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REMARK ON EIGENFUNCTION EXPANSIONS FOR ELLIPTIC OPERATORS WITH CONSTANT COEFFICIENTS

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Introduction.

The purpose of this paper is to establish analogues of well-known results in Fourier series and Fourier integrals associated with the names Fejér, Lebesgue and Riemann for the eigenfunction expansion corresponding to a semi-bounded self-adjoint realization A of an elliptic operator a with constant coefficients in a domain Ω of R^n. We start (Section 2) with the convergence of L_p functions, $1 \leqq p \leqq 2$, and the special case $\Omega = R^n$. Here straight forward estimates of the spectral functions involved lead at once to the desired goal. Applying next (Section 3) a known asymptotic formula for the difference of two spectral functions we obtain as a simple corollary the same result for L_2 functions and general Ω. Finally (Section 4) we exhibit some concrete examples in the case $\Omega = R^n$ which show that the L_1 case is somewhat pathological, compared to the L_2 case. Roughly speaking, we find that the convergence of the eigenfunction expansion for L_1 functions is somewhat influenced by the differential geometry of the surfaces $a(\xi) = \lambda$. In the L_2 case such a phenomenon is not believed to occur although we give no concrete evidence in this direction.

1. Notation.

Let

$$a = a(D) = \sum_{|j| \leqq m} a_j D^j ,$$

where $j = (j_1, \ldots, j_n)$, $|j| = j_1 + \ldots + j_n$,

$$D^j = (-i)^{|j|} (\partial/\partial x_1)^{j_1} \ldots (\partial/\partial x_n)^{j_n}$$

be a formally self-adjoint and formally semi-bounded *elliptic* operator with constant coefficients in R^n; that is $\bar{a}_j = a_j$ and $\sum_{|j|=m} a_j \xi^j > 0$ for any real $\xi = (\xi_1, \ldots, \xi_n) \neq 0$. Let Ω be a domain in R^n. Let A be a semi-

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bounded selfadjoint realization of a in the Hilbert space $L_2(\Omega)$. Replacing, if necessary a by $a+\text{const.}$, we may assume that the lower bound is $\geqq 0$, i.e. that A is positive. For any $f \in L_2(\Omega)$ we define then its Riesz mean of order $\alpha \geqq 0$ (with respect to A) by

$$(1.1) \qquad E^\alpha(\lambda)f = \alpha \int_0^\lambda (1-\mu/\lambda)^{\alpha-1} E(\mu) f \, d\mu$$

where $A = \int_0^\infty \lambda dE(\lambda)$ is the spectral resolution corresponding to A. It is known that $E^\alpha(\lambda)$ is an integral operator, that is $E^\alpha(\lambda)f$ is given by

$$(1.2) \qquad E^\alpha(\lambda)f(x) = \int_\Omega e^\alpha(\lambda,x,y) f(y) \, dy$$

where the kernel $e^\alpha(\lambda,x,y)$, the spectral function of order α, actually belongs to $C^\infty(\Omega \times \Omega)$; therefore $E^\alpha(\lambda)f$ can also be defined for distributions f satisfying proper growth conditions at the boundary of Ω, for example, possibly, for $f \in L_p(\Omega)$.

If $\Omega = R^n$ there is only one realization. In this case $E^\alpha(\lambda)$ is a convolution operator so that $e^\alpha(\lambda,x,y)$ depends on $x-y$. Indeed, we have

$$(1.3) \qquad \begin{aligned} e^\alpha(\lambda,x,y) &= \varphi^\alpha(\lambda,x-y) \\ &= (2\pi)^{-n} \int_{a(\xi)\leqq\lambda} e^{i(x-y)\xi} \, \hat{\varphi}(\lambda,\xi) \, d\xi , \\ \hat{\varphi}^\alpha(\lambda,\xi) &= (1-a(\xi)/\lambda)^\alpha , \end{aligned}$$

where $x\xi = x_1\xi_1 + \ldots + x_n\xi_n$. (Here and in the sequel $\hat{}$ denotes Fourier transforms with respect to x.)

