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ON THE GEOMETRY OF CHOQUET SIMPLEXES

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This paper is a study of certain geometrical properties of Choquet simplexes, especially of their facial structure and of the possibility of decomposition of convex sets into simplexes. Analytically the concept of a closed face is related to a *stable* subset and a set of *determinacy*, and the problem of simplicial decomposition is related to the establishment of *unique* representing boundary measures by passage to appropriate *subsets* of the extreme boundary. (Precise definitions follow in the sequel.)

It turns out that many of the familiar properties of finite dimensional simplexes fail to carry over to the general case. In this connection we shall need a number of counterexamples, which are obtained by a general method to construct convex compact sets K with pre-ascribed affine dependences on the extreme boundary $\partial_e K$. (For precise definitions cf. § 1.) The method is analogous to the presentation of groups by generators and relations. To every compact set X ("generator set") there corresponds a unique simplex ("free group") $\mathfrak{M}_1^+(X)$ with extreme boundary homeomorphic to X, and one may introduce affine dependences ("relations") on X by passage to a quotient $\mathfrak{M}(X)/\mathfrak{N}(X)$ where $\mathfrak{N}(X)$ consists of all measures ("words") corresponding to the desired affine dependences.

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1. Definitions and general properties.

In the sequel E shall be a *locally convex* (Hausdorff) vector space over R and K shall be a *compact convex* subset of E. The extreme points of K form the *extreme boundary*, denoted by $\partial_e K$. The *bounded* (Radon-) *measures* on K form the vector lattice $\mathfrak{M}(K)$, and the *positive normalized* measures ("probability measures") on K form the vaguely compact, convex subset $\mathfrak{M}_1^+(K)$ of $\mathfrak{M}(K)$. The K-restrictions of continuous, affine, real valued functions on E form a vector space $\mathscr{H}$ which is uniformly dense in the vector space $\mathscr{H}$ of all continuous, affine, real valued functions

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on K. By definition $\tilde{\mathscr{H}} = \mathscr{S} \cap (-\mathscr{S})$, where $\mathscr{S}$ is the convex, "sup-closed" cone of continuous convex functions on K.

Every $f \in \mathscr{C}(K)$ admits a greatest l.s.c. convex minorant $\underline{f}$, which is determined by

$$\underline{f}(x) = \sup\{h(x) \mid f \geqq h \in \mathscr{H}\}. \tag{1.1}$$

(The function $\underline{f}$ may be considered as the second conjugate of f in a duality theory of convex functions, cf. [4], [10], [14].)

To every $f \in \mathscr{C}(K)$ there is associated a *boundary set* $B(f)$, defined by

$$B(f) = \{x \mid x \in K, f(x) = \underline{f}(x)\}. \tag{1.2}$$

By a theorem of M. Hervé [11]:

$$\partial_e K = \bigcap_{f \in \mathscr{C}(K)} B(f). \tag{1.3}$$

(In this connection cf. also [8]. Actually (1.3) is implicit in the proof of Theorem 4.1 of [12].)

A member of $\mathfrak{M}(K)$ is said to be a *boundary measure* if its total variation vanishes off any of the G_δ-sets $B(f)$, $f \in \mathscr{C}(K)$. (Recall that $\partial_e K = B(f)$ for some $f \in \mathscr{C}(K)$ provided that K is metrizable [11].)

Choquet's Integral Theorem states that every point x of K is the *barycenter* of a boundary measure μ in $\mathfrak{M}_1^+(K)$. In symbols:

$$x = \int t \, d\mu(t), \tag{1.4}$$

where the integral is taken in the weak sense ([8], cf. also [3], [5], [6], [7], [11], [13]).

An *affine dependence* on $\partial_e K$ is a non-zero (signed) boundary measure μ such that $\mu(K) = 0$, and

$$\int t \, d\mu(t) = 0 \qquad \text{(weak integral)}. \tag{1.5}$$

K is a *simplex* if there is no affine dependence on $\partial_e K$, or briefly if $\partial_e K$ *is affinely independent.* Clearly this requirement is equivalent to *uniqueness* of the measure μ in (1.4).

