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CONGRUENCES FOR THE COEFFICIENTS OF THE MODULAR INVARIANT $j(\tau)$

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1.

The modular invariant $j(\tau)$ is defined by

$$(1)\quad j(\tau) = x^{-1}\prod_1^\infty (1-x^n)^{-24}\left(1+240\sum_1^\infty \sigma_3(n)x^n\right)^3, \qquad x = \exp(2\pi i\tau)\,,$$

where

$$\sigma_k(n) = \sum_{d|n} d^k\,.$$

It is well known that the coefficients in the expansion

$$j(\tau) = \sum_{-1}^\infty c(n)x^n$$

have remarkable divisibility properties. Thus Lehner [7], [8] has shown that

$$(1.1)\qquad c(2^a n) \equiv 0 \pmod{2^{3a+8}}\,,$$

$$(1.2)\qquad c(3^a n) \equiv 0 \pmod{3^{2a+3}}\,,$$

$$(1.3)\qquad c(5^a n) \equiv 0 \pmod{5^{a+1}}\,,$$

$$(1.4)\qquad c(7^a n) \equiv 0 \pmod{7^a}\,,$$

for arbitrary positive integers a,n. The congruences (1.1)–(1.4) have been somewhat improved, we have

$$(1.5)\quad c(2^a n) \equiv -2^{3a+8}3^{a-1}\sigma_7(n) \pmod{2^{3a+13}}, \quad n \text{ odd}\,,$$

$$(1.6)\quad c(3^a n) \equiv \mp 3^{2a+3}10^{a-1}\sigma(n)/n \pmod{3^{2a+6}} \quad \text{if } n \equiv \pm 1 \pmod 3)\,,$$

$$(1.7)\quad c(5^a n) \equiv -5^{a+1}3^{a-1}n\sigma(n) \pmod{5^{a+2}}\,,$$

for $a>0$. Formulas (1.5) and (1.6) are due to Kolberg [2], [3], and (1.7) to the author [1]. Kolberg conjectured that (1.4) could be sharpened in a similar way, and in this note we shall deduce the congruence

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$$(1.8) \qquad c(7^a n) \equiv -7^a 5^{a-1} n\sigma_3(n) \pmod{7^{a+1}},$$

and from (1.8) and the simple congruence

$$n\sigma_3(n) \equiv 0 \pmod 7 \quad \text{if } (n/7) = -1,$$

where $(n/7)$ is Legendre's symbol, we deduce

$$c(7^a n) \equiv 0 \pmod{7^{a+1}} \quad \text{if } (n/7) = -1.$$

From (1.8) it follows especially that 7^a is the exact power of 7 dividing $c(7^a)$, as conjectured by Lehner.

The proofs of (1.5)–(1.7) are in two parts. The first part consists in proving the theorem for $a=1$. The second part of the proof proceeds by induction on a. Here the identity (2.1) for $j(\tau)$ plays an essential role.

In proving (1.8) we shall for the second part use only a slight modification of Lehner's proof. We could have proved (1.8) by the method used in [1], thus including a new proof of (1.4), but the calculations would be very tedious.

2.

Our starting point is the following lemma (see Kolberg [4]): Let p be one of the primes 2, 3, 5, 7, 13 and put

$$\Phi_p(\tau) = x\big(\varphi(x^p)/\varphi(x)\big)^{24/(p-1)}, \qquad \varphi(x) = \prod_1^\infty (1-x^n).$$

Then there exist constants A_{kp} such that

$$(2.1) \qquad j(\tau) = \sum_{k=-1}^{p} A_{kp}\Phi_p(\tau).$$

From this lemma we easily get the identity

$$(2.2) \quad j(\tau) = g^{-1} + 748 + 82\cdot 7^4 g + 176\cdot 7^6 g^2 + 845\cdot 7^7 g^3 + 272\cdot 7^9 g^4 + \\ + 46\cdot 7^{11} g^5 + 4\cdot 7^{13} g^6 + 7^{14} g^7,$$

where $g = \Phi_7(\tau) = x\big(\varphi(x^7)/\varphi(x)\big)^4$.

We introduce an operator L defined by

$$L\sum a(n)x^n = \sum a(7n)x^n.$$

For an arbitrary power series

$$f(\tau) = \sum a(n)x^n, \qquad x = e^{2\pi i\tau},$$

we define the "7-dissection", $f(\tau) = f_0 + f_1 + \ldots + f_6$, by

$$f_j = \sum a(7n+j)x^{7n+j}.$$

To find Lg^{-1} we need some results on the 7-dissection on $f(\tau)=\varphi(x)$, proved by Kolberg in [5]:

$$\varphi(x) = \varphi_0+\varphi_1+\varphi_2+\varphi_5, \qquad \varphi_3 = \varphi_4 = \varphi_6 = 0, \qquad \varphi_2 = -x^2\varphi(x^{49}),$$
$$\varphi_0\varphi_1\varphi_5 = \varphi_2, \qquad \varphi_0^3\varphi_1+\varphi_1^3\varphi_5+\varphi_5^3\varphi_0 = -x\varphi(x^7)^4-8\varphi_2^4 .$$