We recall that if $f \in L_p(\Omega)$ then a Lebesgue point (of power p) of f is a point x such that

$$(1.4) \qquad \left(r^{-n} \int_{|x-y|\leqq r \cap \Omega} |f(y)-f(x)|^p \, dy \right)^{1/p} = o(1), \qquad r \to 0 .$$

We shall use the letters C and c to denote constants, different in different contexts.

2. The case $1 \leqq p \leqq 2$, $\Omega = R^n$.

Our main result reads as follows.

THEOREM 2.1. *Let A be the unique realization of a in $L_2(R^n)$. Let $f \in L_p(R^n)$, $1 \leqq p \leqq 2$.*

i) *Then*

$$E^{\alpha}(\lambda)f(x) = f(x)+o(1), \qquad \lambda \to \infty , \tag{2.1}$$

at any Lebesgue point x of f, provided

$$\alpha > (n-1)/p . \tag{2.2}$$

(Analogue of Fejér–Lebesgue's theorem.)

ii) *Moreover*

$$E^{\alpha}(\lambda)f(x) = o(\lambda^{((n-1)/p-\alpha)/m}), \qquad \lambda \to \infty , \tag{2.3}$$

at any point x such that f vanishes in a neighborhood of x. In particular

$$E^{\alpha}(\lambda)f(x) = o(1), \qquad \lambda \to \infty , \tag{2.4}$$

at such a point x, provided

$$\alpha \geqq (n-1)/p . \tag{2.5}$$

(Analogue of Riemann's localization principle.)

The proof depends on the following

LEMMA 2.1. *There exists a constant C such that*

$$\left(\int\limits_{|x|\geqq r} |\varphi^{\alpha}(\lambda,x)|^{q}\, dx \right)^{1/q} \leqq C \frac{\lambda^{n/(pm)}}{1+(\lambda^{1/m}r)^{\alpha+1/p}}, \qquad 1/p+1/q = 1 , \tag{2.6}$$

when λ is sufficiently large.

For the proof of Lemma 2.1 we need a second

LEMMA 2.2. *Let N be an integer $>\alpha+1/p$. There exist two functions $\varphi_0{}^{\alpha}(\lambda,x,t)$ and $\varphi_1{}^{\alpha}(\lambda,x,t)$, depending besides λ and x thus also on t, $0<t<1$, such that with some constant C:*

$$\varphi^{\alpha}(\lambda,x) = \varphi_0{}^{\alpha}(\lambda,x,t)+\varphi_1{}^{\alpha}(\lambda,x,t) , \tag{2.7}$$

$$\|\hat{\varphi}_0{}^{\alpha}(\lambda,\xi,t)\|_{L_p} \leqq Ct^{\alpha+1/p}\, \lambda^{n/(pm)} , \tag{2.8}$$

$$\|D^{j}\hat{\varphi}_1{}^{\alpha}(\lambda,\xi,t)\|_{L_p} \leqq Ct^{\alpha+1/p-N}\, \lambda^{(n/p-N)/m}, \qquad |j| = N , \tag{2.9}$$

when, in the last two formulas, λ is sufficiently large.

The idea to make this decomposition of $\varphi^{\alpha}(\lambda,x)$, though not very essential, we take over from the theory of interpolation spaces (cf. e.g. [5]).

PROOF THAT LEMMA 2.1 $\Rightarrow$ THEOREM 2.1 (cf. Alexits [1, pp. 240–246]). We consider first the case of *i*).

