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ASYMPTOTIC ESTIMATES FOR THE FINITE PREDICTOR

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1.

Let $f \geqq 0$ be in L^1 of the circle group, $\hat{f}$ its Fourier transform and $D_n(f)$ the determinant of the $(n+1)$-section of the Toeplitz matrix of f; that is

$$D_n(f) = \det\{\hat{f}(i-j)\}_{i,j=0}^n .$$

If $\mu_n = D_n/D_{n-1}$, then it is a well known theorem of G. Szegö [4, p. 44] that

$$\mu_n \to \mu = \exp\frac{1}{2\pi}\int_0^{2\pi} \log f(\theta)\, d\theta ,$$

where the right hand side is to be interpreted as zero if $\log f$ is not summable. It is important to be able to estimate the rate of convergence of μ_n in terms of smoothness properties of f. Various results along this line may be found in work by G. Baxter [2], U. Grenander and M. Rosenblatt [3], U. Grenander and G. Szegö [4, § 10.10], and I. I. Hirschman, Jr. [5]. It is the purpose of this paper to give some results of a general nature which when specialized will yield the results of the above mentioned authors.

2.

Let us begin by recalling some well known facts. A more complete discussion with proofs may be found in [4]. Throughout our discussion we shall always suppose that f is a non-negative summable function with $\log f$ also summable.

We may write $f = |g|^2$ where g is an outer factor in H^2. This means in particular that we can take $\hat{g}(0) > 0$ and if $1/f \in L^1$ then $1/g \in H^2$. Also, it turns out that $\mu = \hat{g}(0)^2$.

The quantity μ_n is given by

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$$\mu_n = \min \frac{1}{2\pi} \int_0^{2\pi} |p|^2 f \, d\theta \,, \tag{1}$$

where the minimum is taken over all n^{th} degree polynomials,

$$p(\theta) = \sum_0^n \hat{p}(k)\, e^{ik\theta}, \qquad \text{with} \quad \hat{p}(0) = 1 \,.$$

If u_n is the minimizing polynomial, then u_n may be characterized as that unique n^{th} degree polynomial with $\hat{u}_n(0) = 1$ for which

$$\int_0^{2\pi} u_n(\theta) e^{-ik\theta} f(\theta)\, d\theta = 0, \qquad 1 \leqq k \leqq n \,. \tag{2}$$

There is another way to characterize u_n which will be important for what follows. If we set $v_n = u_n/\mu_n$ then we claim that

$$\int_0^{2\pi} |1 - \hat{g}(0) v_n g|^2 \, d\theta = \min \int_0^{2\pi} |1 - pg|^2 \, d\theta \,, \tag{3}$$

where the minimum is taken over all n^{th} degree polynomials

$$p(\theta) = \sum_0^n \hat{p}(k)\, e^{ik\theta} \,.$$

Indeed, the unique minimizing polynomial h is characterized by the fact that

$$\frac{1}{2\pi} \int_0^{2\pi} \{1 - hg\} e^{-ik\theta} \bar{g} \, d\theta = 0, \qquad 0 \leqq k \leqq n \,.$$

It is not hard to check that $\hat{g}(0) v_n$ is the polynomial with this property.

Finally we note that

$$1 - \mu/\mu_n = \frac{1}{2\pi} \int_0^{2\pi} |1 - \hat{g}(0) v_n g|^2 \, d\theta \,, \tag{4}$$

which can be checked by a direct computation.

3.

Our object in this section is to prove the following:

THEOREM 1. (a) *If $1/f \in L^1$ and h is any positive trigonometric polynomial of degree n with $h \geqq \gamma > 0$, then*

(5) $$1/\mu - 1/\mu_n \leqq \frac{\nu}{2\pi\gamma}\int_0^{2\pi} |1/f - h|^2 f\, d\theta\,,$$

where

$$\nu = \exp\frac{1}{2\pi}\int_0^{2\pi} \log h\, d\theta\,.$$

(b) *If* $f \geqq \alpha > 0$ *then*

(6) $$\alpha \sum_{k>n} |(1/g)\hat{}(k)|^2 \leqq 1 - \mu/\mu_n\,.$$

(c) *If* $0<\alpha \leqq f \leqq \beta < \infty$ *and if* $s_n = \sum_0^n (1/g)\hat{}(k) e^{ik\theta}$ *with* $|s_n|^2 \leqq \gamma < \infty$ *for all* n, *then*