K will be said to be an *r-simplex* if it is a simplex and if $\partial_e K$ is closed. (The letter r denotes "resolutive", since the r-simplexes are exactly those compact convex sets for which the Dirichlet problem is solvable in $\tilde{\mathscr{H}}$ for every $f \in \mathscr{C}(\partial_e K)$ [1], [2]). By a theorem of H. Bauer, K is an r-simplex if and only if every $x \in K$ is the barycenter of a unique measure $\mu \in \mathfrak{M}_1^+(K)$ supported by $\overline{\partial_e K}$ ([1], [2]). This yields a "concrete" representation of r-simplexes. They are the sets $\mathfrak{M}_1^+(X)$ with vague topology,

X being an arbitrary compact (Hausdorff) space. (Recall that $\partial_e\mathfrak{M}_1^+(X)\cong X$.)

We shall use the symbol Δ_σ to denote the r-simplex $\mathfrak{M}_1^+(\bar{N})$, where $\bar{N}$ is the one-point compactification of the set N of natural numbers. Clearly we may also consider Δ_σ to be the subset of $\mathsf{R}^{\bar{N}}$ consisting of all $a=\{\alpha_n\}_{n\in\bar{N}}$ such that $\sum_{n=1}^{\infty}\alpha_n+\alpha_\infty=1$ and $\alpha_n\geqq 0$ for $n\in\bar{N}$. The extreme boundary of Δ_σ consists of the points $d_m=\{\delta_{m,n}\}_{n\in\bar{N}}$ where $m\in\bar{N}$: Note that the mapping $\{\alpha_n\}_{n\in\bar{N}}\to\{\alpha_n\}_{n\in N}$ is an affine and topological isomorphism of Δ_σ onto the positive part of the unit ball in l^1 equipped with the Tykhonov topology.

An affine manifold M in E is said to be a *supporting manifold* for K if

$$K\cap M\neq\emptyset,\qquad K\setminus M \text{ is convex}\,. \tag{1.6}$$

Clearly, a hyperplane H supports K if and only if it meets K and K is located on one side of H, and a one point set $\{x\}$ supports K if and only if $x\in\partial_e K$.

The traces on K of supporting affine manifolds are called *faces*. Clearly a subset F of K is a face if and only if it is convex and satisfies the requirement:

$$(y,z,\lambda)\in K\times K\times(0,1]\ \ \&\ \ \lambda y+(1-\lambda)z\in F\ \Rightarrow\ y\in F\,. \tag{1.7}$$

An equivalent form of (1.7) is

$$\big(\alpha F-(\alpha-1)K\big)\cap K\subset F\quad\text{for all}\quad \alpha\geqq 1\,. \tag{1.8}$$

It is not hard to prove that if A is a convex subset of K, then the set

$$\bigcup_{\alpha\geqq 1}\big(\alpha A-(\alpha-1)K\big)\cap K \tag{1.9}$$

is also convex. Hence it is the smallest face of K containing A. It will be denoted by $\mathrm{face}_K A$ or simply by $\mathrm{face}\,A$.

If G is a (not necessarily convex) subset of K satisfying (1.7), then $\mathrm{face}\{x\}\subset G$ for all $x\in G$. It follows that *a subset G of K satisfies* (1.7) *if and only if it is a union of faces.*

By finite induction one may replace (1.7) by the equivalent requirement:

$$x\in F\ \ \&\ \ x=\sum_{i=1}^{n}\lambda_i x_i\quad\text{(proper convex combination on } K)\qquad \Rightarrow\ x_i\in F \text{ for } i=1,2,\ldots \tag{1.10}$$

It is known (cf. e.g. the footnote of [8, p. 141]), that every $\mu\in\mathfrak{M}_1^+(K)$ can be vaguely approximated by point-measures $\sum_{i=1}^{n}\lambda_i\varepsilon_{x_i}$ with the same barycenter as μ. By means of this and by (1.10), one can prove that *a closed subset G of K satisfies* (1.7) *if and only if*

$$\mu \in \mathfrak{M}_1^+(K), \int t \, d\mu(t) \in G \Rightarrow \operatorname{Spt}(\mu) \subset G . \tag{1.11}$$

A closed subset G of K is said to be *stable* if it satisfies the requirement (1.11). Now the results of the above discussion may be summed up in the following:

PROPOSITION 1. *A closed subset G of K is stable if and only if it is a union of faces. If G is convex and stable, then it is a face.*

We state two elementary results for later references.

