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ON AN INEQUALITY FOR LAURENT POLYNOMIALS

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In connection with investigation of Toeplitz matrices of Laurent polynomials Spitzer and Schmidt [1] have shown that

$$\overline{\lim_{n\to\infty}} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f^n(e^{i\theta})\, d\theta \right|^{1/n} \leqq \min_r \max_\theta |f(re^{i\theta})| \tag{1}$$

whenever f is a Laurent polynomial in z, that is, a function of the form

$$f(z) = P(z)/z^m , \tag{2}$$

where $P(z)$ is a polynomial and m is a non-negative integer. They have further shown that the inequality (1) is in fact equality in the three cases: $m=0$; $P(z)$ has at most two non-zero coefficients; all the coefficients of $P(z)$ are real and positive. They remarked that this author had found a counter-example to demonstrate that the equality does not hold in (1) for all Laurent polynomials, f. The purpose of the present paper is to exhibit this counter-example.

To this end we first observe that the expression on the left-hand side of (1) is equal to

$$\overline{\lim_{n\to\infty}} \left| \frac{1}{2\pi i} \int_C f^n(z)\, dz \right|^{1/n}$$

for every rectifiable Jordan curve, C, separating the origin from ∞. This expression is in turn less than or equal to

$$M_C(f) = \max_{z\in C} |f(z)| .$$

Introducing the notation

$$M_r(f) = \max_\theta |f(re^{i\theta})| ,$$

we now have

$$\overline{\lim_{n\to\infty}} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f^n(e^{i\theta})\, d\theta \right|^{1/n} \leqq \min_C M_C(f) \leqq \min_r M_r(f) .$$

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It will be shown that the equality does not in general hold in the last inequality.

We will exhibit a function f of the form

$$f(z) = P(z)/z ,$$

where $P(z)$ is a polynomial of degree 3, for which

$$\min_C M_C(f) < \min_r M_r(f) .$$

Let $f(z) = P(z)/z$ where $P(z)$ is a polynomial of degree three with $P(0) \neq 0$ and consider the level curves of $f(z)$, that is, the curves

$$|f(z)| = K .$$

For very large K this curve will consist of two simple closed curves, approximately circles about zero and about infinity. For very small K this curve will consist of three simple closed curves, approximately circles, about the roots of $P(z)$. Intermediate between these two stages there are some values of K for which the topological nature of the level curves changes, i.e., for which the curves have self-intersections. The self-intersections occur at the roots of the derivative of $f(z)$.

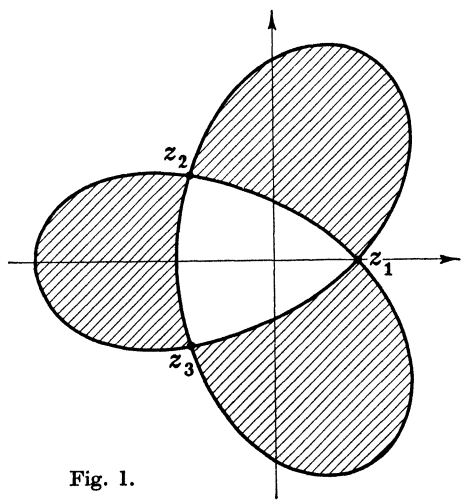

Fig. 1.

In order to construct a counter-example we shall attempt to find a polynomial $P(z)$ of degree three so that the three roots z_1, z_2, z_3 of the derivative of $f(z) = P(z)/z$

(i) are distinct;
(ii) lie on the same level curve of $f(z)$ (that is, for some K_0, $|f(z_n)| = K_0$, $n = 1, 2, 3$);
(iii) do not lie on a circle with center at the origin.

Under these circumstances it is hoped that the level curve $f(z)=K_0$ will have the character exhibited in figure 1, where the shaded region represents $\{z \mid |f(z)| \leqq K_0\}$.

If this situation occurs then we will have our counter-example. It is clear from the figure that any simple closed curve C enclosing the origin on which $|f(z)| \leqq K_0$ must lie entirely in the shaded region and hence must pass through the three points z_1, z_2, z_3.

It will still remain to show that the level curve through z_1, z_2, z_3 actually has the behavior depicted in Fig. 1 instead, say, of that of Fig. 2, which would permit of a circle with center at the origin lying in the shaded region.

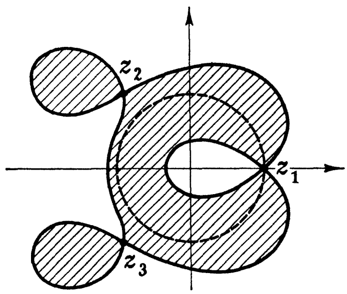

Fig. 2.

To this end it will suffice to show that for $n=1,2,3$, the function $|f(z)|$ has on the ray emanating from the origin through z_n an absolute minimum at z_n. The function

$$f(z) = \frac{22z^3+44z^2-29z+88}{z}$$

has the roots of its derivative located at

$$1, \quad -1+i, \quad -1-i ,$$

so that conditions (i), (iii) are satisfied. Moreover at each of these roots

$$|f(z)| = 125$$

so that (ii) is also satisfied. It now only remains to be verified that

$$|f(tz_n)|, \qquad 0<t<\infty, \ n=1,2,3 ,$$

has an absolute minimum at $t=1$. It will be seen, in fact, that

$$|\mathrm{Re} f(tz_n)| \quad \text{and} \quad |\mathrm{Im} f(tz_n)|$$

each has an absolute minimum at z_n for $n=1,2,3$. For $n=1$, this is clear since $f(t)$, $0<t<\infty$, has 1 as the only root of its derivative and is unbounded in the neighborhood of 0 and of ∞.

In the case $n=2$,

$$f(t(-1+i)) = -44(t+1/t)-29-44i(t^2-t+1/t) ,$$

whence it is easily seen that the absolute values of both the real and imaginary parts have absolute minima at $t=1$. The same result also holds in the case $n=3$ since $z_3=\overline{z_2}$.

REFERENCE

1. P. Schmidt and F. Spitzer, *The Toeplitz matrices of an arbitrary Laurent polynomial*, Math. Scand. 8 (1960), 15–38.

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