Obviously it suffices to consider the case $x=0$. It is also clear that (2.1) holds (regardless of the value of α) when $f \in C_0^{\infty}(R^n)$. Replacing, if necessary, f by $f-f(0)g$ where $g \in C_0^\infty(R^n)$, $g(0)=1$, we may thus assume $f(0)=0$. Thus we have to show that

$$\left(r^{-n} \int_{|x|\le r} |f(x)|^p dx\right)^{1/p} = o(1), \qquad r \to 0\,, \tag{2.10}$$

implies

$$\int \varphi^\alpha(\lambda,x) f(x)\, dx = o(1), \qquad \lambda \to \infty\,. \tag{2.11}$$

Let X be the Banach space of measurable functions f such that

(2.12)
$$\|f\|_X = \max\left(\sup_{r\le 1}\left(r^{-n}\int_{|x|\le r} |f(x)|^p\, dx\right)^{1/p}, \left(\int_{|x|\ge 1} |f(x)|^p\, dx\right)^{1/p}\right) < \infty$$

Every $f \in L_p(R^n)$ satisfying (2.10) belongs to the closure in X of $X \cap C_0^\infty(R^n)$ and (2.11) holds in this sub-space. Therefore we only have to show that

$$\sup\nolimits_\lambda \|\varphi^\alpha(\lambda,x)\|_{X'} < \infty\,, \tag{2.13}$$

where

$$\|g\|_{X'} = \sup_{\|f\|_X \le 1} \left| \int g(x) f(x)\, dx \right|$$

is the norm dual to $\|f\|_X$. Indeed, we easily deduce using Hölder's inequality, (2.6), (2.12), (2.2):

$$\begin{aligned}
&\left|\int \varphi^\alpha(\lambda,x)\, f(x)\, dx\right| \\
&\le \sum_{k<0}\left|\int_{2^k\le|x|\le 2^{k+1}} \varphi^\alpha(\lambda,x)\, f(x)\, dx\right| + \left|\int_{|x|\ge 1} \varphi^\alpha(\lambda,x)\, f(x)\, dx\right| \\
&\le \sum_{k<0}\left(\int_{|x|\ge 2^k} |\varphi^\alpha(\lambda,x)|^q\, dx\right)^{1/q}\left(\int_{|x|\le 2^{k+1}} |f(x)|^p\, dx\right)^{1/p} + \\
&\quad + \left(\int_{|x|\ge 1} |\varphi^\alpha(\lambda,x)|^q\, dx\right)^{1/q}\left(\int_{|x|\ge 1} |f(x)|^p\, dx\right)^{1/p} \\
&\le C\left(\sum_{k<0} \frac{(\lambda^{1/m}2^k)^{n/p}}{1+(\lambda^{1/m}2^k)^{\alpha+1/p}} + \frac{\lambda^{n/(pm)}}{1+\lambda^{(\alpha+1/p)/m}}\right) \le C\,.
\end{aligned}$$

This settles the case of i).

It is now clear how to extend the argument to the case of ii). Instead of X we use the Banach space Y defined by

$$(2.14)\qquad \|f\|_Y = \left(\int_{|x|\geqq R} |f(x)|^p\,dx\right)^{1/p} < \infty\,,\quad f=0 \quad \text{if } |x|<R\,,$$

where R is a fixed number. We leave the details to the reader.

REMARK 2.1. From the above proof follows, in view of Banach–Steinhaus' theorem, that the following condition is necessary and sufficient for the analogue of Fejér–Lebesgue's theorem to hold (cf. [1, pp. 240–246], where the case $p=1$ is considered in detail):

$$(2.15)\quad \sum_{k<0} 2^{k/(np)}\left(\int_{|x|\geqq 2^k} |\varphi^\alpha(\lambda,x)|^q\,dx\right)^{1/q} + \left(\int_{|x|\geqq 1} |\varphi^\alpha(\lambda,x)|^q\,dx\right)^{1/q} \leqq C\,.$$

In the case of Riemann's localization principle the corresponding condition reads:

$$(2.16)\qquad \left(\int_{|x|\geqq R} |\varphi^\alpha(\lambda,x)|^q\,dx\right)^{1/q} \leqq C\,.$$

PROOF THAT LEMMA 2.2 ⇒ LEMMA 2.1. Since a is elliptic, we have

$$(2.17)\qquad \left|\int_{a(\xi)\leqq\lambda} d\xi\right| \leqq C\lambda^{n/m}$$

when λ is sufficiently large, so that, by Hausdorff–Young's theorem, in any case

$$\left(\int |\varphi^\alpha(\lambda,x)|^q\,dx\right)^{1/q} \leqq C\lambda^{n/(mp)}\,.$$