PROPOSITION 2. *If F is a face of K and G is a face of F, then G is a face of K. In particular, if x is extreme in F, then x is extreme in K.*

PROPOSITION 3. *Let φ be a linear mapping of E into another vectorspace E'. A subset F' of $\varphi(K)$ is a face of $\varphi(K)$ if and only if $\varphi^{-1}(F')\cap K$ is a face of K.*

Application of the Krein–Milman Theorem yields the following:

COROLLARY. *Let φ be a continuous linear mapping of E into another locally convex vector space E' and assume that the restriction of φ to $\partial_e K$ is 1-1. A point x of $\varphi(K)$ is extreme in $\varphi(K)$ if and only if $\varphi^{-1}(x)\cap K$ reduces to a single point and this point is extreme in K.*

Let $\mathfrak{N}(K)$ be the linear subspace of $\mathfrak{M}(K)$ which consists of all affine dependences on $\partial_e K$. It is not hard to verify that if $E = \mathsf{R}^n$ and the number of extreme points of K is m (possibly $m = \infty$), then

$$m - n - 1 \leqq \dim \mathfrak{N}(K) \leqq m - 1 . \tag{1.12}$$

In this case, $\mathfrak{N}(K)$ is finite dimensional if and only if K has a finite number of extreme points, or in other words if K is a (convex) polyhedron. In the general case we define K to be a *polyhedron* if $\mathfrak{N}(K)$ is finite dimensional. In particular, *every simplex K is a polyhedron.*

The theory of compact convex sets and simplexes is presented in a functorial setting in a forthcoming paper of Z. Semadeni [15].

2. Closed faces of simplexes.

It is of some interest to decide if the closure of a face is a face. We shall see that the answer is affirmative for r-simplexes, but negative for simplexes in general.

Proposition 4. *Every closed face F of K can be represented in the form*

$$F = \overline{\operatorname{conv} A}, \qquad A \subset \partial_e K . \tag{2.1}$$

If K is an r-simplex, then every set F of the form (2.1) *is a closed face.*

Proof. 1) Let F be closed face of K and define $A = F \cap \partial_e K$. By Proposition 2, $\partial_e F = A$. Hence (2.1) follows in virtue of the Krein–Milman Theorem.

2) Let K be an r-simplex and let A be a subset of $\partial_e K$. Define $F = \overline{\operatorname{conv} A}$, and consider a convex combination

$$x = \lambda y + (1-\lambda) z , \tag{2.2}$$

where $y, z \in K$, $0 < \lambda \leqq 1$, and $x \in F$.

Let ν and ϱ be two measures in $\mathfrak{M}_1^+(K)$ which are supported by the closed set $\partial_e K$ and have barycenters y and z respectively. Define

$$\pi = \lambda \nu + (1-\lambda) \varrho . \tag{2.3}$$

Clearly $\pi \in \mathfrak{M}_1^+(K)$, $\operatorname{Spt}(\mu) \subset \partial_e K$, and π has barycenter x.

On the other hand, there is a measure μ_0 in $\mathfrak{M}_1^+(F)$ which is supported by $\overline{\partial_e F}$ and has barycenter x. Let μ be the canonical image of μ_0 in $\mathfrak{M}_1^+(K)$ (μ vanishes identically off F). It follows by Milman's Theorem, that $\overline{\partial_e F} \subset \bar{A}$. Hence $\operatorname{Spt}(\mu) \subset \bar{A} \subset \partial_e K$. Clearly μ has barycenter x, and by the uniqueness property of r-simplexes, $\mu = \pi$.

Now it follows that $\operatorname{Spt}(\pi) \subset \bar{A}$, and by (2.3) and the fact that $\lambda > 0$, it also follows that $\operatorname{Spt}(\nu) \subset \bar{A}$. Then the barycenter of ν must belong to the closed convex hull of $\bar{A}$, and so $y \in F$. This proves that F satisfies the requirement (1.7), and so it is a face of K.

Theorem 1. *If K is an r-simplex, then the closure of any face of K is a face. The corresponding statement is inexact for general simplexes.*

Proof. 1) Let F be any face of an r-simplex K, and define $\mathscr{F}$ to be the set of all $\mu \in \mathfrak{M}_1^+(K)$ with barycenter in F and with support contained in the closed set $\partial_e K$.