Using these results we get by direct computation

$$\begin{aligned}\big(\varphi(x)^4\big)_1 &= 4(\varphi_0^3\varphi_1+\varphi_1^3\varphi_5+\varphi_5^3\varphi_0)+\varphi_2^4+24\varphi_0\varphi_1\varphi_2\varphi_5\\ &= -4x\varphi(x^7)^4-7x^8\varphi(x^{49})^4 .\end{aligned}$$

Hence

$$Lg^{-1} = -4-7g . \tag{2.3}$$

We note that

$$Lf(\tau) = 7^{-1}\sum_{\lambda=0}^{6} f(\tau+\lambda)/7) ,$$

where $f(\tau)=\sum a(n)x^n$. Putting

$$h = 7^2\Phi_7(\tau/7) = 7^2 g(\tau/7) ,$$

we have from [7]

$$h^7+\sum_{j=1}^{7}(-1)^j p_j h^{7-j} = 0 , \tag{2.4}$$

where

$$(-1)^{j+1}p_j = 7^4\sum_{k=j}^{7} b_k g^{k-j+1} , \tag{2.5}$$

$$\begin{aligned} b_1 &= 82, & b_4 &= 272\cdot 7^5, & b_7 &= 7^{10} .\\ b_2 &= 176\cdot 7^2, & b_5 &= 46\cdot 7^7, & &\\ b_3 &= 845\cdot 7^3, & b_6 &= 4\cdot 7^9 , & & \end{aligned} \tag{2.6}$$

The conjugates in h of (2.4) are clearly

$$h_\lambda = 7^2 g\big((\tau+\lambda)/7\big), \qquad \lambda = 0,1,2,\ldots,6 ,$$

since replacing τ by $\tau+1$ leaves g unaltered. Hence for the sum of the conjugates we have

$$Lg(\tau) = 7^{-1}\sum_0^6 g\big((\tau+\lambda)/7\big) = 7^{-3}\sum_0^6 h_\lambda = 7^{-3}p_1 = 7\sum_1^7 b_k g^k . \tag{2.7}$$

We introduce the symbols

$$P = a_1 7g+a_2 7^2g^2+\ldots+a_r 7^r g^r ,$$
$$Q = b_1 g+b_2 7^2 g^2+\ldots+b_t 7^t g^t ,$$

where the a's, b's, r and t are integers $(r \geqq 1, t > 1)$. P denotes a polynomial of this type, not necessarily the same one at each appearance, likewise for Q. Lehner then proves

$$7^k L g^k = 7Q, \qquad k \geqq 2\,, \tag{2.8}$$

by the aid of Newton's formula for the sums of powers of the roots of an algebraic equation. It is obvious that (2.7) and (2.8) imply

$$LQ = 7Q\,. \tag{2.9}$$

Lehner's proof really implies (cf. [7, p. 147])

$$7^k L g^k = 7P, \qquad k \geqq 2\,. \tag{2.10}$$

An immediate consequence of (2.7) and (2.10) is the equation

$$LP = 7P\,. \tag{2.11}$$

3.

We now rewrite (2.2) and (2.7) using the symbol P:

$$j(\tau) = g^{-1} + 748 + 7^3 P\,. \tag{3.1}$$

$$Lg = 7 \cdot 82 g + 7P\,. \tag{3.2}$$

Comparing (2.3), (2.11) and (3.1) we obtain

$$Lj(\tau) = 744 - 7g + 7^4 P\,.$$

(3.2) and the last equation yield

$$L^2 j(\tau) = 744 - 7^2 \cdot 82 g + 7^2 P\,,$$

and generally

$$L^a j(\tau) = 744 - 7^a (82)^{a-1} + 7^a P\,. \tag{3.3}$$

From the obvious congruence

$$\varphi(x)^7 \equiv \varphi(x^7) \pmod 7\,,$$

it follows that

$$g = x\varphi(x)^{-4}\varphi(x^7)^4 \equiv x\varphi(x)^{24} \pmod 7\,,$$

and using this and the congruence (see Kolberg [6])

$$x\varphi(x)^{24} \equiv \sum_1^\infty n\sigma_3(n)x^n \pmod 7\,,$$

we conclude that

$$\sum_{1}^{\infty} c(7^a n)x^n \equiv -7^a(82)^{a-1} \sum n\sigma_3(n)x^n \pmod{7^{a+1}}.$$

The congruences refer to the coefficients of the power series in x.

Using the identity (2.1) and Lehner's proof for $p=2,3,5$, with slight modifications we get congruences connecting $c(p^a n)$ and $c(pn)$, but the congruences are contained in the proofs of (1.5)–(1.7).

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