Therefore (2.6) is trivial if $r\lambda^{1/m}\leqq 1$ and we may concentrate on the case $r\lambda^{1/m}>1$. We now obtain, using (2.7), the triangle inequality, Hausdorff–Young's theorem, (2.8), (2.9):

$$\begin{aligned}
&\left(\int_{|x|\geqq r} |\varphi^\alpha(\lambda,x)|^q\right)^{1/q}\\
&\leqq \left(\int |\varphi_0{}^\alpha(\lambda,x;t)|^q\,dx\right)^{1/q} + r^{-N}\left(\int |x|^{qN}\,|\varphi_1{}^\alpha(\lambda,x;t)|^q\,dx\right)^{1/q}\\
&\leqq \|\hat{\varphi}_0{}^\alpha(\lambda,\xi;t)\|_{L_p} + Cr^{-N}\sum_{|j|=N}\|D^j\varphi_1{}^\alpha(\lambda,\xi;t\|_{L_p}\\
&\leqq C(t^{\alpha+1/p}\,\lambda^{n/(mp)} + r^{-N}\,t^{\alpha+1/p-N}\,\lambda^{(n/p-N)/m})\,.
\end{aligned}$$

Taking $t=(r\lambda^{1/m})^{-1}$ we get (2.6) for the case $r\lambda^{1/n}>1$.

PROOF OF LEMMA 2.2. We give only an outline of the elementary but rather technical argument. Since a is elliptic, $a(\xi)=\lambda$ implies

$$1/C\lambda^{1/m} \leqq |\xi| \leqq C\lambda^{1/m}$$

when λ is sufficiently large. It follows that then

$$|D^j a(\xi)| \leqq C\lambda^{1-|j|/m} . \tag{2.18}$$

We also note that (cf. (2.17))

$$\left| \int_{\lambda t' \leqq a(\xi) \leqq \lambda t''} d\xi \right| \leqq C(t''-t')^{\lambda n/m}, \qquad \tfrac{1}{2} \leqq t' < t'' \leqq 1 . \tag{2.19}$$

Let now $\varrho(u)$ be a function of one variable such that

$$\begin{cases} \varrho(u) = 1 & \text{if } u < 1-t \\ \varrho(u) = 0 & \text{if } u > 1-\tfrac{1}{2}t \\ |\varrho^{(\nu)}(u)| \leqq Ct^{-\nu}, \ \nu \leqq N, & \text{if } 1-t \leqq u \leqq 1-\tfrac{1}{2}t . \end{cases}$$

Set

$$\omega(\xi) = \varrho\big(a(\xi)/\lambda\big) .$$

Using (2.18) we see that

$$|D^j \omega(\xi)| \leqq C|\lambda|^{-|j|/m}\, t^{-|j|}, \qquad |j| \leqq N . \tag{2.20}$$

Define

$$\begin{cases} \hat{\varphi}_0{}^{\alpha}(\lambda,\xi;t) = (1-\omega(\xi)\, \hat{\varphi}(\lambda,\xi) , \\ \hat{\varphi}_1{}^{\alpha}(\lambda,\xi;t) = \omega(\xi)\, \hat{\varphi}(\lambda,\xi) . \end{cases}$$

Now (2.7) is obvious. Using (2.18), (2.19), (2.20), it is easy to verify (2.8) and (2.9) too. We leave the details to the reader.

The proof of Theorem 2.1 is complete.

REMARK 2.2. If a is homogeneous, the most important case, the above proofs, in particular the proof of Lemma 2.2, become much simpler; for then $\varphi^{\alpha}(\lambda,x)$ is homogeneous in λ^{-m} and x so one can take $\lambda=1$.