Define

$$A_0 = \bigcup_{\mu \in \mathscr{F}} \operatorname{Spt}(\mu) ,$$

and let A be the closure of A_0. Clearly $A \subset \partial_e K$, and by Proposition 4 it suffices to prove that $\bar{F} = \overline{\operatorname{conv} A}$.

For every $x \in F$, there exists a $\mu \in \mathfrak{M}_1^+(K)$ with barycenter x and with support contained in $\partial_e K$. By definition $\mu \in \mathscr{F}$, and hence $\operatorname{Spt}(\mu) \subset A$. This implies $x \in \overline{\operatorname{conv} A}$, and hence $\bar{F} \subset \overline{\operatorname{conv} A}$.

To prove the converse relation, it suffices to verify that $A \subset \bar{F}$, since $\bar{F}$ itself is closed and convex. Let x be an arbitrary point in A_0, and let V be any closed convex neighbourhood of x. By the definition of A_0 there exists a $\mu \in \mathscr{F}$ such that $x \in \mathrm{Spt}(\mu)$, and then $\mu(V) \neq 0$.

If $\mu(V)=1$, then μ is supported by V and hence the barycenter of μ belongs to the closed convex set V. By the definition of $\mathscr{F}$, this barycenter belongs to F, and hence $F \cap V \neq \emptyset$ in this case.

If $\mu(V)<1$, then we write $\mu(V)=\lambda$ and define

$$\mu_1 = \lambda^{-1}\mu_V, \qquad \mu_2 = (1-\lambda)^{-1}\mu_{\complement V} .$$

Now

$$\mu = \lambda\mu_1+(1-\lambda)\mu_2, \qquad 0<\lambda<1 . \tag{2.4}$$

Let y, y_1 and y_2 be the barycenters of μ, μ_1, and μ_2, respectively. By (2.4) one has

$$y = \lambda y_1+(1-\lambda)y_2 .$$

It follows from the definition of $\mathscr{F}$ that $y \in F$, and the characteristic property (1.7) of faces implies that $y_1 \in F$. Being the barycenter of a measure $\mu_1 \in \mathfrak{M}_1{}^+(K)$ supported by V, the point y_1 itself belongs to V. Hence $F \cap V \neq \emptyset$ in this case as well.

It follows that $x \in \bar{F}$. Thus we have proved $A_0 \subset \bar{F}$, and hence $A \subset \bar{F}$, completing the first part of the proof.

2) Let Δ_σ be the r-simplex in $\mathsf{R}^{\bar{N}}$ which is defined in § 1, and let M be a one dimensional subspace of $\mathsf{R}^{\bar{N}}$ generated by the vector $a=\{\alpha_n\}_{n\in\bar{N}}$ for which

$$\alpha_n = \begin{cases} \tfrac{1}{2} & \text{for } n=1,2 , \\ 0 & \text{for } 2<n<\infty , \\ -1 & \text{for } n=\infty . \end{cases} \tag{2.5}$$

Let φ be the canonical mapping of $\mathsf{R}^{\bar{N}}$ onto $E=\mathsf{R}^{\bar{N}}/M$, and let $K=\varphi(\Delta_\sigma)$. Let F_0 be the subset of Δ_σ which consists of all $b=\{\beta_n\}_{n\in\bar{N}}$ for which

$$\beta_1 = \beta_2 = \beta_\infty = 0 , \tag{2.6}$$

and let $F=\varphi(F_0)$. We claim that K is a (compact) simplex in E, that F is a face of K, whereas $\bar{F}$ is no face.

To prove these claims we first observe that φ maps $\partial\Delta_\sigma=\{d_n \mid n \in \bar{N}\}$ biuniquely into K. By the Corollary to Proposition 3, $\partial_e K$ consists of those elements d_n, $n \in \bar{N}$, for which $\varphi^{-1}\varphi(d_n)\cap\Delta_\sigma=\{d_n\}$. It is easily verified that this equality holds for $n \neq \infty$, and so $\varphi(d_n)$ is an extreme point of K for $n=1,2,\ldots$. The point $\varphi(d_\infty)$, however, is non-extreme, since the relation

$$\tfrac{1}{2}d_1+\tfrac{1}{2}d_2-d_\infty = a$$

implies

(2.7) $$\tfrac{1}{2}\varphi(d_1)+\tfrac{1}{2}\varphi(d_2) = \varphi(d_\infty) .$$

Thus we have proved that $\partial_e K=\{\varphi(d_n) \mid n \in N\}$, and it follows that $\partial_e K$ is homeomorphic to N (with discrete topology).

