq v$, $u > v$), so that equation (1) is satisfied. Conversely, since

$$x = \{x+(y \dot{-} x)\} \dot{-} (y \dot{-} x)$$

and

$$y = \{x+(y \dot{-} x)\} \dot{-} (x \dot{-} y)$$

identically, therefore *any* solution x,y of (1) may be obtained from (1*)

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giving u the value y and v the value x. Similarly, the general solution of the equation

(2) $$x \dot- (x \dot- y) = a$$

is

(2*) $$\begin{cases} x = a + (u \dot- v) \\ y = a + (v \dot- u) . \end{cases}$$

We consider next the equation

(3) $$x \dot- y = a .$$

The general solution of (3) is

(3*) $$x = a + \{y \dot- (1 \dot- a)u\} .$$

For if x satisfies 3* then

$$\begin{aligned} x \dot- y &= [\{y \dot- (1 \dot- a)u\} \dot- y] + [a \dot- \{y \dot- (y \dot- (1 \dot- a)u)\}] \\ &= a \dot- \{y \dot- (y \dot- (1 \dot- a)u)\} \\ &= a \end{aligned}$$

(consider in turn the cases $a = 0$, $a \geqq 1$) so that all x given by (3*) satisfy (3). Conversely, since

$$x = (x \dot- y) + \{y \dot- (1 \dot- (x \dot- y))(y \dot- x)\}$$

therefore any solution (x,y) of (3) may be obtained from (3*) giving u the value $y \dot- x$.

It follows that the general solution of

(3.1) $$x_1 \dot- (x_2 \dot- (x_3 \dot- \ldots - (x_n \dot- u_n) \ldots) = u_0$$

is

(3.2) $$x_r = u_{r-1} + \{u_r \dot- (1 \dot- u_{r-1})v_r\}, \qquad 1 \leqq r \leqq n ,$$

where $u_1, u_2, \ldots, u_{n-1}, v_1, v_2, \ldots, v_n$ are arbitrary parameters. For if we write u_{n-1} for $x_n \dot- u_n$, u_{n-2} for $x_{n-1} \dot- u_{n-1}$ and so on up to u_1 for $x_2 \dot- u_2$ then also $x_1 \dot- u_1 = u_0$, so that

$$x_r \dot- u_r = u_{r-1}, \qquad 1 \leqq r \leqq n ,$$

and by (2*) the general solution of this system of equations is (3.2).

We turn next to the equation

(4) $$x \dot- a = y \dot- b$$

of which the general solution is

(4*) $$y = (x \dot- a) + \{b \dot- (1 \dot- (x \dot- a))u\} .$$

We omit the verification that (4*) satisfies (4). That every solution of (4) is contained in (4*) follows from the identity

$$y = (y \dot{-} b) + \{b \dot{-} (1 \dot{-} (y \dot{-} b))(b \dot{-} y)\}$$

which shows that if (x,y) is solution of (4) then this y may be obtained from (4*) by giving u the value $b \dot{-} y$.

Similarly, the equation

(5) $$a \dot{-} x = a \dot{-} y$$

has the general solution

(5*) $$y = \{a \dot{-} (a \dot{-} x)\} + \{1 \dot{-} (a \dot{-} x)\}u\ .$$

The verification that (5*) satisfies (5) is trivial. The generality of the solution follows from the identity

$$y = \{a \dot{-} (a \dot{-} y)\} + \{1 \dot{-} (a \dot{-} y)\}(y \dot{-} a)\ .$$

The solution of the apparantly more general equation

(6) $$a \dot{-} x = b \dot{-} y$$

is readily derived from (5*). For we may suppose, without loss of generality, that $a \leqq b$ and so $a = b \dot{-} (b \dot{-} a)$ whence

$$b \dot{-} y = \big(b \dot{-} (b \dot{-} a)\big) \dot{-} x = b \dot{-} \big(x + (b \dot{-} a)\big)$$

of which the general solution is (by (5*))

(6*) $$\begin{aligned} y &= [b \dot{-} \{b \dot{-} (x + (b \dot{-} a))\}] + [1 \dot{-} \{b \dot{-} (x + (b \dot{-} a))\}]u \\ &= \{b \dot{-} (a \dot{-} x)\} + \{1 \dot{-} (a \dot{-} x)\}u\ . \end{aligned}$$