We also mention the following

THEOREM 2.2. *Let* $f \in L_p(R^n)$, $p>2$.

i) *Then* (2.1) *holds at any Lebesgue point* x *of* f, *provided*

$$\alpha > \tfrac{1}{2}(n-1) . \tag{2.21}$$

ii) *Moreover*

$$E^{\alpha}(\lambda) f(x) = o(\lambda^{(\frac{1}{2}(n-1)-\alpha)/m}), \qquad \lambda \to \infty , \tag{2.22}$$

holds at any point x *such that* f *vanishes in the neighborhood of* x *provided again* (2.21).

The proof is quite similar. We have to use the inequality

$$(2.23)\quad \left(\int\limits_{|x|\geqq r} |\varphi^\alpha(\lambda,x)|^q\,dx\right)^{1/q} \leqq C\,\frac{\lambda^{n/(pm)}}{1+(\lambda^{1/m}r)^{\alpha+\frac{1}{2}-n(\frac{1}{2}-1/p)}}, \qquad 1/p+1/q = 1\,,$$

which follow easily from (2.6). We omit the details.

3. The case $p=2$, Ω arbitrary.

Let A be an arbitrary realization of a in $L_2(\Omega)$ where Ω too is arbitrary. We denote, in this Section, by A_0 the unique realization in $L_2(R^n)$ and use $E_0{}^\alpha(\lambda)$, $e_0{}^\alpha(\lambda,x,y)$ in a similar way. Using a Tauberian theorem Ganelius (unpublished; cf. Bergendal [2], Theorem 3.1.2) has shown that for any $f\in L_2(\Omega)$ holds

$$(3.1)\qquad E^\alpha(\lambda)f(x)-E_0{}^\alpha(\lambda)f(x) = o(\lambda^{(\frac{1}{2}(n-1)-\alpha)/m}), \qquad \lambda\to\infty\,.$$

Combining this with Theorem 2.1 we obtain the following

COROLLARY 3.1. *The conclusions of Theorem* 2.1 *hold for any realization* A *of* a *and any* $f\in L_2(\Omega)$.

This makes more precise some of the results on eigenfunction expansions given by Bergendal in [2, pp. 38–41]). (Actually in [2] is considered also the convergence of derivatives of $E^\alpha(\lambda)f$ etc.; this type of extensions we here disregard entirely.)

REMARK 3.1. An equivalent way of stating this is as follows:

$$(3.2)\qquad \left(\int\limits_{\Omega} |e^\alpha(\lambda,x,y)-e_0{}^\alpha(\lambda,x,y)|^2\,dy\right)^{\frac{1}{2}} = O(\lambda^{(\frac{1}{2}(n-1)-\alpha)/m}), \qquad \lambda\to\infty\,.$$

Indeed, in view of Banach–Steinhaus' theorem, (3.1) implies (3.2). Conversely, (3.1) follows from (3.2) since when $f\in C_0^\infty(\Omega)$, which is a dense sub-set of $L_2(\Omega)$, we have an even stronger result:

$$(3.3)\qquad E^\alpha(\lambda)f(x)-E_0{}^\alpha(\lambda)f(x) = O(\lambda^s), \qquad \lambda\to\infty\,,$$

for *any* s. To prove (3.3) we use the identity

$$\begin{aligned} &E^\alpha(\lambda)f-E_0{}^\alpha(\lambda)f \\ &= \left(E^\alpha(\lambda)f-\left(f-\binom{\alpha}{1}\frac{Af}{\lambda}+\ldots+(-1)^{s-1}\binom{\alpha}{s-1}\frac{A^{s-1}f}{\lambda^{s-1}}\right)\right)- \\ &\quad -\left(E_0{}^\alpha(\lambda)f-\left(f-\binom{\alpha}{1}\frac{A_0f}{\lambda}+\ldots+(-1)^{s-1}\binom{\alpha}{s-1}\frac{A_0{}^{s-1}f}{\lambda^{s-1}}\right)\right). \end{aligned}$$

With the aid of the spectral resolutions of A or A_0 respectively we see that each of the terms is $O(\lambda^s)$ in $D(A^k)$ or $D(A_0{}^k)$ respectively. Taking $k > p/(2m)$ we obtain the pointwise estimate (3.3) by "Sobolev's lemma" (cf. Nilsson [4] where essentially the same idea is used). (Actually, Ganelius proves (3.1) with O instead of o, and the improvement to o is due to Bergendal, who, however, to this end also uses a Tauberian argument, which is thus completely unnecessary.)