An affine dependence on $\partial_e K$ is given by a sequence $\{\mu_n\}_{n\in N}$ such that

(2.8) $$\sum_{n=1}^{\infty} \mu_n = 0, \quad \sum_{n=1}^{\infty} \mu_n \varphi(d_n) = 0 ,$$

where the first sum converges absolutely on R, and the second sum converges in the topology of K. By the continuity of φ, the last sum of (2.8) may be written

$$\varphi\left(\sum_{n=1}^{\infty} \mu_n d_n\right) = 0 ,$$

which is equivalent to

(2.9) $$\sum_{n=1}^{\infty} \mu_n d_n = \lambda a ,$$

for some $\lambda \in$ R.

The relation (2.9) is an equality between members of $\mathrm{R}^{\overline{N}}$. Comparing the ∞-components, we obtain $\lambda=0$. Hence $\mu_n=0$ for $n=1,2,\ldots,$ and so we have proved that there can be no affine dependence on $\partial_e K$. Thus K is a simplex.

Next we observe that F_0 is a face of Δ_σ. It is easily verified that φ maps F_0 biuniquely onto F, and it follows by Proposition 3 that F is a face of K.

The closure of F_0 contains d_∞. By convexity, $\overline{F}_0$ consists of all $b=\{\beta_n\}_{n\in\overline{N}} \in \Delta_\sigma$ such that

(2.10) $$\beta_1 = \beta_2 = 0 .$$

(Note that $\overline{F}_0$ is a closed face of the r-simplex Δ_σ).

It follows by the continuity of φ and the compactness of $\overline{F}_0$, that

(2.11) $$\overline{F} = \varphi(\overline{F}_0) .$$

We claim that $\varphi(d_i) \notin \overline{F}$ for $i=1,2$. In fact if $\varphi(d_1) \in \overline{F}$, then there would be a $b \in \overline{F}_0$ and a $\lambda \in$ R such that

$$d_1 = b+\lambda a .$$

The second component of d_1 vanishes and so does that of b (cf. (2.10)). It follows that $\lambda=0$. This, however, is a contradiction since the first component of d_1 is 1 and the first component of b is 0. A similar contradiction is obtained from $\varphi(d_2) \in \overline{F}$.

It follows from (2.11) that $\varphi(d_\infty) \in \bar{F}$, and the relation (2.7) proves that $\bar{F}$ does not enjoy the characteristic property (1.7) of faces. This completes the proof.

3. Closed faces and sets of determinacy.

We have noted in Section 1 that the faces of K are the traces on K of supporting affine manifolds. Clearly the trace of a closed supporting manifold is a closed face, and it is of some interest to know if every closed face can be obtained in this way. Equivalently we may ask if every closed face F of K satisfies the requirement

$$F = K \cap (\overline{\operatorname{aff} F}), \tag{3.1}$$

where $\operatorname{aff} F$ denotes the affine hull of F.

It follows from the Hahn–Banach Theorem that $K \cap (\overline{\operatorname{aff} F})$ is equal to the set

$$\{x \in K \mid h \in \mathscr{H},\ h \equiv 0 \text{ on } F \Rightarrow h(x) = 0\}. \tag{3.2}$$

This set may naturally be termed the *set of determinacy* by F with respect to the function space $\mathscr{H}$, and the problem is to decide if the concept of a closed face and of a (stable) set of $\mathscr{H}$-determinacy will coalesce. Clearly the answer is affirmative for finite dimensional (Hausdorff) spaces where every affine subspace is closed. In the present paragraph we shall show that the answer is negative in general, even for r-simplexes in Banach-space, but we shall prove that the question has a positive solution if $\mathscr{H}$ is replaced by $\tilde{\mathscr{H}}$. Thus the concept of a closed face and of a (stable) set of $\tilde{\mathscr{H}}$-determinacy will in fact coalesce.

LEMMA. *There exists an infinite dimensional (compact) r-simplex K in the Banach-space l^1 such that $K = \operatorname{face}_K(0)$.*

PROOF. We have seen in Section 1 that the r-simplex Δ_σ may be interpreted as the positive part of the unit ball in l^1. To make it norm compact, we apply the linear transformation

$$\psi(\{\alpha_n\}_{n\in N}) = \{2^{-n}\alpha_n\}_{n\in N}. \tag{3.3}$$

Clearly ψ is an isomorphism of l^1 into itself, and it is continuous from the Tykhonov topology on Δ_σ to the norm topology. In particular, it maps Δ_σ onto a norm compact r-simplex in l^1.