Although the equation

(7) $$x + y = a$$

involves only elementary addition its general solution *in natural numbers* is

(7*) $$\begin{cases} x = a \dot{-} u \\ y = a \dot{-} (a \dot{-} u) \end{cases}$$

and so depends upon the recursive difference function. That (7*) is a solution of (7) for any value of u follows from the identity

(7.1) $$(a \dot{-} u) + \{a \dot{-} (a \dot{-} u)\} = a$$

and that (7*) is the general solution is shown by the identity

$$\{(x+y) \dot{-} y\} + [(x+y) \dot{-} \{(x+y) \dot{-} y\}] = x + y$$

which reveals that any pair (x,y) which satisfy (7) may be obtained from (7*) by giving u the value y.

From 7.1 we readily derive the general solution of

$$(8) \qquad x_1+x_2+\ldots+x_{n+1} = a\,.$$

Write $s_0=0$, $s_{k+1}=s_k+u_k$ so that

$$s_k = u_1+u_2+\ldots+u_k$$

where $u_1,u_2,\ldots,u_n$ are arbitrary parameters, then the general solution of (8) is

$$(8^*) \qquad \begin{cases} x_k = (a \dot{-} s_{k-1}) \dot{-} (a \dot{-} s_k), & 1 \leqq k \leqq n\,, \\ x_{n+1} = a \dot{-} s_n\,. \end{cases}$$

For by (7.1)

$$(a \dot{-} s_k)+\{(a \dot{-} s_{k-1}) \dot{-} (a \dot{-} s_k)\} = a \dot{-} s_{k-1}, \qquad 1 \leqq k \leqq n\,,$$

and so, by addition,

$$\sum_{k=1}^{n}\{(a \dot{-} s_{k-1}) \dot{-} (a \dot{-} s_k)\}+(a \dot{-} s_n) = a$$

which proves that (8*) is a solution of (8) for all $u_1,u_2,\ldots,u_n$. Conversely, writing $X_{k+1}=X_k+x_k$, $X_0=0$, we have

$$\{X_{n+1} \dot{-} X_k\} \dot{-} \{X_{n+1} \dot{-} X_{k+1}\} = x_{k+1}, \qquad 0 \leqq k \leqq n\,,$$

so that any set of values $(x_1,x_2,\ldots,x_{n+1})$ which satisfy (8) may be obtained from (8*) by giving u_k the value x_k, $1 \leqq k \leqq n$.

As a final and rather more difficult example we consider the equation

$$(9) \qquad (x_0 \dot{-} x_1)+(x_1 \dot{-} x_2)+\ldots+(x_n \dot{-} x_0) = a\,.$$

The particular case of (9), with $n=1$,

$$(9.1) \qquad (x \dot{-} y)+(y \dot{-} x) = a$$

has the general solution

$$x = u+a(1 \dot{-} v), \quad y = u+a\{1 \dot{-} (1 \dot{-} v)\}$$

since

$$\{u+a(1 \dot{-} v)\} \dot{-} [u+a\{1 \dot{-} (1 \dot{-} v)\}] = a(1 \dot{-} v)$$

and, if $\min(x,y)=x \dot{-} (x \dot{-} y)$, so that $\min(x,y)$ is equal to the lesser of x and y,

$$\min(x,y) + \{(x \dot{-} y)+(y \dot{-} x)\}\{1 \dot{-} (y \dot{-} x)\} = x\,,$$
$$\min(x,y) + \{(x \dot{-} y)+(y \dot{-} x)\}\{1 \dot{-} \big(1 \dot{-} (y \dot{-} x)\big)\} = y\,.$$