(7) $$\frac{\alpha^2}{2(1+\beta\gamma)} \sum_{|k|>n} |(1/f)\hat{}(k)|^2 \leqq 1 - \mu/\mu_n\,.$$

Proof. To prove (a) we first write $h = |p|^2$, where $p(\theta) = \sum_0^n \hat{p}(k) e^{ik\theta}$, $\hat{p}(0) > 0$ and $1/p \in H^2$. This is just the well known Fejér–Riesz theorem on the factorization of non-negative trigonometric polynomials. Hence, recalling that $f = |g|^2$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} |1/f - h|^2 f\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |p|^2 |1/\overline{pg} - pg|^2\, d\theta$$
$$\geqq \frac{\gamma}{2\pi}\int_0^{2\pi} |1/\overline{pg} - 1/\widehat{pg}(0) + 1/\widehat{pg}(0) - pg|^2\, d\theta\,.$$

Since $1/(pg) \in H^2$, it follows that $1/\overline{pg} - 1/\widehat{pg}(0)$ is orthogonal to $1/\widehat{pg}(0) - pg$. Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} |1/f - h|^2\, d\theta \geqq \frac{\gamma}{2\pi}\int_0^{2\pi} |1/\widehat{pg}(0) - pg|^2\, d\theta$$
$$\geqq \frac{\gamma}{2\pi\nu\mu}\int_0^{2\pi} |1 - \widehat{pg}(0) pg|^2\, d\theta$$
$$\geqq \frac{\gamma}{2\pi\nu\mu}\int_0^{2\pi} |1 - \hat{g}(0) v_n g|^2\, d\theta = [1/\mu - 1/\mu_n]\gamma/\nu\,.$$

This gives our inequality in (a).

To prove (b) we have simply

$$1 - \mu/\mu_n = \frac{1}{2\pi}\int_0^{2\pi} |1 - \hat{g}(0)v_n g|^2\, d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} |g|^2 |1/g - \hat{g}(0)v_n|^2\, d\theta \geqq \alpha \sum_{k>n} |(1/g)\hat{\ }(k)|^2 .$$

Finally, to prove (c) we have

$$\sum_{|k|>n} |(1/f)\hat{\ }(k)|^2 \leqq \frac{1}{2\pi}\int_0^{2\pi} |1/f - |s_n|^2|^2\, d\theta$$
$$\leqq \frac{1}{2\pi\alpha}\int_0^{2\pi} |1/f - |s_n|^2|^2 f\, d\theta \leqq \frac{1}{2\pi\alpha}\int_0^{2\pi} |1/\bar{g} - |s_n|^2 g|^2\, d\theta$$
$$\leqq \frac{1}{\pi\alpha}\left\{\int_0^{2\pi} |1/\bar{g} - \overline{s_n}|^2\, d\theta + \int_0^{2\pi} |gs_n|^2 |1/g - s_n|^2\, d\theta\right\}$$
$$\leqq \frac{1+\beta\gamma}{\pi\alpha}\int_0^{2\pi} |1/g - \hat{g}(0)v_n|^2\, d\theta$$
$$\leqq \frac{1+\beta\gamma}{\pi\alpha^2}\int_0^{2\pi} |1/g - \hat{g}(0)v_n|^2 f\, d\theta = (1 - \mu/\mu_n)\, 2(1+\beta\gamma)/\alpha^2 .$$

4.

We now want to indicate how we can use the previous elementary estimates to obtain most of the results of the previously mentioned authors. We start with the sufficiency part of a result of Grenander and Rosenblatt [3] (see also [4; § 10.10]).

If f has no zeros and its periodic extension is real analytic, then

$$\delta_n = \mu_n - \mu = O(\varrho^n) ,$$

where $0 \leqq \varrho < 1$.

Since f has no zeros, the periodic extension of $1/f$ is analytic. This means that $1/f$ may be considered to be an analytic function on the circle. Indeed, let $\log w$ be any determination of the logarithm function and set $F(w) = 1/f(-i \log w)$ for $w = e^{i\theta}$. The analyticity of $1/f$ implies that F may be extended to be analytic in an open annulus $\{z\colon \varrho < |z| < 1/\varrho\}$ with $0 \leqq \varrho < 1$. We can expand F in a Laurent expansion about zero to get

$$F(z) = \sum_{n=-\infty}^{\infty} (1/f)\hat{\ }(n)\, z^n ,$$

where

$$F_1(z) = \sum_{n=-\infty}^{0} (1/f)^\wedge(n)\, z^n \quad \text{converges for } |z| > \varrho\,,$$

$$F_2(z) = \sum_{n=0}^{\infty} (1/f)^\wedge(n)\, z^n \quad \text{converges for } |z| < 1/\varrho\,.$$

It follows from this that the symmetric partial sums of $1/f$ converge uniformly to $1/f$ and we may apply theorem 1 (a) to get

$$\delta_n = \mu_n - \mu = O\Big(\sum_{|k|>n} |(1/f)^\wedge(k)|^2\Big)\,.$$

But $(1/f)^\wedge(k) = O(\varrho^k)$ implies $\delta_n = O(\varrho^n)$.