Let $x = \{\xi_n\}_{n\in N}$ be some point of Δ_σ such that $\xi_n \neq 0$ for all $n \in N$. It is not hard to verify that for any $a \in \Delta_\sigma$ the line segment $[a,x]$ can be extended beyond x within Δ_σ. Hence the face of x in Δ_σ is all of Δ_σ. Clearly, the point $\psi(x)$ has the corresponding property relatively to

$\psi(\Delta_\sigma)$. Translation yields the norm compact r-simplex $K=\psi(\Delta_\sigma)-\psi(x)$ with the desired property.

THEOREM 2. *There exists an r-simplex K in the Banach-space l^1 possessing a closed face F such that*

$$F \neq K \cap (\overline{\operatorname{aff} F}) . \tag{3.4}$$

PROOF. Let K_0 be an infinite dimensional (compact) r-simplex in l^1 such that $K_0=\operatorname{face}_{K_0}(0)$. Then the affine span M of K_0 is given by

$$M = \bigcup_{n=1}^{\infty} nK_0 . \tag{3.5}$$

By the Baire Theorem, M is non-closed, for otherwise the compact set K_0 would have an interior point contrary to the infinite dimensionality.

Let $x \in \bar{M}\setminus M$, and define

$$K = \overline{\operatorname{conv}(x, K_0)} . \tag{3.6}$$

By a known theorem, K is compact. We claim that K is an r-simplex, and that K_0 is a (closed) face of K.

Using the definition of K and the fact that $x \notin M$, one may prove that $K_0=M\cap K$, and that $K\setminus M$ is convex. Hence K_0 is a face of K (cf. (1.6)).

Since K_0 is a face of K, every extreme point of K_0 is also extreme in K (cf. Proposition 2). It is easily verified that the point x is also extreme in K and that no other point of K can be extreme. Hence

$$\partial_e K = \partial_e K_0 \cup \{x\} . \tag{3.7}$$

By assumption $\partial_e K_0$ is closed, and hence $\partial_e K$ is closed as well.

Let y be an arbitrary point of K. It follows from the fact that $x \notin M$, that y has a *unique* decomposition

$$y = \lambda x+(1-\lambda)z, \qquad z \in K_0, \quad 0 \leqq \lambda \leqq 1 . \tag{3.8}$$

Let $\mu \in \mathfrak{M}_1{}^+(K)$ be a measure supported by $\partial_e K$ and with barycenter y. Since x is an isolated point of $\partial_e K$, there is a $\lambda' \in [0,1]$ and a measure $\mu' \in \mathfrak{M}_1{}^+(K)$ supported by $\partial_e K_0$ such that

$$\mu = (1-\lambda')\mu'+\lambda' \varepsilon_x . \tag{3.9}$$

Denoting the barycenter of μ' by z', one has $z' \in K_0$, and

$$y = (1-\lambda')z'+\lambda' x . \tag{3.10}$$

By the uniqueness of the decomposition (3.8), $\lambda = \lambda'$, and $z = z'$. Since K_0 is an r-simplex, there is a *unique* measure in $\mathfrak{M}_1^+(K_0)$ which is supported by $\partial_e K_0$ and has barycenter z. Hence the measure μ' of (3.9) is uniquely determined, and so is μ. Thus we have verified that K is an r-simplex.

By definition, $x \in \bar{M} \setminus M$, and hence

$$x \in K \cap \overline{(\operatorname{aff} K_0)} \setminus K_0 . \tag{3.11}$$

Thus $F = K_0$ is a closed face of K with the desired property (3.4).