Next we consider the particular case of (9) with $n=2$, namely

$$(x \dot{-} y)+(y \dot{-} z)+(z \dot{-} x) = a\,. \tag{9.2}$$

Let $[x,y,z]$ denote the left hand side of (9.2), let

$$\min(x,y,z) = x \dot{-} \{x \dot{-} (y \dot{-} (y \dot{-} z))\}\,,$$

so that $\min(x,y,z)$ is equal to the least of x, y, z, let

$$\max(x,y) = x+(y \dot{-} x)\,,$$

the greater of x,y, and finally let

$$\mu(x,y,z) = 1 \dot{-} [1 \dot{-} \{x \dot{-} \min(y,z)\}]$$

so that $\mu(x,y,z)=0$ if $x \leqq y$ and $x \leqq z$ and $\mu(x,y,z)=1$ otherwise. Then the general solution of (9.2) is

$$\begin{aligned} x &= t + [a\mu(u,v,w) \dot{-} \{\max(v,w) \dot{-} u\}]\,, \\ y &= t + [a\mu(v,w,u) \dot{-} \{\max(w,u) \dot{-} v\}]\,, \\ z &= t + [a\mu(w,u,v) \dot{-} \{\max(u,v) \dot{-} w\}]\,, \end{aligned} \tag{9.2*}$$

u, v, w not all equal.

To show that (9.2*) satisfy 9.2 we consider in turn the six cases

$$(\alpha)\ u<v\leqq w, \quad (\beta)\ u\leqq w<v, \quad (\gamma)\ v<w\leqq u, \quad (\delta)\ v\leqq u<w,$$
$$(\varepsilon)\ w<u\leqq v, \quad (\xi)\ w\leqq v<u\,.$$

In the first case $\mu(u,v,w)=0$, $\mu(v,w,u)=\mu(w,u,v)=1$ so that $x=t$, $y=t+\{a \dot{-} (w \dot{-} v)\}$, $z=t+a$ and therefore (9.2), with the same result in the remaining cases. That (9.2*) is the general solution follows from the identity

$$x = \min(x,y,z) + \big\{[x,y,z]\mu(x,y,z) \dot{-} \big(\max(y,z) \dot{-} x\big)\big\} \tag{9.3}$$

and the two corresponding results obtained by cyclic interchange of x, y, z. Equation (9.3) shows that if (x,y,z) is any solution of (9.2) then the value of x may be obtained from (9.2*) by giving t the value $\min(x,y,z)$ and u, v, w the values x, y, z respectively.

To prove (9.3) we consider six cases of which

$$y \leqq x \leqq z, \quad y \neq z\,,$$

is typical; in this case the right hand side of (9.3) becomes

$$\begin{aligned} y+\{[x,y,z] \dot{-} (z \dot{-} x)\} &= y+\{(z \dot{-} y) \dot{-} (z \dot{-} x)\} \\ &= y+(x \dot{-} y) = x\,. \end{aligned}$$

The proofs in the remaining cases are similar (or simpler).

Finally, we remark that the general solution of (9) is expressible in the form

$$(9^*)\qquad x_i = t + \left[a\mu_i(u_r, n) \dot{-} \left\{\max_i(u_r, n) \dot{-} u_i\right\}\right], \qquad 0 \leqq i \leqq n\ ,$$

where not all u_r are equal, $0 \leqq r \leqq n$,

$$\mu_i(x_r, n) = 1 \dot{-} \left\{1 \dot{-} \left(x_i \dot{-} \min(x_r, n)\right)\right\},$$

$$\min(x_r, 0) = x_0, \qquad \min(x_r, n+1) = x_{n+1} \dot{-} \left(x_{n+1} \dot{-} \min(x_r, n)\right),$$

$$\max(x_r, 0) = x_0, \qquad \max(x_r, n+1) = x_{n+1} + \left(\max(x_r, n) \dot{-} x_{n+1}\right),$$

$$x_r^i = x_r \quad \text{if } r < i, \qquad x_r^i = x_{r+1} \quad \text{if } r \geqq i\ ,$$

and

$$\max_i(x_r, 0) = x_0, \qquad \max_i(x_r, n+1) = \max(x_r^i, n)\ ,$$

so that $\max_i(x_r, n)$ is the greatest value of x_r, $r \neq i$, $0 \leqq r \leqq n$. The proof follows the same lines as that of (9.2*), the part played by (9.3) being taken by the identity

$$x_i = \min(x_r, n) + \left\{[x_r, n]\mu_i(x_r, n) \dot{-} \left\{\max_i(x_r, n) \dot{-} x_i\right\}\right\}, \qquad 0 \leqq i \leqq n\ ,$$

where $[x_r, n]$ denotes the left hand side of equation (9).

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