Our second example is a result due to Baxter [2].

If $f > 0$ *and* $\sum |\hat{f}(k)|\,|k|^\lambda < \infty$, $\lambda \geqq 0$, *then* $\delta_n = o(n^{-2\lambda})$.

For fixed λ the functions in this class form a Banach algebra with spectrum the unit circle and hence $f > 0$ implies that $1/f$ is in this algebra. If n is sufficiently large we get

$$n^{2\lambda} \sum_{|k|>n} |(1/f)^\wedge(k)|^2 \leqq \sum_{|k|>n} |k|^{2\lambda}\, |(1/f)^\wedge(k)|^2 \leqq \sum_{|k|>n} |k|^\lambda\, |(1/f)^\wedge(k)|\,.$$

The result is now an immediate consequence of theorem 1 (a).

Finally we give an example of a result due to I. I. Hirschman, Jr. [5].

If $0 < \alpha \leqq f \leqq \beta < \infty$ *and* $\lambda > 1$, *then* $\delta_n = o(n^{-\lambda})$ *if and only if*

$$n^\lambda \sum_{|k| \geqq n} |\hat{f}(k)|^2 = o(1)\,.$$

The functions of this class form a subalgebra of the class of summable Fourier series. Under a suitable norm they form a Banach algebra with spectrum the unit circle [5]. Hence the result now follows from Theorem 1, since f is in the algebra if and only if $1/f$ is in the algebra.

REMARKS. (a) *Let* $s_n = \sum_0^n (1/g)^\wedge(k)\, e^{ik\theta}$; *then under the hypothesis of Theorem* 1 (b) *we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |s_n - \hat{g}(0) v_n|^2\, d\theta = O(\delta_n)\,.$$

Indeed, from (4) we may write

$$\{1 - \mu/\mu_n\}^{\frac12} = \left\{\frac{1}{2\pi} \int_0^{2\pi} |g|^2\, |1/g - \hat{g}(0) v_n|^2\, d\theta\right\}^{\frac12}$$

$$\geqq \alpha^{\frac12} \left\{\frac{1}{2\pi} \int_0^{2\pi} |s_n - \hat{g}(0) v_n|^2\, d\theta\right\}^{\frac12} - \alpha^{\frac12} \left\{\sum_{k>n} |(1/g)^\wedge(k)|^2\right\}^{\frac12}.$$

Apply Theorem 1 (b) and we have our result. This result was obtained in special cases by Baxter [2] and Hirschman [5].

(b) If $f=|g|^2$, g outer in H^2, it is in general an undecided question as to which smoothness properties of f carry over to g. Our Theorem 1 sheds a small amount of light on this problem. For example, if $f\in \mathrm{Lip}(\lambda,2)$, then it is a well known result [1; p. 171] that

$$\sum_{|k|\geqq n} |\hat{f}(k)|^2 = O(n^{-2\lambda}) .$$

The converse is also true. Hence if $0<\alpha\leqq f\leqq \beta<\infty$ and if s_n are the partial sums of the Fourier expansion of f with $s_n\geqq\gamma>0$, then an application of Theorem 1 (a) tells us $\delta_n(1/f)=O(n^{-2\lambda})$ which in turn, by Theorem 1 (b) tells us that

$$\sum_{k\geqq n} |\hat{g}(k)|^2 = O(n^{-2\lambda}) .$$

This means we also have $g\in \mathrm{Lip}(\lambda,2)$.

5.