THEOREM 3. *Every closed face F of an r-simplex K is its own set of determinacy with respect to the function space $\tilde{\mathscr{H}}$, that is, for every $x \in K \setminus F$ there exists an $h \in \tilde{\mathscr{H}}$ such that $h \equiv 0$ on F and $h(x) \neq 0$. The corresponding statement is inexact with $\mathscr{H}$ in the place of $\tilde{\mathscr{H}}$.*

PROOF. 1) For every $y \in K$ let μ_y be the (unique) measure in $\mathfrak{M}_1^+(K)$ which is supported by the closed set $\partial_e K$ and has barycenter y. If $x \in K \setminus F$, then $\operatorname{Spt}(\mu_x) \not\subset F$; and by the definition of support, there is an $f \in \mathscr{C}(\partial_e K)$ such that $f \equiv 0$ on $\partial_e F = F \cap \partial_e K$ (cf. Proposition 2) and such that $\int f \, d\mu_x \neq 0$.

By a theorem of H. Bauer [1] there exists a function $h \in \tilde{\mathscr{H}}$ which extends f. To fix the ideas, we recall that

$$h(y) = \int f \, d\mu_y, \qquad y \in K . \tag{3.12}$$

Here, h is continuous and affine by virtue of the vague compactness of $\mathfrak{M}_1^+(\partial_e K)$ and by the continuity and linearity of the mapping $\mu \to \int t \, d\mu(t)$.

By assumption, $h \equiv 0$ on $\partial_e F$. A known maximum principle (based on the Krein–Milman Theorem) implies that $h \equiv 0$ on all of $F = \overline{\operatorname{conv} \partial_e F}$. However, by (3.12)

$$h(x) = \int f \, d\mu_x \neq 0 . \tag{3.13}$$

2) The last statement follows immediately from Theorem 2, since $h \in \mathscr{H}$, $h \equiv 0$ on F and $x \in \overline{\operatorname{aff} F}$ implies $h(x) = 0$.

4. Existence of irreducible polyhedra.

A classical theorem of Carathéodory states that every point x of a convex compact set in $\mathbf{R}^n$ can be expressed as a convex combination of (at most $n+1$) affinely independent extreme points. (cf. e.g. [9]).

In the terminology of the present paper this may be rephrased as follows: *Every convex compact set K in $\mathbf{R}^n$ can be decomposed as*

$$K = \bigcup \{\Delta \mid \Delta \text{ simplex}, \ \partial_e \Delta \subset \partial_e K\} . \tag{4.1}$$

We shall see that the corresponding statement is inexact in general, even for polyhedra. In fact one has the following:

THEOREM 4. *There exists a non-simplicial compact polyhedron K with a point x which is contained in no proper subset A such that*

(i) *A is closed and convex,*
(ii) *$\partial_e A \subset \partial_e K$.*

PROOF. We consider the r-simplex Δ_σ, which we take to be the positive part of the closed unit ball of l^1 (cf. § 1).

In the space l^1 one can easily find two vectors $b = \{\beta_n\}_{n \in N}$ and $x_0 = \{\xi_n\}_{n \in N}$ such that

$$\sum_{n=1}^{\infty} \beta_n = 0, \qquad \beta_n \neq 0 \quad \text{for all } n \in N , \tag{4.2}$$

$$\sum_{n=1}^{\infty} \xi_n < 1, \qquad \xi_n > 0 \quad \text{for all } n \in N , \tag{4.3}$$

$$\lim_{n \to \infty} (\xi_n / \beta_n)^+ = \lim_{n \to \infty} (\xi_n / \beta_n)^- = 0 . \tag{4.4}$$

Let M be the one-dimensional subspace generated by b, and let φ be the canonical mapping of l^1 onto $E = l^1/M$. We claim that the set $K = \varphi(\Delta_\sigma)$ and the point $x = \varphi(x_0)$ will have the desired properties stated in the Theorem.

To prove this claim, we first note that $\partial_e \Delta_\sigma$ consists of all the points $d_m = \{\delta_{m,n}\}_{n \in N}$, $m \in N$, and of the origin in l^1 (the latter corresponds to the extreme point d_∞ when Δ_σ is interpreted as a subset of $\mathsf{R}^{\bar{N}}$). Any element of the form $d_m + \alpha b$, $m \in N$, or of the form αb, has a series of negative components unless $\alpha = 0$. Hence φ maps $\partial_e \Delta_\sigma$ biuniquely into K, and

$$\varphi^{-1}\varphi(d_m) \cap \Delta_\sigma = \{d_m\} \qquad \text{for } m \in N ,$$

and $\varphi^{-1}\varphi(0) \cap \Delta_\sigma = \{0\}$. It follows by the Corollary to Proposition 3, that φ maps $\partial_e \Delta_\sigma$ homeomorphically onto $\partial_e K$. In particular $\partial_e K$ is homeomorphic to $\bar{N}$ (the one-point compactification of N).