REMARK 3.2. We do not know whether the analogues of Corollary 3.1 or formulas (3.1) and (3.2) hold true in L_p or not. For $p=1$ the following weaker result is known (cf. [2, Theorem 3.1ĺ])

$$(3.4) \qquad \sup_K |e^\alpha(\lambda,x,y)-e_0{}^\alpha(\lambda,x,y)| = o(\lambda^{(n-1-\alpha)/m}), \qquad \lambda \to \infty,$$

where K is any compact sub-set of Ω.

4. On the case $p=1$, $\Omega = R^n$: Examples,

We return to the situation of Section 2, taking $p=1$. Denote by $\varkappa_1$ ("the critical index") the greatest lower bound on α for the conclusions of Theorem 1.1 to hold. We consider the dependence of $\varkappa_1$ on a. Clearly by Theorem 1.1 we have $\varkappa_1 \leqq n-1$ for any a. It is now a quite remarkable fact that for some particular operators a it may happen that $\varkappa_1 < n-1$ or even $\varkappa_1 = \frac{1}{2}(n-1)$ (which was the bound on α given by Theorem 1.1 in the case $p=2$; in this case such a phenomenon is believed not to occur). Indeed we shall see that $\varkappa_1$ *depends in general on the differential geometry of the surfaces* $a(\xi)=\lambda$. We shall not formulate this as a precise result but rather look at some illustrative examples.

EXAMPLE 4.1. We take

$$(4.1) \qquad a(\xi) = |\xi|^2 = \xi_1{}^2 + \ldots + \xi_n{}^2;$$

thus $a = -\Delta$ where Δ is Laplace operator. Here, as we shall see, $\varkappa_1 = \frac{1}{2}(n-1)$ and we get thus back a classical result on "spherical" summability of multiple Fourier integrals due to Bochner [3] (cf. also the recent survey article by Shapiro [6]). We have apparently to prove the following estimate

$$(4.2) \qquad |\varphi^\alpha(\lambda,x)| \leqq C\frac{\lambda^{n/m}}{1+(\lambda^{1/m}|x|)^{\alpha'+1}}, \qquad \alpha' = \alpha + \tfrac{1}{2}(n-1),$$

which is thus sharper than (2.6), with $q=\infty$. The proof of (4.2) is usually (cf. e.g. [5]) obtained by expressing $\varphi^\alpha(\lambda,x)$ in terms of the Bessel function $J_\beta(u)$, herewith $\beta = \alpha + \frac{1}{2}n$, and then applying the well-known asymptotic formula

(4.3) $$J_\beta(u) = cu^{-\frac{1}{2}} \cos(u - \pi/4 - \beta\pi/2) + O(u^{-\frac{3}{2}}), \qquad u \to \infty .$$

Here, following a suggestion of Lars Gårding, we shall give another more direct method, which will be used also in the other examples below. Since, in this special case, $\varphi^\alpha(\lambda, x)$ is rotation invariant in x, and homogeneous in x and λ^{-m}, it is sufficient to estimate it on just one ray issuing from 0, say the (positive) x_1 axis, and for $\lambda = 1$. We write (1.3), with $a(\xi)$ given by (4.1), as

$$\varphi^\alpha(1,x) = (2\pi)^{-n} \int_{-1}^{1} e^{ix_1\xi_1} \left(\int_{\xi_2^2+\ldots+\xi_n^2 \leq 1-\xi_1^2} (1-|\xi|^2)^\alpha \, d\xi_2 \ldots d\xi_n \right) d\xi_1 ,$$

$x = (x_1, 0, \ldots, 0)$. Making a change of variables we see that the inner integral becomes of the form $c(1-\xi_1^2)^{\alpha'}$. Thus