We would now like to sharpen and complete the results we have previously obtained. We shall show that if $\delta_n=\mu_n-\mu$ goes to zero sufficiently rapidly, then f^{-1} has a summable Fourier series. Specifically we shall prove the following:

THEOREM 2. *If* $\sum_{k=0}^{\infty}\{2^k\delta_{2k}\}^{\frac{1}{2}}<\infty$ *and* $\log f$ *is summable, then* $1/f$ *has a summable Fourier series.*

Roughly speaking this result says that, unless f has no zeros and is very smooth most of the time, then δ_n cannot go to zero very much faster than $1/n$. This is to be compared with the results of the next section. Note that if $f\geqq\alpha>0$, then theorem 2 is an immediate consequence of theorem 1 (b). Indeed we get

$$\sum_{2^n+1}^{2^{n+1}} |(1/g)^\wedge(k)| \leqq 2^{\frac{1}{2}n}\left[\sum_{2^n+1}^{2^{n+1}} |(1/g)^\wedge(k)|^2\right]^{\frac{1}{2}} \leqq \alpha^{-1}2^{\frac{1}{2}n}\delta_{2n}^{\frac{1}{2}} .$$

Summing both sides over n we get the result.

In case f is bounded above, it is not hard to show that theorem 2 is a consequence of Theorem 1.1 of Baxter [2a]. Indeed, the general case can be obtained by an application of an idea developed in this same paper [2a]. This was pointed out to us by I. I. Hirschman.

For the sake of completeness we shall briefly review this material (see also [5]). Let H be the Hilbert space generated by the one-sided trigonometric polynomials $p(\theta)=\sum_o^m \hat{p}(k)e^{ik\theta}$ in the L^2 norm given by the measure $fd\theta$, and let H_n be the subspace generated by polynomials of degree n. Using the notation of our previous sections, we find that the polynomial $v_n - v_{n-1}$ is in the one-dimensional space $H_n \ominus H_{n-1}$. It is a simple matter to check that the polynomial $e^{in\theta}\bar{v}_n$ also is in this latter space. Hence, there is a constant α_n so that

$$v_n - v_{n-1} = \alpha_n e^{in\theta} \bar{v}_n \,, \tag{8}$$

and therefore

$$v_n = \sum_{k=0}^{n} \alpha_k e^{ik\theta} \bar{v}_k \,.$$

Now, the polynomials $\mu_n^{\frac{1}{2}} e^{in\theta}\bar{v}_n$ are the orthonormal Szegö polynomials associated with f and hence from (3) and (4) we get

$$1 - \mu/\mu_n = \mu \sum_{k=n+1}^{\infty} |\alpha_k|^2/\mu_k \,. \tag{9}$$

Proof of theorem 2. Let $\|\cdot\|_1$ be the l^1 norm; i.e. for any function h with summable Fourier series we write

$$\|h\|_1 = \sum_{-\infty}^{\infty} |\hat{h}(k)| \,.$$

From (8) we get $\|v_n\|_1 - \|v_{n-1}\|_1 \leqq |\alpha_n|\,\|v_n\|_1 \leqq \|v_n\|_1 + \|v_{n-1}\|_1$. Consequently,

$$\big|1 - |\alpha_n|\big|\, \|v_n\|_1 \leqq \|v_{n-1}\|_1 \,,$$

and repeated iteration of this result gives

$$\left|\prod_{m+1}^{n} (1 - |\alpha_k|)\right| \|v_n\|_1 \leqq \|v_m\|_1 \,.$$

From the fact that $\sum \{2^k \delta_{2k}\}^{\frac{1}{2}} \leqq \infty$, it follows from (9), using exactly the same kind of argument as used after the statement of theorem 2, that $\sum_1^\infty |\alpha_k| < \infty$. Therefore $\prod_1^n (1 - |\alpha_k|)$ converges and the sequence $\{\|v_n\|_1\}$ is uniformly bounded by a constant C.

Returning to (8) we see that

$$\|v_{n+p} - v_n\|_1 \leqq C \sum_{n+1}^{n+p} |\alpha_k|$$

and hence $\{v_n\}$ is Cauchy in the l^1 norm. From (4) it follows immediately that the limit of this sequence in the l^1 norm is $[\bar{g}(0) g]^{-1}$. Therefore, $1/g$ and hence $1/f = 1/|g|^2$ have summable Fourier series.

Remarks (a) The result we have just obtained contains the necessity part of the Grenander–Rosenblatt result. Indeed, if $\delta_n = O(\varrho^n)$, $0 \leqq \varrho < 1$, then f is bounded away from zero and we may apply theorem 1 (b) to show that $(1/g)\hat{}(n) = O(\varrho^n)$. This, in turn, shows that the periodic extension of $1/g$ is an analytic function of θ and hence the periodic extension of $1/f = 1/|g|^2$ is an analytic function of θ. Now, $1/f$ can have no zeros since this would preclude the possibility of f being summable. Hence f is real analytic with no zeros.