An affine dependence on $\partial_e K$ is given by a sequence $\{\mu_n\}_{n \in \bar{N}}$ such that

$$\sum_{n=1}^{\infty} \mu_n + \mu_\infty = 0, \qquad \sum_{n=1}^{\infty} \mu_n \varphi(d_n) + \mu_\infty \varphi(0) = 0 , \tag{4.5}$$

where the first sum converges absolutely on R, and the second sum converges in the topology of K. By the continuity of φ, the last sum of (4.5) may be written

$$\varphi\left(\sum_{n=1}^{\infty}\mu_n d_n\right) = 0,$$

which is equivalent to $\sum_{n=1}^{\infty}\mu_n d_n = \lambda b$ for some $\lambda \in \mathrm{R}$, and this in turn is equivalent to

$$\mu_n = \lambda\beta_n, \qquad \text{for all } n \in N. \tag{4.6}$$

It follows that $\mathfrak{N}(K)$ is one-dimensional, and so K *is a polyhedron, but no simplex.*

Let ν be some measure in $\mathfrak{M}_1{}^+(K)$ which is supported by $\partial_e K$ and has barycenter x. Writing $\nu_n = \nu(\varphi(d_n))$ for $n \in N$, and $\nu_\infty = \nu(\varphi(0))$, one has

$$x = \sum_{n=1}^{\infty}\nu_n\varphi(d_n) + \nu_\infty\varphi(0).$$

By definition of x, and by the continuity of φ,

$$\varphi\left(x_0 - \sum_{n=1}^{\infty}\nu_n d_n\right) = 0.$$

Hence for some $\varrho \in \mathrm{R}$,

$$\xi_n - \nu_n = \varrho\beta_n, \qquad \text{for all } n \in N. \tag{4.7}$$

We claim that $\nu_n \neq 0$ for all $n \in \bar{N}$.

Summing over n, and making use of (4.2), one obtains

$$\sum_{n=1}^{\infty}\xi_n = \sum_{n=1}^{\infty}\nu_n.$$

Hence

$$\nu_\infty = 1 - \sum_{n=1}^{\infty}\nu_n = 1 - \sum_{n=1}^{\infty}\xi_n > 0.$$

Next we assume that $\nu_k = 0$ for some $k \neq \infty$. Then by (4.7), $\varrho = \xi_k/\beta_k$ and so

$$\xi_n - \frac{\xi_k}{\beta_k}\beta_n = \nu_n, \qquad \text{for all } n \in N,\ n \neq k. \tag{4.8}$$

Let $J^+ = \{n \mid n \in N, \beta_n > 0\}$, $J^- = \{n \mid n \in N, \beta_n < 0\}$, and assume first $k \in J^+$. Then the positivity of ν_n entails

$$\frac{\xi_n}{\beta_n} \geqq \frac{\xi_k}{\beta_k}, \qquad \text{for all } n \in J^+. \tag{4.9}$$

If $k \in J^-$, then

$$(4.10) \qquad \frac{\xi_n}{|\beta_n|} \geqq \frac{\xi_k}{|\beta_k|}, \qquad \text{for all } n \in J^- .$$

Either of the alternatives (4.9), (4.10) contradicts the assumption (4.4). This completes the verification that $\nu_n \neq 0$ for all $n \in \bar{N}$.

Finally, let A be some convex compact set such that $x \in A$, $\partial_e A \subset \partial_e K$. The Choquet Theorem for A yields a measure $\nu \in \mathfrak{M}_1{}^+(K)$ which is supported by $\partial_e K$, has barycenter x, and satisfies

$$(4.11) \qquad \nu(\partial_e K \setminus \partial_e A) = 0 .$$

By the above discussion $\partial_e K \setminus \partial_e A = \emptyset$, and the proof is complete.

Theorem 4 shows the impossibility of "triangulation" of general polyhedra. However, it would be of some interest to find sufficient conditions for a convex compact set K to admit a decomposition of the type (4.1). To every point x of a metrizable convex compact set with this property there would correspond a (non-unique) *subset* $A = \partial_e \Delta$ of $\partial_e K$ such that x has a *unique* representing measure μ_x (cf. (1.4)) concentrated on A.

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