(4.4)
$$\varphi^\alpha(1,x) = c \int_{-1}^{1} e^{ix_1\xi_1} (1-\xi_1^2)^{\alpha'} \, d\xi_1, \quad x = (x_1, 0, \ldots, 0), \quad \alpha' = \alpha + \tfrac{1}{2}(n-1) .$$

Applying now Lemma 2.1 and Lemma 2.2, in the case of one variable (or, alternatively, using Bessel functions including formula (4.3)), we get

$$|\varphi^\alpha(1,x)| \leqq C \frac{1}{1+|x|^{\alpha'+1}}, \qquad x = (x_1, 0, \ldots, 0) ,$$

from which, passing back to general x and λ, follows the estimate (4.2).

Example 4.2. Nothing essential is changed if we replace (4.1) by

(4.5) $$a(\xi) = |\xi|^m = (\xi_1^2 + \ldots + \xi_n^2)^{m/2} \quad (m \text{ even});$$

thus $a = (-\Delta)^{m/2}$. Again $\varkappa_1 = \frac{1}{2}(n-1)$, and the proof is quite similar as in the case $m=2$ (Example 4.1). We therefore omit the details.

Example 4.3. Much more interesting is the case

(4.6) $$a(\xi) = \xi_1^m + \ldots + \xi_n^m \quad (m \text{ even}) .$$

Now we get, in the same way as in Example 4.1, instead of (4.4),

(4.7) $$\varphi(1,x) = c \int_{-1}^{1} e^{ix_1\xi_1} (1-\xi_1^m)^{\alpha'} \, d\xi_1,$$

$$x = (x_1, 0, \ldots, 0) , \qquad \alpha' = \alpha + (n-1)/m ,$$

and thus instead of (4.2)

(4.8)
$$|\varphi^\alpha(\lambda,x)| \leqq C \frac{\lambda^{n/m}}{1+(\lambda^{1/m}|x|)^{\alpha'+1}}, \qquad x = (x_1, 0, \ldots, 0), \qquad \alpha' = \alpha + (n-1)/m .$$

Write now (4.7) as

$$\varphi^{\alpha}(1,x) = c\int_{-1}^{1} e^{ix_1\xi_1}(1-\xi_1^2)^{\alpha'}\,d\xi_1 +$$
$$+\,c\int_{-1}^{1} e^{ix_1\xi_1}(1-\xi_1^2)^{\alpha'}\left((1+\xi_1^2+\ldots+\xi_1^{2(\frac{1}{2}m-1)})^{\alpha'}-(\tfrac{1}{2}m)^{\alpha'}\right)d\xi_1 .$$

One sees readily, using a technique similar to the one based on Lemma 2.1 and Lemma 2.2, that the first term is the dominant one so, using (4.3) we see that the exponent $\alpha'+1=\alpha+(n-1)/m+1$ in the denominator of (4.8) cannot replaced by a larger one. Therefore, in view of Remark 2.1, we have $\varkappa_1 \geqq (n-1)(1-1/m)$. From the proof we also see how this is connected with the fact that the surface $\xi_1^m+\ldots+\xi_n^m=1$ is very "flat" at the points $(\pm 1,0,\ldots,0)$ if m is large. This justifies our previous statement on the influence of the differential geometry of the surfaces $a(\xi)=\lambda$.

REFERENCES

1. G. Alexits, *Konvergenzprobleme der Orthogonalreihen*, Berlin, 1960.
2. G. Bergendal, *Convergence and summability of eigenfunction expansions connected with elliptic differential operators*, Thesis, Lund, 1959 (=Medd. Lunds Univ. Mat. Sem. 14 (1959), 1–63).
3. S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. 40 (1936), 175–207.
4. N. Nilsson, *Some estimates for eigenfunction expansions and spectral functions corresponding to elliptic differential operators*, Math. Scand. 9 (1961), 107–121.
5. J. Peetre, *Espaces d'interpolation, généralisations, applications*, Rend. Sem. Mat. Fis. Milano 34 (1964), 2–34.
6. V. Shapiro, *Fourier series in several variables*, Bull. Amer. Math. Soc. 70 (1964), 48–93.

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