(b) The statement $\sum_0^\infty \{2^k \delta_{2^k}\}^{\frac{1}{2}} < \infty$ is clearly equivalent with the statement $\sum_0^\infty \{\delta_n/n\}^{\frac{1}{2}} < \infty$. Following the lead of Hirschman [5], it is natural to conjecture that the class of functions which satisfy the condition

$$\|h\|_2 = \sum_{n=0}^{\infty} \left\{2^n \sum_{|k| \geq 2^n} |\hat{h}(k)|^2\right\}^{\frac{1}{2}} < \infty$$

is a Banach algebra under the norm

$$\|h\| = c\{\|h\|_1 + \|h\|_2\} ,$$

where c is a suitably chosen constant and $\|\cdot\|_1$ is the l^1 norm. This is indeed the case and moreover, the spectrum of this algebra is the unit circle. The proof requires only a slight modification of the proof given by Hirschman in a special case. It is easy to see that one gets an equivalent norm by taking

$$\|h\|_2 = \sum_{n=1}^{\infty} \left\{n^{-1} \sum_{|k| \geq n} |\hat{h}(k)|^2\right\}^{\frac{1}{2}} .$$

One interest in knowing that we have a Banach algebra stems from the possibility of being able to get asymptotic estimates for δ_n in terms of f rather than in terms of $1/f$.

6.

It was pointed out by Grenander and Rosenblatt [3] that if f has zeros, then in general we cannot expect δ_n to go to zero faster than $1/n$. As they pointed out, a function for which δ_n goes to zero at precisely this rate is $f(\theta) = |1 - e^{i\theta}|^2$. It is the purpose of this section to generalize their results. If for any non-negative f with $\log f$ summable we set $\partial_n(f) = 1 - \mu/\mu_n$ then we have the following:

Theorem 3. *If $f = f_1|e^{i\theta} - 1|^2$, where f and f_1 are non-negative and summable and $\log f_1$ is summable, then*

$$\partial_n^{\frac{1}{2}}(f) \leq (1/r)^{\frac{1}{2}} + 2\partial_s^{\frac{1}{2}}(f_1), \qquad r + s = n . \tag{10}$$

Proof. Let $f = |g|^2$ and $f_1 = |g_1|^2$ where g and g_1 are outer factors in H^2. From (4) we have

$$\partial_n(f) = \frac{1}{2\pi}\int_0^{2\pi} |1-\hat{g}(0)v_n g|^2\, d\theta\,.$$

Let w_s be the polynomial such that

$$\partial_s(f_1) = \frac{1}{2\pi}\int_0^{2\pi} |1-\hat{g}_1(0)w_s g_1|^2\, d\theta\,,$$

and set

$$p_r(\theta) = \sum_{k=0}^{r-1}(1-k/r)e^{ik\theta}\,.$$

From (3) it follows that v_n has a minimizing property and hence

$$\partial_n^{\frac{1}{2}}(f) \leqq \left\{\frac{1}{2\pi}\int_0^{2\pi} |1-p_r(1-e^{i\theta})\hat{g}_1(0)w_s g_1|^2\, d\theta\right\}^{\frac{1}{2}}$$

(11)
$$\leqq \left\{\frac{1}{2\pi}\int_0^{2\pi} |1-p_r(1-e^{i\theta})|^2\, d\theta\right\}^{\frac{1}{2}} +$$

$$+\frac{1}{2\pi}\left\{\int_0^{2\pi} |p_r(1-e^{i\theta})|^2\,|1-\hat{g}_1(0)w_s g_1|^2\, d\theta\right\}^{\frac{1}{2}}\,.$$

Now, it is easily computed that

$$p_r(1-e^{i\theta}) = 1-(1/r)\sum_{k=1}^{r} e^{ik\theta}\,.$$

Therefore,

$$|p_r(1-e^{i\theta})| \leqq 2\,,$$

$$\frac{1}{2\pi}\int_0^{2\pi} |1-p_r(1-e^{i\theta})|^2\, d\theta = 1/r\,.$$

If we use these estimates in (11) we get our result.

COROLLARY. *If* $f=f_1\prod_{j=1}^{k}|e^{i\theta}-e^{i\theta_j}|^{2\lambda_j}$, *and* $\lambda=\sum\lambda_j$, *then*

$$\delta_n(f) = O\big(1/n+\delta_{[n/2^\lambda]}(f_1)\big)\,.$$

This is obtained by iterating (10) λ times. At the first stage choose $r=[(n+1)/2]$ and $s=[n/2]$, say, and then continue in this way with $\partial_{[n/2]}$.

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